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The nonclassical mixed domination Ramsey numbers

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Abstract

The nonclassical mixed domination Ramsey number $v(m, G)$ is the smallest integer p such that in every 2-coloring of the edges of K_p with color red and blue, either $\Gamma(B) \geq m$ or there exists a blue copy of graph G, where B is the subgraph of K_p induced by blue edges. $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of a graph G. We give exact values for numbers $v(m, K_3 - e)$, $v(3, P_m)$, $v(3, C_m)$. In addition, we give exact values and bounds for numbers $v(3, K_n - e)$, where $n \geq 3$.

1 Introduction

Our notation comes from papers [3] and [4]. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ of order $p = |V(G)|$ and edge set $E(G)$. If v is a vertex in $V(G)$, then the open neighborhood of v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood of v is $N_G[v] = \{v\} \cup N_G(v)$. The open neighborhood of a set S of vertices is $N_G(S) = \bigcup_{v \in S} N_G(v)$, and the closed neighborhood is $N_G[S] = N_G(v) \cup S$.

A set $S \subseteq V(G)$ is an *independent set* in G if no two vertices of S are adjacent in G. A set $S \subseteq V(G)$ is an *irredundant set* if for each $s \in S$ there is a vertex w in G such that $N_G[w] \cap S = \{s\}$. A set $S \subseteq V(G)$ is a *dominating set* in S if each vertex v of G belongs to S or is adjacent to some vertex in S.

If S is an irredundant set in G and $v \in S$, the set $N[v] - N[S - \{v\}]$ is nonempty and is called the set of *private neighbors of* v in G (relative to S), denoted by $pn_G(v, S)$ or simply by $pn(v, S)$. The *independence number* of G, denoted by $\beta(G)$, is the maximum cardinality among all independent sets of vertices of G. The *upper irredundance number* of G, denoted by $IR(G)$, is the maximum cardinality of an irredundant set of G. The *upper domination number* of G, denoted by $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of G. A minimal dominating set of cardinality $\Gamma(G)$ is called a $\Gamma(G)$-set. Similarly, an irredundant set of cardinality $IR(G)$ is called an $IR(G)$-set.

It is apparent that any independent set is also irredundant. Since every minimal dominating set is an irredundant set, we have $\Gamma(G) \leq IR(G)$ for every graph G. Futhermore, since every maximum independent set is also a dominating set, we have $\beta(G) \leq \Gamma(G)$ for every graph G. Hence the parameters $\beta(G)$, $\Gamma(G)$, $IR(G)$ are related by the following inequalities, which were observed by Cockayne and Hedetniemi.

Theorem 1 *For every graph G, $\beta(G) \leq \Gamma(G) \leq IR(G)$.*

Let $G_1, G_2, \ldots, G_t$ be an arbitrary t-edge coloring of K_n, where for each $i \in \{1, 2, \ldots, t\}$, G_i is the spanning subgraph of K_n whose edges are colored with color i. The classical *Ramsey number* $r(n_1, n_2, \ldots, n_t)$ is the smallest value of n such that for every t-edge coloring $G_1, G_2, \ldots, G_t$ of K_n, there is an $i \in \{1, 2, \ldots, t\}$ for which $\beta(\overline{G_i}) \geq n_i$, where $\overline{G}$ is the complement of G. The *irredundant Ramsey number* denoted by $s(n_1, n_2, \ldots, n_t)$, is the smallest n such that for every t-edge coloring $G_1, G_2, \ldots, G_t$ of K_n, there is at least one $i \in \{1, 2, \ldots, t\}$ for which $IR(\overline{G_i}) \geq n_i$. Since any independent set is irredundant, the irredundant Ramsey numbers exist by Ramsey's theorem and they satisfy $s(n_1, n_2, \ldots, n_t) \leq r(n_1, n_2, \ldots, n_t)$ for all n_i, where $i = 1, 2, \ldots, t$. The *upper domination Ramsey number* $u(n_1, n_2, \ldots, n_t)$ is defined as the smallest n such that for every t-edge coloring $G_1, G_2, \ldots, G_t$ of K_n, there is at least one $i \in \{1, 2, \ldots, t\}$ for which $\Gamma(\overline{G_i}) \geq n_i$.

