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Ortho-radial drawings of graphs

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Abstract

By an ortho-radial drawing of a graph we mean a planar drawing on concentric circles such that each edge is an alternating sequence of circular and radial segments, where a circular segment is a part of a circle and a radial segment is a part of a half-line starting at the center of the circles. Ortho-radial drawings are topologically an extension of orthogonal drawings to drawings on a cylinder.

We study the relationship between ortho-radial drawings and orthogonal drawings, then we prove necessary and sufficient conditions for a path, cycle or a theta graph to have an ortho-radial drawing consistent with a C-shape (cylindrical shape) which is a specification of the direction in which each edge must be drawn. Furthermore, we present an example of a C-shape of a graph such that all of its cycles have an ortho-radial drawing but the graph itself does not have any ortho-radial drawing with this C-shape. This is in contrast to the properties of orthogonal drawings on the plane.

1 Introduction

Orthogonal drawings are planar drawings of graphs in which every edge is represented by a chain of horizontal and vertical segments. Orthogonal drawings are of much interest because of their application in VLSI design and layout. An orthogonal shape, P-shape (planar-shape), of a graph specifies in which of the four horizontal or vertical directions each edge must be drawn [8]. Vijayan and Wigderson [8] found a necessary and sufficient condition for orthogonal shapes to have an orthogonal drawing. Many algorithms for orthogonal drawings are based on this condition [1, 5, 7].

Drawing of an orthogonal shape in three dimensional space is a challenging problem. Several studies have been conducted on this issue [2, 3, 4, 6].

We call a grid composed of concentric circles with center S and half-lines starting at S an *ortho-radial grid.* An *ortho-radial drawing* of a graph G is a planar drawing of G in an ortho-radial grid such that each edge is an alternating sequence of circular and radial segments where a circular segment is a connected part of a circle and a radial segment is a connected part of a half-line, not including S.

Since in an ortho-radial drawing no vertex is drawn at S and no edge passes through S, an ortho-radial drawing is topologically a drawing in a grid on a cylinder or a drawing in a grid on a sphere in which no vertex is drawn on either the North pole or the South pole and no edge passes through either pole (see Figure 1).

Figure 1: An ortho-radial drawing of a graph and its corresponding drawings on a cylinder and a sphere.

In an ortho-radial grid, a directed radial segment has direction *Down* if it points towards S and direction *Up* if it points away from S. A directed circular segment has direction *Clockwise* if S is on its right side and direction *Anticlockwise* (Counterclockwise) if S is on its left side (see Figure 2).

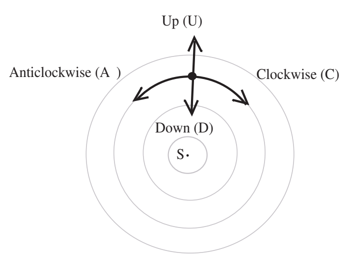

Figure 2: The four directions on an ortho-radial grid

In a graph, the term *dart* is used for each of the two possible orientations (u, v) and (v, u) of an undirected edge uv.

Let G be a graph with labels from the set {*U (Up), D (Down), C (Clockwise), A (Anticlockwise)*} assigned to its darts. The assignment of labels is called a *C-shape* of G. Does an ortho-radial drawing of G exist such that each dart is drawn as a single radial or circular segment with direction consistent with its associated label? We study this problem for some classes of graphs.

Note that the labels of the darts of G imply a cyclic order of the darts leaving each vertex, and therefore imply a combinatorial embedding of G. We will only be concerned with the case that this embedding is planar. Vijayan and Wigderson [8] proved that a biconnected graph has an orthogonal drawing with a given P-shape if and only if its faces have orthogonal drawings with the induced P-shapes. This theorem is not true in the case of C-shapes. The graph shown in Figure 3(a) does not have any ortho-radial drawing with the C-shape shown but all its faces have ortho-radial drawings with the induced C-shapes. In the example in Figure 3(b), all the cycles of the graph have ortho-radial drawings, but the graph itself does not have an ortho-radial drawing with the given C-shape.

