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On the size of graphs whose cycles have length divisible by a fixed integer

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Abstract

Let G be a simple graph of order n and size m which is not a tree. If $\ell \geqslant 3$ is a natural number and the length of every cycle of G is divisible by ℓ, then $m \leqslant \frac{\ell}{\ell-2}(n-2)$, and the equality holds if and only if the following hold: (i) ℓ is odd and G is a cycle of order ℓ or (ii) ℓ is even and G is a generalized θ-graph with paths of length $\frac{\ell}{2}$. It is shown that for a (0 mod ℓ)-cycle graph, $\frac{m}{n} < \frac{\ell}{\ell-1}$, if ℓ is odd, and for a given $\varepsilon > 0$, there exists a (0 mod ℓ)-cycle graph G with $\frac{m}{n} > \frac{\ell}{\ell-1} - \varepsilon$. Also $\frac{m}{n} < \frac{\ell}{\ell-2}$, if ℓ is even, and for a given $\varepsilon > 0$, there exists a (0 mod ℓ)-cycle graph G with $\frac{m}{n} > \frac{\ell}{\ell-2} - \varepsilon$.

1 Introduction

In this article we follow all definitions and terminologies of [2]. Throughout this paper all graphs are simple with no loops and no multiple edges. Let G be a graph. The set of vertices and the set of edges of G are denoted by $V(G)$ and $E(G)$, respectively. The number of vertices and the number of edges of G are called the *order* of G and the *size* of G, respectively. We denote the cycle and the complete graph of order

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n, by C_n and K_n, respectively. A graph G is said to be an *$(r$ mod $\ell)$-cycle graph* if the length of every cycle of G is r modulo of ℓ. Clearly, a graph is bipartite if and only if it is a (0 mod 2)-cycle graph. An *arc* of a graph G is a path in G whose internal vertices have degree 2 in G. We recall that an *ear* of G is a maximal arc of G. For instance for every $e \in E(C_n)$, $C_n \backslash \{e\}$ is an ear of C_n. Note that every ear of a graph G has the form uPv, where u and v are end vertices and P is a path. A *block* of G is a maximal subgraph of G which has no cut vertex. Let G be a connected graph with blocks, $B_1, \ldots, B_r$. A block B_i of G is called a *leaf block*, if $|V(B_i) \bigcap \bigcup_{j=1, j\neq i}^{r} V(B_j)| = 1$. A *generalized θ-graph*, denoted by θ_m, is a graph consisting of m internally disjoint (u, v)-paths, where $m \geqslant 2$.

(0 mod ℓ)-cycle graphs have been studied by several authors; see [1]. Let $\ell \geqslant 3$ be a natural number. In this paper we study the maximum size of a (0 mod ℓ)-cycle graph. We show that these graphs are sparse.

2 Results

The main goal of this paper is to show that for $\ell \geqslant 3$, the size of (0 mod ℓ)-cycle graphs cannot be large. More precisely, we prove that if G is a (0 mod ℓ)-cycle graph of order n and size m with odd ℓ, then $\frac{m}{n} < \frac{\ell}{\ell-1}$, and for each $\epsilon > 0$, there exists a (0 mod ℓ)-cycle graph such that $\frac{m}{n} > \frac{\ell}{\ell-1} - \epsilon$. On the other hand, if G is a (0 mod ℓ)-cycle graph and if ℓ is even, then $\frac{m}{n} < \frac{\ell}{\ell-2}$, and for each $\epsilon > 0$, there exists a (0 mod ℓ)-cycle graph such that $\frac{m}{n} > \frac{\ell}{\ell-2} - \epsilon$.

We note that for $\ell = 2$, there are (0 mod 2)-cycle graphs for which m/n can be arbitrary large (m is the size and n is the order of graph). For instance for the complete bipartite graph $K_{r,r}$, we have $\frac{m}{n} = \frac{r}{2}$.

Lemma 1. *Let G be a 2-connected (0 mod ℓ)-cycle graph with at least 3 vertices, where $\ell \geqslant 2$ is a natural number. Then the following hold:*
(i) *If ℓ is odd and $G \neq C_\ell$, then G has an arc of length $k\ell$, for some natural number k.*
(ii) *If ℓ is even, then G has an arc of length $\frac{k\ell}{2}$, for some natural number k.*

Proof. (i) If G is a cycle, then clearly the assertion holds. If G is not a cycle, then consider an ear decomposition for G; see [2, p.163]. Let uPv be the last ear in this ear decomposition. Since $G \setminus V(P)$ has an ear decomposition, by Theorem 4.2.8 of [2], $G \setminus V(P)$ is a 2-connected graph. Using Menger's Theorem [2, p.167], there are two internally disjoint paths Q and T between u and v in $G \backslash V(P)$. Suppose that uPv has length y, and Q and T have lengths x and z, respectively.