In the case where $t = 2$, $r(m, n)$ is the smallest integer p such that for every 2-coloring of the edges of K_p with colors red (R) and blue (B), $\beta(B) \geq m$ or $\beta(R) \geq n$. Similarly, the irredundant Ramsey number $s(m, n)$ is the smallest integer p such that in every 2-coloring of the edges of K_p with colors red (R) and blue (B), satisfies $IR(B) \geq m$ or $IR(R) \geq n$. The upper domination Ramsey number $u(m, n)$ is the smallest integer p such that in every 2-coloring of the edges of K_p with colors red (R) and blue (B), satisfies $\Gamma(B) \geq m$ or $\Gamma(R) \geq n$. Henning and Oellermann [4] defined another kind of domination Ramsey number — *mixed domination Ramsey number* denoted by $v(m, n)$. The mixed domination Ramsey number $v(m, n)$ is the smallest integer p such that in every 2-coloring the edges of K_p with color red and blue, either $\Gamma(B) \geq m$ or $\beta(R) \geq n$.

It follows from Theorem 1, that for all m, n,

$$s(m, n) \leq u(m, n) \leq v(m, n) \leq r(m, n).$$

The *nonclassical Ramsey number* $r(m, G)$ is the smallest integer p such that in every 2-coloring the edges of K_p with color red (R) and blue (B), either $\beta(B) \geq m$ or there exists a blue copy of G. Now we introduce a new mixed Ramsey number. We define the *nonclassical mixed domination Ramsey number* $v(m, G)$ to be smallest integer p such that in every 2-coloring the edges of K_p with colors red (R) and blue (B), either $\Gamma(B) \geq m$ or there exists a blue copy of G. The nonclassical Ramsey number is the upper bound for the nonclassical mixed domination Ramsey number, so we have that

$$v(m, G) \leq r(m, G).$$

The next definitions will be very useful in further considerations. A 2-coloring of K_n is called a $(m, G; n)$-coloring, if $\Gamma(B) < m$ and it does not contain a blue copy of G. A $(m, G; n)$-coloring is said to be critical if $n = v(m, G) - 1$. Finally, a 2-coloring of K_n is called a $(G_1, G_2; n)$-coloring if it contains neither a red G_1 nor a blue G_2.

2 Known results

All known exact values of mixed domination Ramsey numbers are summarized in the following Table 1 [4].

n	3	4	5	6
$v(3, n)$	6	9	12	15

Table 1: Known results of $v(3, n)$

Table 2 contains all known values (and bounds) of nonclassical Ramsey numbers $r(m, G)$, where G is the complete graph K_k minus an edge and $3 \leq k \leq 11$.

G m	$K_3 - e$	$K_4 - e$	$K_5 - e$	$K_6 - e$	$K_7 - e$	$K_8 - e$	$K_9 - e$	$K_{10} - e$	$K_{11} - e$
3	5	7	11	17	21	25	31	37-38	42-47
4	7	11	19	27-36	37-52				
5	9	16	30-34	43-67	$\leq$ 112				
6	11	21	37-55	$\leq$ 119	$\leq$ 205				
7	13	28-34	51-88	$\leq$ 204					

Table 2: Known results of $r(m, K_k - e)$

3 Values of $v(m, G)$, where G is $K_3 - e$ or C_m, or P_m

First, we state the following:

Theorem 2 $v(m, K_3 - e) = v(m, P_3) = 2m - 1.$

Proof. Let us consider a complete graph H with $m-1$ independent blue edges (the remaining edges of H are colored with color red). The maximum cardinality of a minimal dominating set of a subgraph of H induced by blue edges is $m-1$. Since H does not contain a blue $K_3 - e$, we obtain that $v(m, K_3 - e) > 2(m-1) = 2m - 2$.