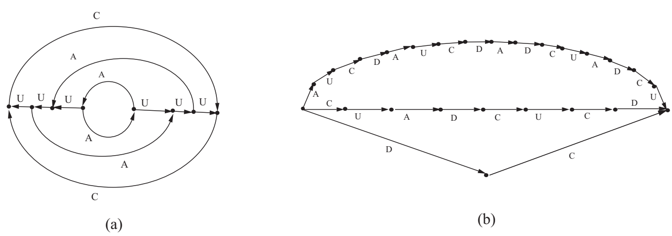

Figure 3: The examples which are in contrast to the properties of orthogonal drawings on the plane

The remainder of this paper is organized as follows: Section 2 presents the exact definition of C-shapes, the relationship between orthogonal and ortho-radial drawings, and necessary and sufficient conditions for paths and cycles to have ortho-radial drawings with given C-shapes. Section 3 deals with the problem for biconnected graphs and presents necessary and sufficient conditions for theta graphs. Section 4 summarizes the results and gives directions for further work.

2 Ortho-radial drawings and C-shapes

2.1 C-shapes

For an undirected graph G, a *C-shape* is a labeling of the darts of G such that:

1. The labels are in the set $\{C, A, D, U\}$.
2. The labels of darts (u, v) and (v, u) are opposite. (The labels C, A, D and U are the opposites of A, C, U and D, respectively.)
3. No two consecutive darts (u, v) and (v, w) have opposite labels.

Let γ be a C-shape of G. A *drawing of G with C-shape γ* is an ortho-radial drawing Γ of G where each dart consists of a single radial or circular segment with direction consistent with its label. A C-shape γ of a graph G is called *drawable* if there is a drawing of G with C-shape γ.

Let $C = v_1, v_2, ..., v_{n+1} = v_1$ be a cycle with C-shape γ and $\sigma = \sigma_1\sigma_2...\sigma_n$ be the labels of darts (v_1, v_2), $(v_2, v_3), ..., (v_n, v_{n+1})$. We call σ a *C-shape cycle*. For a path $p = v_1, v_2, ..., v_{n+1}$ with C-shape γ, let $\sigma = \sigma_1\sigma_2...\sigma_n$ be the labels of darts $(v_1, v_2), (v_2, v_3), ..., (v_n, v_{n+1})$. We call σ a *C-shape path*. If σ is a C-shape path or

C-shape cycle, a dart (v_i, v_{i+1}) for $1 \leq i \leq n$ is a *forward dart* of σ. We say a path p is labeled with X if all of its forward darts have the same label X.

For a C-shape path or cycle $\sigma = \sigma_1\sigma_2...\sigma_n$, we define $\bar{\sigma} = \bar{\sigma}_n\bar{\sigma}_{n-1}...\bar{\sigma}_1$ where $\bar{\sigma}_i$ is the opposite of σ_i.

If $e_1 = (u, v)$ and $e_2 = (v, w)$, $u \neq w$, are two darts of a graph G with a given C-shape, the angle between these darts is the angle ($\pi/2$, π or $3\pi/2$) on the left when we move from u to w. Let l_1 and l_2 be the labels of e_1 and e_2. The function $turn(l_1, l_2)$ is defined as follows:

$$turn(l_1, l_2) = \begin{cases} 1 & \text{if the angle between } e_1 \text{ and } e_2 \text{ is } \frac{\pi}{2} \\ 0 & \text{if the angle between } e_1 \text{ and } e_2 \text{ is } \pi \\ -1 & \text{if the angle between } e_1 \text{ and } e_2 \text{ is } \frac{3\pi}{2} \end{cases} \quad (1)$$

Let $\sigma = \sigma_1\sigma_2...\sigma_n$ be a C-shape path. The *rotation* of σ, $rot(\sigma)$, is defined as follows:

$$rot(\sigma) = \sum_{i=1}^{n-1} turn(\sigma_i, \sigma_{i+1}) \quad (2)$$

and for a C-shape cycle $\sigma = \sigma_1\sigma_2...\sigma_n$, the rotation of σ is

$$rot(\sigma) = \sum_{i=1}^{n-1} turn(\sigma_i, \sigma_{i+1}) + turn(\sigma_n, \sigma_1). \quad (3)$$

For a graph G with C-shape γ, without loss of generality we can contract two consecutive darts with the same label into a single dart if the degree of their common vertex is two.