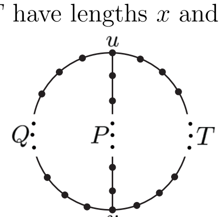

Figure 1

Since G is a $(0 \bmod \ell)$-cycle graph we have

$$x + y = y + z = x + z = 0 \pmod{\ell}. \quad (*)$$

This implies that $\ell \mid 2y$ and since ℓ is odd, $\ell \mid y$ and (i) is proved.
(ii) Similarly, the equations in $(*)$ yield $\frac{\ell}{2} \mid y$ and the proof is complete. □

Using the proof of Lemma 1 we obtain the following lemma.

Lemma 2. *Let $\ell \geq 2$ be a natural number and G be a $(0 \bmod \ell)$-cycle graph. Then the following hold:*
(i) *If G is not a cycle, then the last ear in the ear decomposition of G can be considered as the arc given in Lemma 1.*
(ii) *If $u, v \in V(G)$ and there are three internally disjoint paths of lengths x, y and z between u and v, then x, y and z are divisible by $\frac{\ell}{(\ell,2)}$, where $(\ell, 2)$ denotes the greatest common divisor of ℓ and 2.*

Lemma 3. *If ℓ is an odd number, then every 2-connected $(0 \bmod \ell)$-cycle graph, except C_ℓ contains $C_{r\ell}$ for some $r \geqslant 2$.*

Proof. If G is a cycle, then we are done. Thus assume that G is not a cycle. Now, consider an ear decomposition for G. Hence G contains a cycle C and an ear uPv for some $u, v \in V(C)$. Now, by Lemma 2, Part (ii), the proof is complete. □

Theorem 1. *Let G be a graph of order n and size m. If $\ell \geqslant 3$ is a natural number and G is a 2-connected $(0 \bmod \ell)$-cycle graph, then the following hold:*
(i) *If ℓ is odd and $G \neq C_\ell$, then $m \leqslant \frac{\ell}{\ell-1}(n-2)$. The equality holds if and only if G is a generalized θ-graph with paths of length ℓ.*
(ii) *If ℓ is even, then $m \leqslant \frac{\ell}{\ell-2}(n-2)$. The equality holds if and only if G is a generalized θ-graph with paths of length $\frac{\ell}{2}$.*

Proof. (i) We prove this part by induction on m. By Lemma 3, $C_{r\ell}$ is a subgraph of G for some $r \geqslant 2$. Thus $C_{2\ell}$ is the smallest graph which satisfies the assumption of Part (i). Thus $m \geqslant 2\ell$. Evidently, the assertion holds for $C_{2\ell}$. If G is a cycle, then we are done. Hence assume that G is not a cycle. By Lemma 2, Part (i), the length of the last ear in the ear decomposition of G is divisible by ℓ. If this ear is uPv, where P is a path, then $H_1 = G \setminus V(P)$ is a 2-connected $(0 \bmod \ell)$-cycle graph. By Lemma 2, Part (ii), $H_1 \neq C_\ell$. Now, by induction hypothesis if $|V(H_1)| = n_1$ and $|E(H_1)| = m_1$, then we have $m_1 \leqslant \frac{\ell}{\ell-1}(n_1 - 2)$. By Lemma 2, Part (ii), the length of uPv is $k\ell$, for some natural number k, and so we find

$$m \leqslant \frac{\ell}{\ell-1}(n_1 - 2) + k\ell = \frac{\ell}{\ell-1}(n_1 - 2 + k\ell - k) = \frac{\ell}{\ell-1}(n - k - 1) \leqslant \frac{\ell}{\ell-1}(n-2) \; (**)$$

and we are done. It is not hard to see that the equality holds for all generalized θ-graphs with paths of length ℓ. Now, assume that $m = \frac{\ell}{\ell-1}(n-2)$. If G is a cycle, then $G = C_{2l}$. Otherwise, since G is 2-connected, G has an ear decomposition with at least one ear, say tQw, which has length $s\ell$. Let $H_2 = G\backslash V(Q)$. If we consider the relations in $(**)$ for H_2 instead of H_1, then noting that $m = \frac{\ell}{\ell-1}(n-2)$, both inequalities are indeed equality. Therefore $s = 1$ and $m_2 = \frac{\ell}{\ell-1}(n_2 - 2)$, where $n_2 = |V(H_2)|$ and $m_2 = |E(H_2)|$. Since H_2 is a 2-connected (0 mod ℓ)-cycle graph, by induction hypothesis, H_2 is a generalized θ-graph whose paths have length ℓ. If H_2 is a cycle, then clearly we are done. Therefore one may assume that G has the following form:

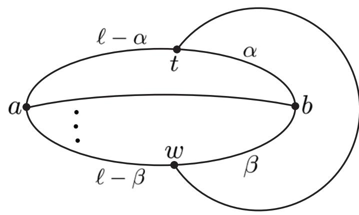

Figure 2

Noting the cycles $tQwbt$ and $wQtabw$, we have $\ell \mid \beta \pm \alpha$. This yields that $\ell \mid 2\beta$, and since ℓ is odd and $\alpha, \beta \leqslant \ell$, we have $\alpha = \ell$ and $\beta = 0$ or, $\alpha = 0$ and $\beta = \ell$. Hence G is a generalized θ-graph with paths of length ℓ, as desired.
(ii) The proof is similar to Part (i). □

Theorem 2. *Let G be a graph of order n and size m. If $\ell \geqslant 3$ is an odd natural number and G is a (0 mod ℓ)-cycle graph, then $m \leqslant \frac{\ell}{\ell-1}(n-1)$. The equality holds if and only if G is a connected graph every block of which is C_ℓ.*

Proof. First assume that G is a connected graph. We prove the theorem by induction on m. If $m = 1$, then obviously the assertion holds. Now, suppose that G is a graph and $m \geqslant 2$. If $G \neq C_\ell$ and G is a 2-connected graph then by Theorem 1, the assertion holds. If $G = C_\ell$, clearly we are done. Thus suppose that G is not a 2-connected graph. Assume that G has the following form where B is a leaf block of G.

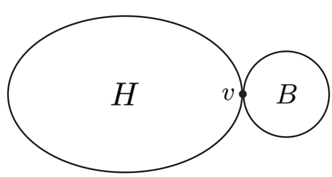

Figure 3

Let $H = G\backslash(V(B)\backslash\{v\})$. Since H is a (0 mod ℓ)-cycle graph by induction hypothesis we have $m_H \leqslant \frac{\ell}{\ell-1}(n_H - 1)$ and $m_B \leqslant \frac{\ell}{\ell-1}(n_B - 1)$, where $m_H = |E(H)|$, $n_H = |V(H)|$, $m_B = |E(B)|$, and $n_B = |V(B)|$. Thus $m \leqslant \frac{\ell}{\ell-1}(n-1)$ as desired. Now, assume that G is not a connected graph and $G_1, \ldots, G_k$ $(k \geqslant 2)$ are the connected components of G. Let $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$. We have

$$m = \sum_{i=1}^{k} m_i \leqslant \sum_{i=1}^{k} \frac{\ell}{\ell-1}(n_i - 1) = \frac{\ell}{\ell-1}(n-k) < \frac{\ell}{\ell-1}(n-1).$$

Now, we would like to verify the equality case. If G is a connected graph whose every block is C_ℓ, then using induction on the number of blocks we get the equality. For the other side suppose that $m = \frac{\ell}{\ell-1}(n-1)$. By the above inequalities, G is a connected graph. If G is a 2-connected graph, then by Theorem 1, $G = C_\ell$. Thus suppose that G is not a 2-connected graph and B' is a leaf block of G. Assume that G has the following form:

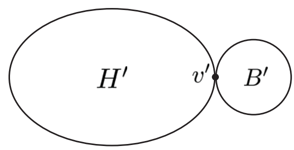

Figure 4

Let $H' = G\backslash(V(B')\backslash\{v'\})$. We have $m_{H'} \leqslant \frac{\ell}{\ell-1}(n_{H'}-1)$ and $m_{B'} \leqslant \frac{\ell}{\ell-1}(n_{B'}-1)$, where $m_{H'} = |E(H')|$, $n_{H'} = |V(H')|$, $m_{B'} = |E(B')|$ and $n_{B'} = |V(B)|$. Since $m = \frac{\ell}{\ell-1}(n-1)$, then $m_{H'} = \frac{\ell}{\ell-1}(n_{H'}-1)$ and $m_{B'} = \frac{\ell}{\ell-1}(n_{B'}-1)$. Now, by induction the proof is complete. □

Theorem 3. *Let G be a graph of order n and size m which is not a tree. If $\ell \geq 3$ is a natural number and G is a $(0 \bmod \ell)$-cycle graph, then $m \leqslant \frac{\ell}{\ell-2}(n-2)$, and the equality holds if and only if the following hold:*
(i) ℓ is odd and $G = C_\ell$,
(ii) ℓ is even and G is a generalized θ-graph with paths of length $\frac{\ell}{2}$.