If a graph H has $2m-1$ vertices, then to avoid a blue $K_3 - e$, it contains at most $m - 1$ blue independent edges and at least one isolated vertex. For such a graph, $\Gamma(B) \geq m$. This leads us to the conclusion that if the graph has $2m-1$ vertices, then it contains a blue $K_3 - e$ or its subgraph induced by the blue edges has a minimal dominating set of order at least m. We obtain that $v(m, K_3 - e) \leq 2m - 1$ and the proof is complete. □

To prove our next results, we will use the following:

Theorem 3 (Burr et al. [2]) *$r(3, G) = 2|V(G)| - 1$ for any connected graph G on at least 4 vertices and with at most $\frac{17|V(G)|+1}{15}$ edges, in particular for $G = P_i$ and $G = C_i$, for all $i \geq 4$.*

From this theorem, we immediately obtain the following:

Corollary 4 *$r(3, C_m) = 2m - 1$, $m \geq 4$.*

Now, we can prove the next result of this section.

Theorem 5 *$v(3, C_3) = 6$ and $v(3, C_m) = 2m - 1$, where $m \geq 4$.*

Proof. Since $v(3, 3) = 6$ [4], we obtain that $v(3, C_3) = v(3, 3) = 6$. By using the inequality $v(m, G) \leq r(m, G)$ and Corollary 4, we have that $v(3, C_m) \leq 2m - 1$. The critical $(3, C_m; 2m-2)$-coloring is very simple (two complete subgraphs of order $m - 1$ colored with blue and remaining edges colored with red), so we have that $v(3, C_m) > 2m-2$. These observation leads us to the result that $v(3, C_m) = 2m-1$, where $m \geq 4$. □

Now, let us present the last theorem of this section.

Theorem 6 *$v(3, P_m) = 2m - 1$.*

Proof. By Theorem 3 and the inequality $v(3, P_m) \leq v(3, C_m) = 2m - 1$, we obtain that $v(3, P_m) \leq 2m - 1$. The critical $(3, P_m; 2m - 2)$-coloring is identical to that in the proof of Theorem 3, so we immediately obtain the result. □

4 Values of $v(3, K_n - e)$

In this section, we establish the values of $v(3, K_3 - e)$, $v(3, K_4 - e)$, $v(3, K_5 - e)$ and $v(3, K_6 - e)$.

Theorem 7 $v(3, K_3 - e) = 5$.

Proof. It is easy to see that $v(3, K_3 - e) = v(3, P_3)$ and by Theorem 6 we immediately obtain the result. □

Theorem 8 $v(3, K_4 - e) = 7$.

Proof. We know that $r(3, K_4 - e) = 7$ (see Table 2). By the inequality $v(m, G) \leq r(m, G)$, we have that $v(3, K_4 - e) \leq 7$. It is easy to show a critical $(3, K_4 - e; 6)$-coloring (analogous to that in the proof of Theorem 3), so we have that $v(3, K_4 - e) > 6$. This observations lead us to the conclusion that $v(3, K_4 - e) = 7$. □

Theorem 9 $v(3, K_5 - e) = 11$.

Proof. We know that $r(3, K_5 - e) = 11$ (see Table 2) and $v(m, G) \leq r(m, G)$, so $v(3, K_5 - e) \leq 11$. By using the critical $(3, K_5 - e; 10)$-coloring (blue edges of this coloring are presented below in Figure 1) we obtain that $v(3, K_5 - e) > 10$, so we immediately have the result.

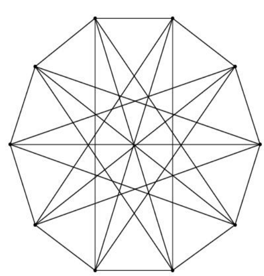

Figure 1: An example of $(3, K_5 - e; 10)$-coloring

□

Theorem 10 $v(3, K_6 - e) = 13$.