2.2 Ortho-radial and orthogonal drawings

In an ortho-radial drawing of a graph two cases may happen:

1. The point S is in the external face. This is a *type-1* ortho-radial drawing (see Figure 4(a)).

2. The point S is in an internal face. This is a *type-2* ortho-radial drawing (see Figure 4(b)).

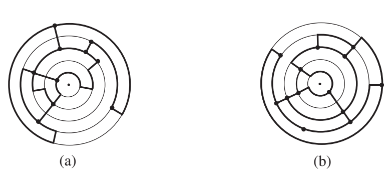

Figure 4: (a) A type-1 ortho-radial drawing, (b) A type-2 ortho-radial drawing

There is a close relationship between orthogonal drawings and type-1 ortho-radial drawings. We present this relationship formally in the next theorem.

Theorem 1 *Every type-*1 *ortho-radial drawing of a graph G can be transformed into an orthogonal drawing in the plane in such a way that each vertical segment becomes a radial segment and each horizontal segment becomes a circular segment, and vice versa.*

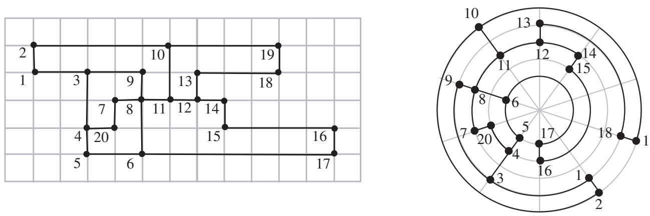

Figure 5: An orthogonal drawing and a corresponding type-1 ortho-radial drawing

In an ortho-radial drawing, a *bend* is a point at which an edge changes direction. Theorem 1 proves that for each orthogonal drawing there is an ortho-radial drawing with the same number of bends (see Figure 5). In fact, there are some graph families whose orthogonal drawings have at least a linear number of bends, but have ortho-radial drawings with no bends. Figure 6 illustrates an example of such a graph family. Since minimizing the number of bends is an important aesthetic criterion, ortho-radial drawings can be of more importance than just a new style of graph drawing.

Note that every graph with an orthogonal drawing or ortho-radial drawing has maximum degree at most 4.

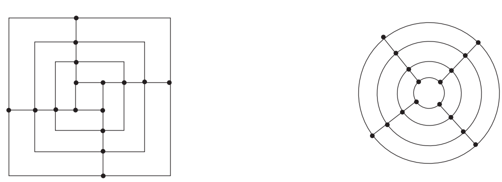

Figure 6: This family of graphs has ortho-radial drawings with no bends, but no orthogonal drawings with less than a linear number of bends.

The formal definition of P-shapes is the same as that of C-shapes, the difference being that in P-shapes the labels are in the set $\{L, R, D, U\}$, where L, R, D and U are abbreviations for Left, Right, Down and Up, respectively (see [8]).

To state the main results of this paper, we need to present some results about P-shapes. We regard the *contour* of an internal face of a biconnected plane graph as an anticlockwise cycle formed by the edges on the boundary of the face, and we regard the *contour* of the external face of a biconnected plane graph as a clockwise cycle formed by the edges on the boundary of the face. The rotation of a P-shape cycle (path) is defined similarly to that of C-shapes [1]. This is a useful property of the rotation:

Property 1. [1]. If $\sigma = \sigma_1\sigma_2...\sigma_n$ is the P-shape of a face f in an orthogonal drawing of a biconnected graph, then

$$rot(\sigma) = \begin{cases} 4 & \text{if } f \text{ is an internal face} \\ -4 & \text{if } f \text{ is the external face} \end{cases} \tag{4}$$

In the following we provide two lemmas which are powerful tools for proving the main results of this paper. For a plane graph G, we denote by C_0 the contour of the external face.

Lemma 1 *Let G be a biconnected plane graph with a drawable P-shape γ. Suppose C_0 contains three consecutive paths p_1, p_2 and p_3 labeled with U, R and D (D, L and U), respectively. Then there is a drawing of G where the vertices of p_2 have the largest (smallest) y-coordinate.*

Proof: Let u and v be the first and last vertex of path p_2, respectively. We construct a graph G' from G by inserting a path q with length 5 from v to u in the external face of G. Define γ' to be the P-shape of G' with the darts of q labeled with R, D, L, U and R, respectively and other darts labeled with the same label as in γ. It is easy to check that γ' is a drawable P-shape and in every drawing of G' with P-shape γ', the vertices of p_2 have the largest y-coordinate. So, by removing the path q from a drawing of G', we reach a drawing of G where the vertices of p_2 have the largest y-coordinate. □

Let G be a plane graph with a drawable P-shape γ, let p_1 be a maximal path labeled by R on C_0 and let p_2 be a maximal path labeled by L on C_0. We say p_1 and p_2 are *admissible* if G has a drawing in which vertices of p_1 have the largest y-coordinate and vertices of p_2 have the smallest y-coordinate. In Figure 7, p_1 and p_3 are admissible, but p'_1 and p_3 are not admissible.