Proof. If G is a forest, then $m \leqslant n-2 \leqslant \frac{\ell}{\ell-2}(n-2)$. So suppose that G contains a cycle. This implies that $\ell \leqslant n$. First assume that G is a connected graph. If ℓ is odd, then by Theorem 2,

$$m \leqslant \frac{\ell}{\ell-1}(n-1) \leqslant \frac{\ell}{\ell-2}(n-2).$$

If $m = \frac{\ell}{\ell-2}(n-2)$, then $\ell = n$ and $G = C_\ell$. Evidently, if $G = C_\ell$, then the equality in the statement of theorem holds.

Now, assume that ℓ is even. In this case by induction on the number of blocks of G we prove the assertion. If G is a 2-connected graph, then by Theorem 1, we are done. Hence one can assume that G has at least two leaf blocks. Clearly, G has a block B, such that $H = G\backslash(V(B)\backslash\{v\})$ is not a tree, see Figure 3. By induction hypothesis $m_H \leqslant \frac{\ell}{\ell-2}(n_H-2)$, where n_H and m_H denote the order and the size of H, respectively. If $B = K_2$, then we find $m = m_H + 1 \leqslant \frac{\ell}{\ell-2}(n_H-2)+1 < \frac{\ell}{\ell-2}(n-2)$. If $B \neq K_2$, then by induction hypothesis we have

$$m = m_H + m_B \leqslant \frac{\ell}{\ell-2}(n_H-2) + \frac{\ell}{\ell-2}(n_B-2) < \frac{\ell}{\ell-2}(n-2),$$

where $m_B = |E(B)|$ and $n_B = |V(B)|$. Now, if $m = \frac{\ell}{\ell-2}(n-2)$, then G is a 2-connected graph and by Theorem 1, G is a generalized θ-graph with paths of length

$\frac{\ell}{2}$. Obviously, if G is a generalized θ-graph with paths of length $\frac{\ell}{2}$, then the equality holds in the statement of theorem.

Now, assume that G is not a connected graph and $G_1, \dots, G_k$ $(k \geqslant 2)$ are the connected components of G. Let $v_i \in V(G_i)$, $i = 1, \dots, k$. Join v_i to v_{i+1} for every i, $i = 1, \dots, k-1$ and call the resultant graph by S. Since S is a (0 mod ℓ)-cycle connected graph, we find $m < m + k - 1 = m_S \leqslant \frac{\ell}{\ell-2}(n-2)$, where m_S is the size of S. The proof is complete. □

Remark 1. If ℓ, $3 \leqslant \ell \leqslant n$, is a natural number, then the condition not being tree in the previous theorem is superfluous.

Corollary 1. *Let G be a graph of order n and size m, and $\ell \geqslant 3$ be a natural number. If ℓ is odd, then $\frac{m}{n} < \frac{\ell}{\ell-1}$ and for every $\varepsilon > 0$, there exists a (0 mod ℓ)-cycle graph G with $\frac{m}{n} > \frac{\ell}{\ell-1} - \varepsilon$. If ℓ is even, then $\frac{m}{n} < \frac{\ell}{\ell-2}$ and for every $\varepsilon > 0$, there exists a (0 mod ℓ)-cycle graph G satisfying $\frac{m}{n} > \frac{\ell}{\ell-2} - \varepsilon$.*

Proof. If ℓ is odd, Theorem 2 implies that every (0 mod ℓ)-cycle graph G satisfies $\frac{m}{n} \leqslant \frac{\ell}{\ell-1} \times \frac{n-1}{n} < \frac{\ell}{\ell-1}$. Moreover, the theorem also provides infinitely many (0 mod ℓ)-cycle graphs G satisfying $\frac{m}{n} = \frac{\ell}{\ell-1} \times \frac{n-1}{n}$. Since $\lim_{n\to\infty} \frac{\ell}{\ell-1} \times \frac{n-1}{n} = \frac{\ell}{\ell-1}$, we see that for every $\varepsilon > 0$, there exists a (0 mod ℓ)-cycle graph G with $\frac{m}{n} > \frac{\ell}{\ell-1} - \varepsilon$. Similarly, if ℓ is even, Theorem 3 implies that every (0 mod ℓ)-cycle graph G satisfies $\frac{m}{n} < \frac{\ell}{\ell-2}$, and for every $\varepsilon > 0$, there exists a (0 mod ℓ)-cycle graph G satisfying $\frac{m}{n} > \frac{\ell}{\ell-2} - \varepsilon$. □

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