Proof. In 1990 Radziszowski [5] proved that $r(K_3, K_8 - e) = 25$ and $r(K_3, K_9 - e) = 31$. In the proof he presented all critical $(K_3, K_6 - e; n)$-colorings, where $n \leq 16$ $(r(K_3, K_6 - e) = 17)$. He gave exactly 22 nonisomorphic $(K_3, K_6 - e; 13)$-colorings. Note that for every such coloring we have that $\beta(B) < 3$ and there is no a blue $K_6 - e$. For remaining colorings of the edges of K_{13} we immediately have a red K_3 (in this case $\Gamma(B) \geq \beta(B) \geq 3$) or a blue $K_6 - e$. By simple observations we easily obtained that for every coloring of type $(K_3, K_6 - e; 13)$, we have that $\Gamma(B) \geq 3$ (see available on-line Appendix which contains these colorings and $\Gamma(B)$-sets [1]). This property establishes that $v(3, K_6 - e) \leq 13$.

Now, we prove that $v(3, K_6 - e) > 12$. All we need is to present a coloring of type $(3, K_6 - e; 12)$. We used one among 354 $(K_3, K_6 - e; 12)$-colorings which are given by Radziszowski in [5]. This coloring contains 37 blue and 29 red edges and is presented in Figure 2 (we present only blue edges of this coloring of 2-colored K_{12}).

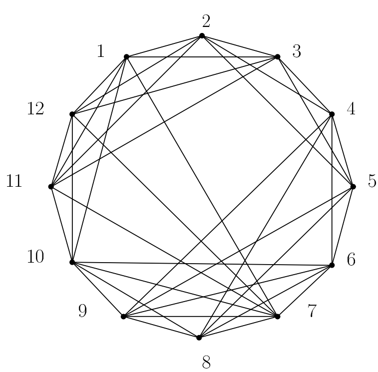

Figure 2: An example of $(3, K_6 - e; 12)$-coloring

We will prove that this is a coloring of type $(3, K_6 - e; 12)$. Since such coloring is also a $(K_3, K_6 - e; 12)$-coloring, we have that it does not contain a blue $K_6 - e$ and we know that $\beta(B) \leq 2$. Let us denote by T the minimal dominating set of the maximum cardinality of the blue subgraph of K_{12} colored as above. We have to show that $|T| \leq 2$. First, one can observe that every two nonadjacent vertices dominate all vertices of K_{12}; in the opposite case we immediately have that $\beta(B) \geq 3$, a contradiction. This means that if $|T| > 2$, then the set T contains only complete subgraphs of order 3, 4 or 5. There are three complete subgraphs of order 5 on the following sets of vertices: $\{1, 2, 3, 11, 12\}$, $\{4, 5, 6, 8, 9\}$ and $\{6, 7, 8, 9, 10\}$. It is easy to check that none of them dominate all vertices of K_{12}. Next, we have to consider all subgraphs K_4 which are not the subgraphs of these K_5's. There are 5 such subgraphs on vertices $\{1, 7, 10, 11\}$, $\{1, 7, 10, 12\}$, $\{1, 7, 11, 12\}$, $\{1, 10, 11, 12\}$, $\{2, 3, 4, 5\}$. By simple observations we checked that none of them gives us a minimal dominating set of the cardinality at least 3 because none of them dominate all vertices of K_{12}. Since all triangles are subgraphs of those K_4's and K_5's, we obtain that $|T| = 2$. This leads us to the conclusion that we have a good $(3, K_6 - e; 12)$-coloring, so $v(3, K_6 - e) > 12$

and since $v(3, K_6 - e) \leq 13$, we obtain that $v(3, K_6 - e) = 13$. This completes the proof. □

Theorem 11 *For all integers $n, m \geq 3$*

$$v(m, K_n - e) > (m-1)(n-1).$$

Proof. For every graph K_n we can give a $(m, K_n - e; (n-1)(m-1))$-coloring. Such a coloring is very simple, it contains exactly $m-1$ independent complete subgraphs on $n-1$ vertices which all edges are colored with color blue (the remaining edges of $K_{(m-1)(n-1)}$ are colored with color red). □

References

[1] Appendix, online at `http://ajc.maths.uq.edu.au/appendices/AJCvol45pp109-115Appendix.pdf` .

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