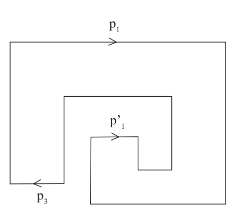

Figure 7: Paths p_1 and p_3 are admissible, but p'_1 and p_3 are not admissible.

Lemma 2 *Let G be a plane graph with a drawable P-shape γ. Suppose that p_1, p_2, p_3 and p_4 are four paths of C_0 such that $C_0 = p_1p_2p_3p_4$, p_1 is a maximal path labeled with R and p_3 is a maximal path labeled with L, the first and last darts of p_2 are labeled with D and the first and last darts of p_4 are labeled with U. Let σ_2 and σ_4 be the P-shapes of p_2 and p_4, respectively. Then p_1 and p_3 are admissible if and only if $rot(\sigma_2) = rot(\sigma_4) = 0$.*

Proof: Suppose that p_1 and p_3 are admissible. Let Γ be a drawing of G in which vertices of p_1 have the largest y-coordinates and vertices of p_3 have the smallest y-coordinates. By inserting two paths with lengths three in the external face as in Figure 8, we can conclude $rot(\sigma_2) = rot(\sigma_4) = 0$.

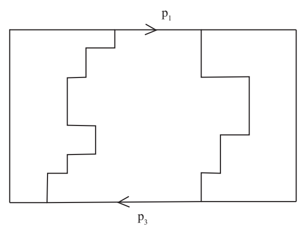

Figure 8: Illustration of the proof of Lemma 2

Suppose that $rot(\sigma_2) = rot(\sigma_4) = 0$. Let u and v be the first and the last vertices of p_1 and let z and w be the first and the last vertices of p_3. G' is the graph obtained from G by inserting two directed paths q_1 and q_2, q_1 with length 3 from v to z and q_2 with length 3 from w to u. Let γ' be the P-shape obtained from γ by labeling darts of q_1 with labels R, D and L and darts of q_2 with labels L, U and R, respectively. It is easy to check that γ' is a drawable P-shape and in every drawing of G' with P-shape γ', vertices of p_1 have the largest y-coordinate and vertices of p_3 have the smallest y-coordinate. By removing the paths q_1 and q_2 from a drawing of G', we obtain a drawing of G where vertices of p_1 have the largest y-coordinate and vertices of p_3 have the smallest y-coordinate. □

2.3 Drawing of C-shape paths and C-shape cycles

In this section we prove necessary and sufficient conditions for a C-shape path or cycle to have an ortho-radial drawing.

By Theorem 1 and the fact that every P-shape path is drawable, we can conclude that every C-shape path is drawable. Theorem 1 and Property 1 prove that a C-shape cycle σ has a type-1 drawing if and only if $rot(\sigma) = \pm 4$. Theorem 2 characterizes C-shape cycles with type-2 drawings.

Theorem 2 *A C-shape cycle σ has a type-2 drawing if and only if $rot(\sigma) = 0$ and one of the following cases happens:*

1. *All the labels of σ are C or all the labels are A.*

2. *σ contains at least one D and one U label.*

Proof: Suppose that σ has a type-2 drawing Γ and σ is the C-shape of the external face of Γ. (If σ is the C-shape of the internal face then we apply the following argument for $\bar{\sigma}$.)

If all the labels of σ are the same, then all of them are A or C and $rot(\sigma) = 0$.

Otherwise, if (u, v) is a dart labeled with D then in any drawing of σ, v is closer to S than u is. Clearly the cycle can not be completed unless a dart labeled U is also present.

Now we will show that $rot(\sigma) = 0$. Let $e = (u, v)$ be the forward dart of σ corresponding to the farthest circular segment of Γ from the point S. Without losing generality we can suppose that there is no other edge in the circle where e is drawn.

