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On the spectrum of minimal covers by triples

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Abstract

A Minimal Cover by Triples is an ordered pair (V, T) where V is a finite set and T is a collection of three-element subsets of V with the properties that every pair of elements of V appear together in at least one element of T and if any element of T is removed, the first property no longer holds. In this paper we explore the possible values $|T|$ can take on for a given $|V|$. In addition, construction techniques are given to construct a Minimal Cover by Triples for given values of $|V|$ and $|T|$.

1 Introduction

We will refer to the ordered pair (V, T) as a *Collection of Triples (CoT)* whenever V is a finite non-empty set and T is a collection of three-element subsets of V. The elements of T will be referred to as *triples*. A *Steiner Triple System* is a CoT (V, T) with the additional property that every pair of elements of V appears together in exactly one triple of T. It is well known that a Steiner Triple System exists on V if and only if $|V| \equiv 1$ or 3 (mod 6).

The natural next question of how close you can come to a Steiner triple system when when $|V| \not\equiv 1$ or 3 (mod 6) is also well known in two forms. One can look at maximum packings or minimum covers. A *Packing of Triples* is a CoT (V, T) with the property that every pair of elements of V appears together in at most one triple

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of T. A Packing of Triples (V, T) is a *maximum packing* if, for every other Packing of Triples (V, T'), we have that $|T'| \leq |T|$. A *Cover by Triples* is a CoT (V, T) with the property that every pair of elements of V appears together in at least one triple of T. A Cover by Triples (V, T) is a *minimum cover* if, for every Cover by Triples (V, T'), we have that $|T| \leq |T'|$. More detailed information on maximum packings and minimum covers can be found in [1].

The purpose of this paper is to take the idea of minimum covers one step further and look at minimal covers. A *Minimal Cover by Triples* (MCT) is a Cover by Triples, (V, T), with the property that removal of any triple from T no longer produces a Cover by Triples. A (v, b) MCT is a MCT with $|V| = v$ and $|T| = b$.

We will address specifically how many triples can be in a minimal cover by triples for a given value for v. An attempt, albeit unsuccessful, to do so is made in [2] (that paper also considers the case of block size 4). When we compared our results to the results in [2] they did not agree and on close inspection we found errors in that paper.

We state our main claim, which will be proved in this discourse, as follows:

Theorem 1.1 *There is a (v, b) MCT if and only if*

$$\left\lceil \frac{v}{3} \left\lceil \frac{v-1}{2} \right\rceil \right\rceil \leq b \leq \binom{v-1}{2} - 2, \text{ or } b = \binom{v-1}{2} \tag{1}$$

with the following exceptions:

1. *there is a $(5, 5)$ MCT;*
2. *there is no $(7, 8)$ MCT.*

2 Bounds and the gap

The lower bound of equation (1) is well known and can be obtained from information found in [1] where there is a section devoted to minimum covers with triples. The upper bound and missing value will be dealt with in this section.

2.1 The Upper Bound

It is often useful to consider this problem in graph theoretical terms. If we consider our minimal cover to be a cover of the complete graph with v vertices using copies of K_3, then we can use this notion to help prove the upper bound. With this in mind, we will refer to any edge that appears in exactly one triple of the cover to be a *unique edge*, and any edge that appears in at least two triples of the cover will be called a *common edge*. In any minimal cover, each triple will have at least one unique edge. Choose a unique edge from each triple and denote those edges as *special edges*.

Note that each triple then has one special edge and two non-special edges. We also consider the fact that each non-special edge can appear in at most $v - 2$ triples. Thus we have b special edges and at least $\frac{2b}{v-2}$ non special edges giving us that

$$b + \frac{2b}{v-2} \leq \binom{v}{2}. \tag{2}$$

Applying some algebra yields the inequality

$$b \leq \binom{v-1}{2} \tag{3}$$

To show that this bound is sharp, simply choose an arbitrary point of V, name it ∞, and let $T = \{\{\infty, u, v\} \mid u \neq v,\ u, v \in V \setminus \{\infty\}\}$.

2.2 Solving the Mystery of the Missing Value

Possibly the most interesting result that contributes to the main theorem is the following:

Theorem 2.1 *There exists a $(v, \binom{v-1}{2} - 1)$ MCT if and only if $v = 5$.*

If we let $V = \{0,1,2,3,4\}$ and $T = \{\{0,1,3\},\{1,2,4\},\{2,3,0\},\{3,4,1\},\{4,0,2\}\}$, then we have a $(5,5)$ MCT. So it remains to be shown that a $(v, \binom{v-1}{2} - 1)$ MCT cannot be obtained if $v \neq 5$.

Suppose (V,T) is a (v,b) MCT. As in Section 2.1 we will choose one unique edge of each triple in T and label it as a special edge. All other edges will be considered non-special edges. Let G be the subgraph of K_v induced by the set of non-special edges. Note that $|E(G)| = \binom{v}{2} - b$.

For each $x, y \in V$ with $x \neq y$ and $xy \notin E(G)$ there is a path P_{xy} of length 2 from x to y in G. Since xy is not in G it was a special edge so we will take the non-special edges from the triple it appears in to make up the path P_{xy}. Defining the path P_{xy} in such a way, guarantees that each edge of G is in at least one P_{xy}.

With the existence of all the P_{xy}'s we have that every pair of vertices is either adjacent or distance 2 apart in G. Hence the diameter of G is 2 and it is connected. It also follows that $v - 1 \leq |E(G)|$ and thus $b \leq \binom{v}{2} - (v-1) = \binom{v-1}{2}$. (A fact that we have previously proved in Section 2.1.)

Suppose that $b = \binom{v-1}{2} - 1$. Then we have that $|E(G)| = v$ and it follows that removing any edge of G either disconnects it or makes it acyclic. Hence G contains exactly one cycle of some length k. See Figure 1 for an example of such a graph. If we were to remove the edges of the cycle from G, we would be left with a forest. We will refer to the subgraph induced by the edges of this forest as the fringe of G. So let's consider what possible values we can have for k.

If $k \geq 6$, then G would have diameter at least 3; hence $k < 6$. Suppose $k = 5$ or 4. If there is a fringe edge, then again the diameter will be at least 3; in these cases G can only be a cycle. If $k = 3$ and the fringe is anything other than a star then the diameter will again be at least 3. So the possible forms of G will be as in Figure 2.

If $k = 3$ then there is no way for edge w_1w_2 to be in any P_{xy} so $k \neq 3$. If $k = 4$ then we will have P_{uz} and P_{wt}. Without loss of generality, assume that P_{uz} is the path utz. If P_{wt} is the path wut then the edge wz is in no P_{xy} and if P_{wt} is the path wzt then the edge wu is in no P_{xy}. So it follows that $k \neq 4$. Thus if $b = \binom{v-1}{2} - 1$ then $v = 5$.

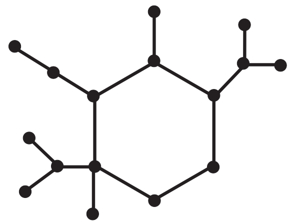

Figure 1: An example of a graph with exactly one cycle of length 6

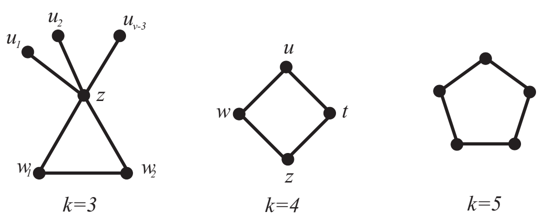

Figure 2: Narrowing down G when $b = \binom{v-1}{2} - 1$

3 Needed Constructions

Note that minimum covers exist for all v (see [1]) and that we have shown our upper bound of $\binom{v-1}{2}$ to be sharp. Therefore, by by the result of Theorem 2.1, we need only prove that a (v, b) MCT exists for all b such that $\lceil \frac{v}{3} \lceil \frac{v-1}{2} \rceil \rceil < b < \binom{v-1}{2} - 1$ to complete our proof of Theorem 1.1. In this section we introduce some needed constructions that we will use to do just that in the following section.

3.1 Minimum cover when $v \equiv 0 \pmod 6$

We will make use of the "1-factor covering Construction" found in [1]. It will be used not only as the way to obtain the minimum number of triples in its respective case but as a starting point to obtain most of the other possible numbers of triples. The construction makes use of a pairwise balanced design with one block of size 5 and the rest of size 3. A *pairwise balanced design* (or PBD for short) is an ordered pair (X, B) were X is a finite nonempty set and B is a collection of subsets of X, called blocks, with the property that every pair of elements of X appears in exactly one block of B. How to construct the required PBD for the 1-factor covering construction is also found in [1].

We reproduce the 1-factor covering construction here with a few minor changes in labeling.

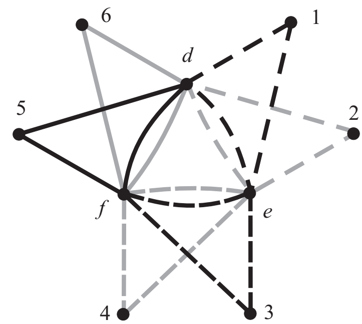

Figure 3: Handling the common edges

Construction 3.1 (The 1-factor covering Construction)

Let $v = 6n$ and let (X, B) be a PBD of order $v-1$ with one block $\{d, e, f, x_0, y_0\}$ of size 5 and the remaining blocks of size 3. Denote by T the collection of blocks of size 3. Let $S = \{\infty\} \cup X$ and let $\pi = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{3n-3}, y_{3n-3}\}\}$ be any partition of $X \setminus \{d, e, f, x_0, y_0\}$. Let $\pi(\infty) = \{\{\infty, x_1, y_1\}, \{\infty, x_2, y_2\}, \ldots, \{\infty, x_{3n-3}, y_{3n-3}\}\}$ and $F(\infty) = \{\{\infty, d, f\}, \{\infty, e, f\}, \{\infty, x_0, y_0\}, \{x_0, y_0, f\}, \{d, e, x_0\}, \{d, e, y_0\}\}$. Then (S, T^*) is a minimum covering of order v where $T^* = T \cup \pi(\infty) \cup F(\infty)$.

3.2 One more triple than a minimum cover when $v \equiv 1$ or $3 \pmod 6$

For a given Cover by Triples, we will define the multigraph obtained by taking all the edges of the triples of the cover as the *cover graph*. Every vertex of this graph will be of even degree. When $v \equiv 1$ or $3 \pmod 6$ the cover graph for a minimum cover is the complete graph. So if we are interested in one more triple than there is in a minimum cover the associated cover graph will be a complete graph plus three extra edges. Since a complete graph with $v \equiv 1$ or $3 \pmod 6$ has even degree it follows that the extra edges must form a K_3. The only way to do this and have a minimal cover is for no two of the three common edges to appear in a triple together. This forces us to have the structure in Figure 3. Since this requires at least nine vertices, it follows that we can not have a $(7, 8)$ MCT.

If we are looking to obtain a $(9, 13)$ MCT, we can first add the triples $\{d, 3, 4\}$, $\{e, 5, 6\}$ and $\{f, 1, 2\}$ to what we have in Figure 3. For future reference we will refer to this collection of triples as α. What we are then left with is the edges of a K_6 minus a matching (denoted as $K_6 - M$ for brevity from here on) which can easily be partitioned into triples. An example of a $(9, 13)$ MCT can be found in Appendix A.1.

In order to obtain a $(13, 27)$ MCT, we again start with α. What remains is the graph we have in Figure 4. We then 3-edge-color the K_4 using the vertex labels of the $\overline{K_3}$ as colors, and we 4-edge-color the $K_6 - M$ using the vertex labels of the K_4 as colors. Then for each edge $\{u, w\}$ of either the K_4 or $K_6 - M$, we add the triple

$\{u, w, c\}$, where c is the color of $\{u, w\}$. An example of a $(13, 27)$ MCT constructed in such a manner can be found in Appendix A.2.

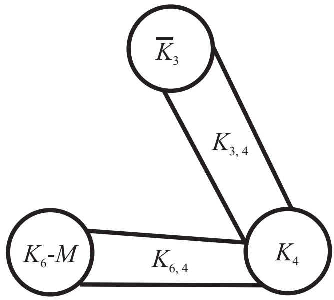

Figure 4: Building a $(13, 27)$ MCT

To obtain a $(15, 36)$ MCT we again start with α, after which we are left with the graph in Figure 5. This time we will need to make use of an equitable edge coloring. For our purposes, an edge coloring is *equitable* if, for any two colors i and j, the number of edges colored i is equal to the number of edges colored j. We then 6-edge-color the $K_6 - M$ equitably, using the vertex labels of the K_6 as colors, in such a way that the color pairs missing at each vertex form a 2-regular subgraph of the K_6 when taken as edges. We will call this 2-regular subgraph G. Next we 3-edge-color the $K_6 - G$ using the labels of the $\overline{K_3}$ as colors. For each edge $\{u, w\}$ that has been colored we add the triple $\{u, w, c\}$ where c is the color of the edge. Finally, for each edge $\{u, w\}$ of G, if x is the vertex of the $K_6 - M$ that has no incident edge colored u or w, then add the triple $\{u, w, x\}$. An example of a $(15, 36)$ MCT constructed in such a manner can be found in Appendix A.2.

For any $v \equiv 1$ or $3 \pmod 6$ with $v > 18$ we can start by embedding any STS(9) in a STS(v) using the process found in [3]. Replacing the triples of the STS(9) with the triples of our $(9, 13)$ MCT will give us a $(v, \frac{v(v-1)}{6} + 1)$ MCT.

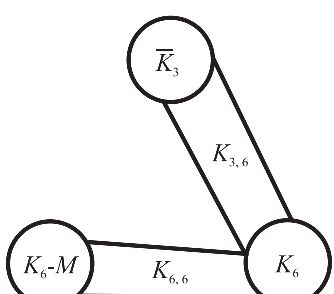

Figure 5: Building a $(15, 36)$ MCT

3.3 One more triple than a minimum cover when $v \equiv 2$ or $4 \pmod 6$ and $v \geq 10$

We will make use of a modified version of the "tripole covering Construction" found in [1]. The unmodified version of the construction gives rise to a minimum cover for these orders. Instead of starting with a Steiner triple system of order $v-1$, we will start with the minimal cover (V', T') of order $v-1$ with one extra triple constructed in Section 3.2. We begin by renaming the elements of V'. Choose any triple of T' that does not contain any of the points $d, e, f, 2, 4$ or 6 and rename its points u, s and w.

In order to assure that such a triple exists first note that we can use the $(9, 13)$ MCT in Appendix A.1 or the $(13, 27)$ MCT in Appendix A.2. For larger values of v consider that the number of triples maximum the six vertices in question can collectively appear in is $6(\frac{v-1}{2}) + 3$. The vertices d, e and f each appear in 5 triples of α and the vertices 2,4 and 6 each appear in 2. The total number of triples that will be in the MCT will be $\frac{v(v-1)}{6} + 1$, nine of which are α. Since for $v > 13$ we have that $6(\frac{v-1}{2}) - 18 \leq \frac{v(v-1)}{6} - 8$, it follows that the desired triple is guaranteed to exist.

Continuing with the renaming process we let $d = x_1$, $e = x_2$, $f = x_3$, $2 = y_3$, $4 = y_1$, $6 = y_2$. Rename the remaining points in any fashion with the labels $x_4, \ldots, x_{\frac{1}{2}v-2}, y_4, \ldots, y_{\frac{1}{2}v-2}$.

Once we have renamed the points, we let $V = V' \cup \{\infty\}$ and construct T in the following manner. Let A be all the triples of $T' \setminus \{\{u, s, w\}\}$ (properly renamed); let $B = \{\{\infty, u, s\}, \{\infty, s, w\}, \{\infty, u, w\}\}$; let $C = \{\{\infty, x_i, y_i\} \mid 1 \leq i \leq \frac{1}{2}v - 2\}$. Finally $T = A \cup B \cup C$. We then have the desired MCT.

3.4 Minimum cover when $v \equiv 5 \pmod 6$

We will be making use of the "double edge covering Construction" found in [1].

Construction 3.2 (The double edge covering Construction)

Let $v = 6n + 5$ and let (S, B) be a PBD of order v with one block $\{a, c, d, e, f\}$ of size 5 and the remaining blocks of size 3. Denote by T the collection of blocks of size 3, and let $T^* = \{\{a, f, c\}, \{a, f, d\}, \{a, f, e\}, \{c, d, e\}\}$. Then $(S, T \cup T^*)$ is a minimum cover by triples of order v.

4 Proof of Main Theorem

The techniques we will use to construct minimal covers for all possible values of b will start with a MCT with either the least possible number of triples or one more than the least possible. It will entail choosing one special point (akin to choosing the point in the construction of a MCT with the largest possible number of triples). Then in a strategic manner we will remove triples that do not contain the special point. For each unique edge from a triple that has been removed, we will add a triple containing that edge and the special point. When we do this the only edges we cause to become duplicated are edges incident to the special point. The only danger in this is if there are triples whose unique edges are incident to the special point. In the cases where this is a problem, we will carefully remove triples in a manner that will not cause those critical unique edges to be duplicated until after the common edge of the triple they are in which is not incident to the special point has become a unique edge.

4.1 $v \equiv 0 \pmod{6}$

Let $v = 6n$, $n \geq 1$. The minimum value, $6n^2$, can be obtained using Construction 3.1. To get the remaining values, we will next separate the triples of this initial minimal cover into 4 classes. For notational purposes, if we are referring to a vertex x_i or y_i and it is not important whether it is an x or a y but the subscript is important, we will refer to it as z_i. We categorize the triples as follows:

- **Type 0:** The triples $\{\infty, e, f\}$, $\{\infty, d, f\}$ and the triples of the form $\{\infty, x_i, y_i\}$ for $0 \leq i \leq 3n-3$.
- **Type 1:** The triple $\{x_0, y_0, f\}$ and the triples of the form $\{x_i, y_i, v_i\}$ where $1 \leq i \leq 3n-3$ and $v_i \in V - \{\infty\}$.
- **Type 2:** Any triple of the form $\{z_i, z_j, z_k\}$ where $0 \leq i < j < k \leq 3n-3$ and any triple of the form $\{u, z_i, z_j\}$ where $u \in \{d, e, f\}$ and $1 \leq i < j \leq 3n-3$
- **Type 3:** The triples $\{d, e, x_0\}$ and $\{d, e, y_0\}$

Obviously, we have $3n$ Type 0, $3n-2$ Type 1 and two Type 3 triples; this leaves us $6n^2 - 6n$ Type 2 triples. We now proceed to show how to use this starting point as a way to construct a minimal cover with b triples for all b, $6n^2 < b < 18n^2 - 9n$.

We also want to take note that the Type 0 triples of the form $\{\infty, x_i, y_i\}$ have unique edges that are adjacent to the point ∞. Each one shares the edge $\{x_i, y_i\}$ with a Type 1 triple. So we need to take care how we remove triples until all the Type 1 triples are removed.

If $b \leq 6n^2 + 3n - 2$, a minimal cover with b triples can be obtained simply by removing $b - 6n^2$ Type 1 triples and replacing them with new triples in the following manner:

1. Remove the triple $\{x_0, y_0, f\}$ and replace it with the triples $\{\infty, x_0, f\}$ and $\{\infty, y_0, f\}$.
2. Set $i = 1$.
3. Remove the Type 1 triple $\{x_i, y_i, v_i\}$ and replace it with the triples $\{\infty, x_i, v_i\}$ and $\{\infty, y_i, v_i\}$.
4. If $v_i = z_j$ for some j and the Type 1 triple containing x_j and y_j has not been removed yet, set $i = j$; otherwise, set i to the lowest possible value such that $\{x_i, y_i, v_i\}$ has not been removed.

5. Go to step 3 unless we now have enough triples or there are no more Type 1 triples left.

When we remove the triple $\{x_i, y_i, v_i\}$ with $v_i = z_j$ and if $\{x_j, y_j, v_j\}$ has not yet been removed, we are in danger of causing a triple to contain no unique edges. Assume without loss of generality that $v_i = x_j$. At this point in the Type 0 triple $\{\infty, x_j, y_j\}$ the only unique edge is $\{\infty, y_i\}$. If there is also a Type 1 triple $\{x_k, y_k, v_k\}$ where $v_k = y_j$ and we remove it and add the corresponding triples it will result in the triple $\{\infty, x_j, y_j\}$ no longer having a unique edge. Hence we no longer have a MCT. If we first remove the triple $\{x_j, y_j, v_j\}$ we can avoid this problem. This is the reason for Step 4 being as it is.

We need not worry about the removal of a single Type 1 triple causing a problem. The removal of $\{x_i, y_i, v_i\}$ with $v_i = z_j$ (without loss of generality assume $z_j = x_j$) can only cause the edges $\{\infty, x_i\}$, $\{\infty, y_i\}$, and $\{\infty, x_j\}$ to become common edges. This is not a problem because $\{x_i, y_i\}$ has just become a unique edge and if $\{\infty, y_j\}$ is common, then the triple $\{x_j, y_j, v_j\}$ would have previously been removed. Therefore, $\{x_j, y_j\}$ will be a unique edge.

If $b > 6n^2 + 3n - 2$ we first remove and replace all the Type 1 triples in the above described manner. We then choose $\lfloor \frac{b-(6n^2+3n-2)}{2} \rfloor$ Type 2 triples. For each such triple $\{u, v, w\}$ chosen, we remove it and replace it with the triples $\{\infty, u, v\}$,$\{\infty, v, w\}$ and $\{\infty, u, w\}$. If we now have a system containing b triples, we are done; otherwise we have a system containing $b-1$ triples and then remove the triple $\{d, e, x_0\}$ and replace it with the triples $\{\infty, x_0, d\}$ and $\{\infty, x_0, e\}$.

Example 4.1 **($v = 18, b = 66$)**
Suppose we wish to construct an $(18, 66)$ MCT. Let $V = \{d, e, f, x_0, x_1, \ldots, x_6, y_0, y_1, \ldots, y_6\}$, and let T contain the following triples:

- **Type 0:** $\{\infty, d, f\}$, $\{\infty, e, f\}$, and $\{\infty, x_i, y_i\}$ for all i, $0 \le i \le 6$;
- **Type 1:** $\{x_0, y_0, f\}$, $\{x_1, y_1, x_3\}$, $\{x_2, y_2, y_3\}$, $\{x_3, y_3, y_6\}$, $\{x_4, y_4, x_5\}$, $\{x_5, y_5, y_2\}$, $\{x_6, y_6, x_1\}$;
- **Type 2:** $\{x_0, x_3, x_5\}$, $\{x_0, y_3, x_1\}$, $\{x_0, y_5, y_1\}$, $\{x_0, x_4, x_6\}$, $\{x_0, y_4, x_2\}$, $\{x_0, y_6, y_2\}$, $\{y_0, x_5, x_1\}$, $\{y_0, y_3, y_5\}$, $\{y_0, x_4, y_1\}$, $\{y_0, x_6, x_2\}$, $\{y_0, y_4, y_6\}$, $\{y_0, x_3, y_2\}$, $\{d, x_3, x_6\}$, $\{d, y_3, x_5\}$, $\{d, x_4, y_6\}$, $\{d, y_4, y_5\}$, $\{d, x_1, y_2\}$, $\{d, y_1, x_2\}$, $\{e, x_5, x_2\}$, $\{e, y_5, x_1\}$, $\{e, x_6, y_2\}$, $\{e, y_6, y_1\}$, $\{e, x_3, y_4\}$, $\{e, y_3, x_4\}$, $\{f, x_1, y_4\}$, $\{f, y_1, y_3\}$, $\{f, x_2, x_3\}$, $\{f, y_2, x_4\}$, $\{f, x_5, y_6\}$, $\{f, y_5, x_6\}$, $\{x_3, x_4, y_5\}$, $\{x_5, x_6, y_1\}$, $\{x_4, x_1, x_2\}$, $\{y_3, y_4, x_6\}$, $\{y_5, y_6, x_2\}$, $\{y_4, y_1, y_2\}$;
- **Type 3:** $\{d, e, x_0\}$ and $\{d, e, y_0\}$.

We begin with $b = 54$. First we must remove the Type 1 triples. We start by removing $\{x_0, y_0, f\}$ and replacing it with $\{\infty, x_0, f\}$ and $\{\infty, y_0, f\}$. Next we remove $\{x_1, y_1, x_3\}$ and replace it with $\{\infty, x_1, x_3\}$ and $\{\infty, y_1, x_3\}$. At this point we are supposed to set $i = 3$, but let us consider what happens if we do not bother to and just remove $\{x_2, y_2, y_3\}$ and replace it with $\{\infty, x_2, y_3\}$ and $\{\infty, y_2, y_3\}$. If we do that, T now has as a subset $\{\{\infty, x_1, x_3\}, \{\infty, x_2, y_3\},$ $\{\infty, x_3, y_3\}, \{x_3, y_3, y_6\}\}$ which means that $\{\infty, x_3, y_3\}$ has no unique edge.

So we proceed as necessary and set $i = 3$ and then remove $\{x_3, y_3, y_6\}$ and replace it with $\{\infty, x_3, y_6\}$ and $\{\infty, y_3, y_6\}$. Next we need to set $i = 6$ and then remove $\{x_6, y_6, x_1\}$ and replace it with $\{\infty, x_6, x_1\}$ and $\{\infty, y_6, x_1\}$. This time, since we have already removed $\{x_1, y_1, x_3\}$, we set $i = 2$. We then remove $\{x_2, y_2, y_3\}$ and replace it with $\{\infty, x_2, y_3\}$ and $\{\infty, y_2, y_3\}$. Next we set $i = 4$ and remove $\{x_4, y_4, x_5\}$ and replace it with $\{\infty, x_4, x_5\}$ and $\{\infty, y_4, x_5\}$. Finally, we set $i = 5$ and remove $\{x_5, y_5, y_2\}$ and replace it with $\{\infty, x_5, y_2\}$ and $\{\infty, y_5, y_2\}$.

At this point we now have $b = 61$ so we need to add five more triples. Dividing by two and rounding down tells us to pick two Type 2 triples for removal. So lets remove $\{x_0, x_3, x_5\}$ and $\{x_0, y_3, x_1\}$. We replace them with $\{\infty, x_0, x_3\}$, $\{\infty, x_3, x_5\}$, $\{\infty, x_0, x_5\}$, $\{\infty, x_0, y_3\}$, $\{\infty, y_3, x_1\}$, and $\{\infty, x_0, x_1\}$. At this point we now have $b = 65$, so finally we remove $\{d, e, x_0\}$ and replace it with $\{\infty, x_0, d\}$ and $\{\infty, x_0, e\}$. So following this process we have constructed an $(18, 66)$ MCT.

4.2 $v \equiv 1$ or $3 \pmod 6$

We will cover all possibilities with $v \geq 9$. The case when $v = 7$ will be handled separately in Section B.2. Assume $v = 6n + 1$ with $n \geq 2$ or that $v = 6n + 3$ with $n \geq 1$. A Steiner Triple System of order v is a minimum cover by triples in this case [1]. We will obtain the rest of the values in the range by starting with the $(v, \frac{v(v-1)}{6} + 1)$ MCT we constructed in Section 3.2 and building larger covers by replacing select triples with multiple replacement triples.

Noting that $V = \{d, e, f, 1, 2, \ldots, v-3\}$ and $\alpha \subset T$, we will start by categorizing the triples in the following manner:

- **Type 0:** All triples of T of the form $\{d, u, w\}$ where $u, w \in V - \{d\}$,
- **Type 1:** The triples $\{e, f, 3\}$ and $\{e, f, 4\}$,
- **Type 2:** All triples of T of the form $\{x, y, u\}$ where $x, y \in V - \{d, e, f\}$ and $u \in V - \{d\}$.

It follows that we have $\frac{v+1}{2}$ Type 0 triples and two Type 1 triples. Thus it leaves us either $6n^2 - 2n - 2$ or $6n^2 + 2n - 2$ Type 2 triples depending on whether $v = 6n + 1$ or $6n + 3$, respectively.

We see that in this case, each of the Type 0 triples have a unique edge that is not incident to d, so the construction process will be much less complicated. Suppose we want to construct a (v, b) MCT where $\frac{v(v-1)}{6} + 1 < b < \frac{(v-1)(v-2)}{2} - 1$. First we choose $\left\lfloor \left\{ b - \left(\frac{v(v-1)}{6} + 1 \right) \right\} / 2 \right\rfloor$ type 2 triples. For each such triple $\{x, y, u\}$ chosen, we remove it and replace it with the triples $\{d, x, y\}$, $\{d, y, u\}$ and $\{d, u, x\}$. At this point our T either contains b or $b - 1$ triples. In the first case we are done; otherwise, we simply remove the triple $\{e, f, 3\}$ and replace it with the triples $\{d, e, 3\}$ and $\{d, f, 3\}$.

Example 4.2 ($v = 9, b = 20$)

Suppose we wish to construct a $(9, 20)$ MCT. We start with the $(9, 13)$ MCT from Section A.1. Categorizing the triples we have:

- **Type 0:** $\{d, e, 1\}$, $\{d, e, 2\}$, $\{d, f, 5\}$, $\{d, f, 6\}$,and $\{d, 3, 4\}$;
- **Type 1:** $\{e, f, 3\}$ and $\{e, f, 4\}$;
- **Type 2:** $\{e, 5, 6\}$, $\{f, 1, 2\}$, $\{1, 3, 5\}$, $\{1, 4, 6\}$, $\{2, 3, 6\}$, and $\{2, 4, 5\}$.

We need to add 7 triples, so we start by removing three Type 2 triples. We remove $\{e, 5, 6\}$, $\{f, 1, 2\}$, and $\{1, 3, 5\}$. Next we replace them with the triples $\{d, e, 5\}$, $\{d, e, 6\}$, $\{d, 5, 6\}$, $\{d, f, 1\}$, $\{d, f, 2\}$, $\{d, 1, 2\}$, $\{d, 1, 3\}$, $\{d, 1, 5\}$, and $\{d, 3, 5\}$. At this point we have a $(9, 19)$ MCT. All we need to do is remove $\{e, f, 3\}$ and replace it with $\{d, e, 3\}$ and $\{d, f, 3\}$, giving us a $(9, 20)$ MCT.

4.3 $v \equiv 2$ or $4 \pmod 6$

We will cover all possibilities with $v \geq 10$. The case when $v = 8$ will be handled separately in Section B.3. When $v = 4$ the results are trivial since $\left\lceil \frac{4}{3} \left\lceil \frac{3}{2} \right\rceil \right\rceil = \binom{3}{2}$. Assume $v = 6n + 2$ with $n \geq 2$ or that $v = 6n + 4$ with $n \geq 1$. A construction for a minimum cover with $b = \frac{1}{3}\left(\frac{v^2}{2} + 1\right)$, which is $6n^2 + 4n + 1$, (respectively $6n^2 + 8n + 3$) can be found in [1]. We will make use of the construction from Section 3.3 that creates a MCT with one extra triple as a starting point.

We categorize the triples in the following manner:

- **Type 0:** The triples $\{\infty, u, s\}$, $\{\infty, u, w\}$, $\{\infty, s, w\}$ and all triples of the form $\{\infty, x_i, y_i\}$ for all i, $1 \leq i \leq \frac{1}{2}v - 2$;
- **Type 1:** the triples of the form $\{x_i, y_i, \alpha_i\}$ for all i, $1 \leq i \leq \frac{1}{2}v - 2$ where $\alpha_i \in V \setminus \{x_i, y_i, \infty\}$;
- **Type 2:** the triples of the form $\{z_i, z_j, z_k\}$ or $\{z_i, z_k, \gamma\}$ where $1 \leq i < j < k \leq \frac{1}{2}v - 2$ and $\gamma \in \{u, s, w\}$ except for the triples described in the next type;
- **Type 3:** the triples $\{x_1, x_2, y_3\}$, $\{x_1, x_2, \beta_1\}$, $\{x_2, x_3, y_1\}$, $\{x_2, x_3, \beta_3\}$, $\{x_1, x_3, y_2\}$ and $\{x_1, x_3, \beta_5\}$ where β_i is the point that i was renamed to in Section 3.3.

We are starting with a total of $6n^2 + 4n + 2$ (respectively $6n^2 + 8n + 4$) triples. Clearly there are $3n + 2$ (respectively $3n + 3$) Type 0 , $3n - 1$ (respectively $3n$) Type 1 and six Type 3 triples; this leaves us with $6n^2 - 2n - 5$ (respectively $6n^2 + 2n - 5$) Type 2 triples. We now proceed to show how to use this starting point as a way to construct a minimal cover with b triples for all b, $6n^2 + 4n + 2 < b < 18n^2 + 3n - 1$ (respectively $6n^2 + 8n + 4 < b < 18n^2 + 15n + 2$).

If $b \leq 6n^2 + 7n + 1$ (respectively $b \leq 6n^2 + 11n + 4$) it can be obtained simply by removing $b - (6n^2 + 4n + 2)$ (respectively $b - (6n^2 + 8n + 4)$) Type 1 triples and replacing them with new triples in the following manner:

1. Set $i = 1$.
2. Remove the Type 1 triple $\{x_i, y_i, \alpha_i\}$ and replace it with the triples $\{\infty, x_i, \alpha_i\}$ and $\{\infty, y_i, \alpha_i\}$.
3. If $\alpha_i = z_j$ for some j and the Type 1 triple containing x_j and y_j has not been removed yet, set $i = j$; otherwise set i to the lowest possible value such that $\{x_i, y_i, \alpha_i\}$ has not been removed.
4. Go to Step 2 unless we no longer need any additional triples or there are no more Type 1 triples to remove.

As in the construction in Section 4.1 we have to take care while removing triples if there are still Type 1 triples. Thus we have taken similar precautions in our algorithm for their removal.

If $6n^2+7n+1 < b \leq 18n^2+3n-9$ (respectively $6n^2+11n+4 < b \leq 18n^2+15n-6$), we first remove and replace all the Type 1 triples in the above described manner. We then choose $\lfloor \frac{b-(6n^2+7n+1)}{2} \rfloor$ (respectively $\lfloor \frac{b-(6n^2+11n+4)}{2} \rfloor$) Type 2 triples. For each such triple $\{p, q, r\}$ chosen, we remove it and replace it with the triples $\{\infty, p, q\}$,$\{\infty, q, r\}$ and $\{\infty, p, r\}$. If we now have a system containing b triples we are done; otherwise, we have one containing $b-1$ triples and then remove the triple $\{x_1, x_2, y_3\}$ and replace it with the triples $\{\infty, y_3, x_1\}$ and $\{\infty, y_3, x_2\}$.

If $b > 18n^2 + 3n - 9$ (respectively $b > 18n^2 + 15n - 6$) we first remove all Type 1 triples followed by all Type 2 triples using the methods described above. At this point the number of triples we need to add is between 1 and 7; let this number be b'. We then add the remaining number of triples in the following manner:

- **if $\mathbf{b'} = \mathbf{1}$,** remove $\{x_1, x_2, y_3\}$ and replace it with $\{\infty, y_3, x_1\}$ and $\{\infty, y_3, x_2\}$;
- **if $\mathbf{b'} = \mathbf{2}$,** proceed as if $b' = 1$ and then remove $\{x_1, x_3, y_2\}$ and replace it with $\{\infty, y_2, x_1\}$ and $\{\infty, y_2, x_3\}$;
- **if $\mathbf{b'} = \mathbf{3}$,** proceed as if $b' = 1$ and then remove $\{x_1, x_2, \beta_1\}$ and replace it with $\{\infty, x_1, x_2\}$, $\{\infty, x_1, \beta_1\}$ and $\{\infty, x_2, \beta_1\}$;
- **if $\mathbf{b'} = \mathbf{4}$,** proceed as if $b' = 3$ and then remove $\{x_1, x_3, y_2\}$ and replace it with $\{\infty, y_2, x_1\}$ and $\{\infty, y_2, x_3\}$;
- **if $\mathbf{b'} = \mathbf{5}$,** proceed as if $b' = 4$ and then remove $\{x_2, x_3, y_1\}$ and replace it with $\{\infty, y_1, x_2\}$ and $\{\infty, y_1, x_3\}$;
- **if $\mathbf{b'} = \mathbf{6}$,** proceed as if $b' = 4$ and then remove $\{x_1, x_3, \beta_5\}$ and replace it with $\{\infty, x_1, x_3\}$, $\{\infty, x_1, \beta_5\}$ and $\{\infty, x_3, \beta_5\}$;
- **if $\mathbf{b'} = \mathbf{7}$,** proceed as if $b' = 6$ and then remove $\{x_2, x_3, y_1\}$ and replace it with $\{\infty, y_1, x_2\}$ and $\{\infty, y_1, x_3\}$.

Example 4.3 ($v = 10, b = 30$)
Suppose we wish to construct a $(10, 30)$ MCT. We will first start with the $(10, 18)$ MCT constructed using the technique from Section 3.3. We have $V = \{\infty, u, s, w, x_1, x_2, x_3, y_1, y_2, y_3\}$ and T had the following triples:

- **Type 0:** $\{\infty, u, s\}$, $\{\infty, u, w\}$, $\{\infty, s, w\}$, $\{\infty, x_1, y_1\}$, $\{\infty, x_2, y_2\}$, and $\{\infty, x_3, y_3\}$;
- **Type 1:** $\{x_1, y_1, s\}$, $\{x_2, y_2, w\}$, and $\{x_3, y_3, u\}$;
- **Type 2:** $\{u, y_1, y_2\}$, $\{y_3, s, y_2\}$, and $\{y_3, y_1, w\}$;
- **Type 3:** $\{x_1, x_2, y_3\}$, $\{x_1, x_2, u\}$, $\{x_2, x_3, y_1\}$, $\{x_2, x_3, s\}$, $\{x_1, x_3, y_2\}$ and $\{x_1, x_3, w\}$

So initially we have $b = 18$. We start by removing $\{x_1, y_1, s\}$, replacing it with $\{\infty, x_1, s\}$ and $\{\infty, y_1, s\}$. Next we set $i = 2$ and remove $\{x_2, y_2, w\}$, replacing it with $\{\infty, x_2, w\}$ and $\{\infty, y_2, w\}$. Now we set $i = 3$ and remove $\{x_3, y_3, u\}$, replacing it with $\{\infty, x_3, u\}$ and $\{\infty, y_3, u\}$. After removing all the Type 1 triples we now have $b = 21$.

We will need to remove all the Type 2 triples. After removing them we replace them with the triples $\{\infty, u, y_1\}$, $\{\infty, u, y_2\}$, $\{\infty, y_1, y_2\}$, $\{\infty, y_3, s\}$, $\{\infty, y_3, y_2\}$, $\{\infty, s, y_2\}$, $\{\infty, y_3, y_1\}$, $\{\infty, y_3, w\}$, and $\{\infty, y_1, w\}$. At this point we now have $b = 27$ this means that $b' = 3$.

So, proceeding as directed, we now remove $\{x_1, x_2, y_3\}$ and replace it with $\{\infty, y_3, x_1\}$ and $\{\infty, y_3, x_2\}$ making $b = 28$. Finally, we remove $\{x_1, x_2, u\}$ and replace it with $\{\infty, x_1, x_2\}$, $\{\infty, x_1, u\}$ and $\{\infty, x_2, u\}$, giving us a $(10, 30)$ MCT.

4.4 $v \equiv 5 \pmod 6$

We will handle all possibilities for $v \geq 11$. The case when $v = 5$ will be handled in Section B.1. Assume that $v = 6n + 5$; the minimum value is obtained using Construction 3.2 and will be used as a starting point to obtain the remaining values needed. As done previously we will build a MCT with the desired value for b by carefully replacing select triples with multiple replacement triples.

We start by choosing any point of $V \setminus \{a, c, d, e, f\}$ and renaming it ∞. Noting that we start with $6n^2 + 9n + 4$ triples, we then categorize the triples in the following manner:

- **Type 0:** All triples of T of the form $\{\infty, u, w\}$ where $u, w \in V - \{\infty\}$,
- **Type 1:** The triples $\{a, f, c\}$, $\{a, f, d\}$ and $\{a, f, e\}$,
- **Type 2:** All triples of T of the form $\{x, y, u\}$ where $x, y \in V - \{a, f, \infty\}$ and $u \in V - \{\infty\}$.

It follows that we have $3n+2$ Type 0 triples and three Type 1 triples. Thus it leaves us $6n^2+6n-1$ Type 2 triples. We now proceed to show how to use this to obtain a (v,b) MCT when $6n^2+9n+4 < b < 18n^2+21n+5$.

If $b \leq 18n^2+21n+3$ we choose $\lfloor \frac{b-(6n^2+9n+4)}{2} \rfloor$ Type 2 triples. For each such triple $\{u,s,w\}$, we remove it and replace it with the triples $\{\infty,u,s\}$, $\{\infty,u,w\}$ and $\{\infty,s,w\}$. Our MCT now has either b or $b-1$ triples, so either we are done or we remove $\{a,f,c\}$ and replace it with $\{\infty,a,c\}$ and $\{\infty,f,c\}$. If $b=18n^2+21n+4$, we replace all the Type 2 triples as above, and then remove $\{a,f,c\}$ and $\{a,f,d\}$ and replace them with $\{\infty,a,c\}$, $\{\infty,f,c\}$, $\{\infty,a,d\}$ and $\{\infty,f,d\}$.

Example 4.4 ($v=11, b=25$)

Suppose we wish to construct a $(11,26)$ MCT. We will start with the $(11,19)$ MCT we get from the construction in Section 3.2. We have $V=\{a,c,d,e,f,6,7,8,9,10,\infty\}$ and T is made up of:

- **Type 0:** $\{\infty,a,10\}$, $\{\infty,f,7\}$, $\{\infty,c,6\}$, $\{\infty,d,8\}$, and $\{\infty,e,9\}$;
- **Type 1:** $\{a,f,c\}$, $\{a,f,d\}$, and $\{a,f,e\}$;
- **Type 2:** $\{a,6,7\}$, $\{a,8,9\}$, $\{f,6,9\}$, $\{f,8,10\}$, $\{c,7,8\}$, $\{c,9,10\}$, $\{d,e,c\}$, $\{d,6,10\}$, $\{d,7,9\}$, $\{e,6,8\}$, and $\{e,7,10\}$.

We first need to remove three Type 2 triples, so we remove $\{a,6,7\}$, $\{a,8,9\}$, and $\{f,6,9\}$. We then replace them with $\{\infty,a,6\}$, $\{\infty,a,7\}$, $\{\infty,6,7\}$, $\{\infty,a,8\}$, $\{\infty,a,9\}$, $\{\infty,8,9\}$, $\{\infty,f,6\}$, $\{\infty,f,9\}$, and $\{\infty,6,9\}$. This leaves us with $b=25$, so we remove $\{a,f,c\}$ and replace it with $\{\infty,a,c\}$ and $\{\infty,f,c\}$. This gives us a $(11,25)$ MCT.

A Some Specific MCT's

A.1 $(9,13)$ MCT

$V=\{d,e,f,1,2,3,4,5,6\}$, $T=\{\{d,e,1\},\{d,e,2\},\{e,f,3\},\{e,f,4\},\{d,f,5\},\{d,f,6\},\{d,3,4\},\{e,5,6\},\{f,1,2\},\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\}$.

A.2 $(13,27)$ MCT

$V=\{d,e,f,1,2,3,4,5,6,7,8,9,10\}$, $T=\{\{d,e,1\},\{d,e,2\},\{e,f,3\},\{e,f,4\},\{d,f,5\},\{d,f,6\},\{d,3,4\},\{e,5,6\},\{f,1,2\},\{d,7,8\},\{d,9,10\},\{e,7,9\},\{e,8,10\},\{f,7,10\},\{f,8,9\},\{1,3,7\},\{1,4,8\},\{1,5,9\},\{1,6,10\},\{2,3,10\},\{2,4,9\},\{2,5,7\},\{2,6,8\},\{3,5,8\},\{3,6,9\},\{4,5,10\},\{4,6,7\}\}$.

A.3 $(15,36)$ MCT

$V=\{d,e,f,1,2,3,4,5,6,7,8,9,10,11,12\}$, $T=\{\{d,e,1\},\{d,e,2\},\{e,f,3\},\{e,f,4\},\{d,f,5\},\{d,f,6\},\{d,3,4\},\{e,5,6\},\{f,1,2\},\{d,7,10\},\{d,8,12\},\{d,9,11\},\{e,7,11\},\{e,8,10\},\{e,9,12\},\{f,7,12\},\{f,8,11\},\{f,9,10\},\{1,3,10\},$

$\{1,4,9\}, \{1,5,12\}, \{1,6,11\}, \{1,7,8\}, \{2,3,12\}, \{2,4,7\}, \{2,5,8\}, \{2,6,9\}, \{2,10,11\}, \{3,5,11\}, \{3,6,7\}, \{3,8,9\}, \{4,5,10\}, \{4,6,8\}, \{4,11,12\}, \{5,7,9\}, \{6,10,12\}\}$.

B v = 5, 7 or 8

B.1 $v = 5$

$V = \{a, c, d, e, f\}$

- $\mathbf{b = 4}$: $T = \{\{a,f,c\}, \{a,f,d\}, \{a,f,e\}, \{c,d,e\}\}$
- $\mathbf{b = 5}$: $T = \{\{a,f,c\}, \{f,c,d\}, \{c,d,e\}, \{d,e,a\}, \{e,a,f\}\}$
- $\mathbf{b = 6}$: $T = \{\{a,f,c\}, \{a,f,d\}, \{a,f,e\}, \{a,c,d\}, \{a,c,e\}, \{a,d,e\}\}$

B.2 $v = 7$

$V = \{1, 2, 3, 4, 5, 6, 7\}$

- $\mathbf{b = 7}$: $T = \{\{1,2,4\}, \{1,3,7\}, \{1,5,6\}, \{2,3,5\}, \{2,6,7\}, \{3,4,6\}, \{4,5,7\}\}$
- $\mathbf{b = 9}$: $T = \{\{1,2,4\}, \{1,3,7\}, \{1,5,6\}, \{1,2,3\}, \{1,2,5\}, \{1,3,5\}, \{2,6,7\}, \{3,4,6\}, \{4,5,7\}\}$
- $\mathbf{b = 10}$: $T = \{\{1,2,5\}, \{1,3,6\}, \{1,4,7\}, \{2,3,5\}, \{2,3,6\}, \{2,4,5\}, \{2,4,7\}, \{3,4,6\}, \{3,4,7\}, \{5,6,7\}\}$
- $\mathbf{b = 11}$: $T = \{\{1,2,4\}, \{1,3,7\}, \{1,5,6\}, \{1,2,3\}, \{1,2,5\}, \{1,3,5\}, \{1,2,6\}, \{1,2,7\}, \{1,6,7\}, \{3,4,6\}, \{4,5,7\}\}$
- $\mathbf{b = 12}$: $T = \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,2,7\}, \{2,3,5\}, \{2,4,5\}, \{2,5,6\}, \{2,6,7\}, \{2,5,7\}, \{3,4,6\}, \{3,4,7\}\}$
- $\mathbf{b = 13}$: $T = \{\{1,2,4\}, \{1,3,7\}, \{1,5,6\}, \{1,2,3\}, \{1,2,5\}, \{1,3,5\}, \{1,2,6\}, \{1,2,7\}, \{1,6,7\}, \{1,3,4\}, \{1,3,6\}, \{1,4,6\}, \{4,5,7\}\}$
- $\mathbf{b = 15}$: $T = \{\{1,2,4\}, \{1,3,7\}, \{1,5,6\}, \{1,2,3\}, \{1,2,5\}, \{1,3,5\}, \{1,2,6\}, \{1,2,7\}, \{1,6,7\}, \{1,3,4\}, \{1,3,6\}, \{1,4,6\}, \{1,4,5\}, \{1,4,7\}, \{1,5,7\}\}$

B.3 $v = 8$

$V = \{\infty, 1, 2, 3, 4, 5, 6, 7\}$

- $\mathbf{b = 11}$: $T = \{\{\infty,1,2\}, \{\infty,1,4\}, \{\infty,2,4\}, \{\infty,3,5\}, \{\infty,6,7\}, \{2,3,5\}, \{3,4,6\}, \{4,5,7\}, \{5,6,1\}, \{6,7,2\}, \{7,1,3\}\}$
- $\mathbf{b = 12}$: $T = \{\{\infty,1,2\}, \{\infty,1,4\}, \{\infty,2,4\}, \{\infty,3,5\}, \{\infty,6,7\}, \{\infty,2,3\}, \{\infty,2,5\}, \{3,4,6\}, \{4,5,7\}, \{5,6,1\}, \{6,7,2\}, \{7,1,3\}\}$

- $\mathbf{b=13}: T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,3,4\},$ $\{\infty,3,6\},\{\infty,4,6\},\{2,3,5\},\{2,6,7\},\{4,5,7\},\{5,6,1\},\{7,1,3\}\}$
- $\mathbf{b=14}: T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,3,4\},$ $\{\infty,3,6\},\{\infty,4,6\},\{\infty,2,3\},\{\infty,2,5\},\{2,6,7\},\{4,5,7\},\{5,6,1\},$ $\{7,1,3\}\}$
- $\mathbf{b=15}: T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,3,4\},$ $\{\infty,3,6\},\{\infty,4,6\},\{\infty,2,3\},\{\infty,2,5\},\{\infty,2,6\},\{\infty,2,7\},\{4,5,7\},$ $\{5,6,1\},\{7,1,3\}\}$
- $\mathbf{b=16}: T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,3,4\},$ $\{\infty,3,6\},\{\infty,4,6\},\{\infty,2,3\},\{\infty,2,5\},\{\infty,5,6\},\{\infty,5,1\},\{\infty,6,1\},$ $\{2,6,7\},\{4,5,7\},\{7,1,3\}\}$
- $\mathbf{b=17}: T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,3,4\},$ $\{\infty,3,6\},\{\infty,4,6\},\{\infty,2,3\},\{\infty,2,5\},\{\infty,5,6\},\{\infty,5,1\},\{\infty,6,1\},,$ $\{\infty,2,6\},\{\infty,2,7\},\{4,5,7\}\{7,1,3\}\}$
- $\mathbf{b=18}: T = \{\{\infty,1,4\},\{\infty,1,5\},\{\infty,1,6\},\{\infty,2,4\},\{\infty,2,5\},\{\infty,2,6\},$ $\{\infty,2,7\},\{\infty,3,4\},\{\infty,3,5\},\{\infty,3,6\},\{\infty,4,5\},\{\infty,4,6\}\{\infty,4,7\},$ $\{\infty,5,6\},\{\infty,5,7\},\{\infty,6,7\},\{1,3,7\},\{1,2,3\}\}$
- $\mathbf{b=19}: T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,1,5\},\{\infty,1,6\},\{\infty,2,3\},\{\infty,2,4\},$ $\{\infty,2,5\},\{\infty,2,6\},\{\infty,2,7\},\{\infty,3,4\},\{\infty,3,5\},\{\infty,3,6\},\{\infty,4,5\},$ $\{\infty,5,6\},\{\infty,4,6\}\{\infty,4,7\},\{\infty,5,7\},\{\infty,6,7\},\{1,3,7\}\}$
- $\mathbf{b=21}: T = \{\{\infty,1,2\},\{\infty,1,3\},\{\infty,1,4\},\{\infty,1,5\},\{\infty,1,6\},\{\infty,1,7\},$ $\{\infty,2,3\},\{\infty,2,4\},\{\infty,2,5\},\{\infty,2,6\},\{\infty,2,7\},\{\infty,3,4\},$ $\{\infty,3,5\},\{\infty,3,6\},\{\infty,3,7\},\{\infty,4,5\},\{\infty,4,6\}\{\infty,4,7\},\{\infty,5,6\},$ $\{\infty,5,7\},\{\infty,6,7\}\}$

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