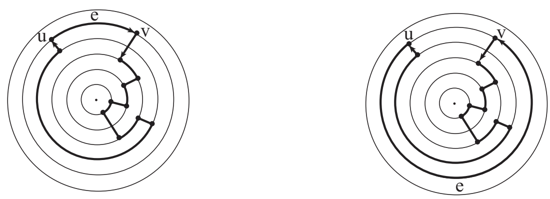

Figure 9: Illustration of the proof of Theorem 2

Since σ is the C-shape of the external face of Γ, the darts before and after e have labels U and D respectively, and e has label C. Thus, $\sigma = \sigma' UCD$. If we remove e and add an anticlockwise circular segment from u to v, then the resulting drawing is a type-1 drawing Γ' of $\tau = \sigma' UAD$ in which τ is the C-shape of the internal face of Γ' (see Figure 9). By Theorem 1 and Property 1, $rot(\tau) = 4$. Therefore, $rot(\sigma)$ and $rot(\tau)$ satisfy the following equality:

$$\begin{aligned} rot(\tau) &= rot(\sigma) - turn(U, C) - turn(C, D) + turn(U, A) + turn(A, D) \\ &= rot(\sigma) - (-1) - (-1) + 1 + 1. \end{aligned}$$

Hence $rot(\sigma) = 0$.

Conversely, suppose that σ is a C-shape cycle such that $rot(\sigma) = 0$. If all labels of σ are the same then σ has a type-2 drawing which is a circle. Otherwise, since σ has at least one D and one U by the third property in the definition of C-shapes, it contains some labels C or A. Thus, σ contains some C-shape paths UAD or C-shape paths UCD. If σ contains a C-shape path UAD, then we can consider $\sigma = \sigma' UAD$. Let $\tau = \sigma' UCD$ then,

$$\begin{aligned} rot(\tau) &= rot(\sigma) - turn(U, A) - turn(A, D) + turn(U, C) + turn(C, D) \\ &= 0 - 1 - 1 + (-1) + (-1) = -4. \end{aligned}$$

So τ has a type-1 drawing with τ as the C-shape of the external face. Let (u, v) be the dart labeled with C. By Lemma 1 and Theorem 1, τ has a drawing Γ such that e is the farthest segment of Γ from the point S. If we remove e and add an anticlockwise radial segment from u to v, then the resulting drawing is a type-2 drawing of σ where σ is the C-shape of the internal face.

In the last case, if σ contains no C-shape path UAD, it contains a C-shape path UCD. $\bar{\sigma}$ is a C-shape cycle containing C-shape path UAD and $rot(\bar{\sigma}) = 0$. By the above discussion $\bar{\sigma}$ is a drawable C-shape and there is a drawing Γ of $\bar{\sigma}$ with $\bar{\sigma}$ as the C-shape of the internal face. In Γ, σ is the C-shape of the external face of Γ. Hence σ has a type-2 drawing. This completes the proof. $\square$

Let σ be a drawable C-shape cycle such that $rot(\sigma) = 0$. Then σ has a drawing such that σ is the C-shape of the external face if and only if σ contains a C-shape path UCD or all its labels are C, and σ has a drawing such that σ is the C-shape of the internal face if and only if σ contains a C-shape path UAD or all its labels are A. If there is a drawing Γ where σ is the C-shape of the external face of Γ, we call σ an *E-shape cycle* (external-shape cycle), and if there is a drawing Γ where σ is the C-shape of the internal face of Γ, we call σ an *I-shape cycle* (inner-shape cycle). Note that a drawable C-shape cycle with rotation 0 is either an I-shape cycle or an E-shape cycle and it is possible that it is both.

3 Ortho-radial drawings of biconnected graphs with given C-shapes

A connected graph G is *biconnected* if the removal of each vertex does not make it disconnected. A *theta graph* is a biconnected graph consisting of three disjoint paths of length at least two between two vertices of degree three.

Let G be a graph with C-shape γ. The labels of γ define a cyclic order of the darts leaving each vertex; that is, they define a combinatorial embedding of the graph on some surface. We call this *the embedding induced by* γ. This is a planar embedding if the number of faces obeys Euler's formula.

The next theorem is a clarification of Theorem 5.1 of [8], where the need for planarity of the induced embedding is not clearly stated.

Theorem 3 *Let G be a biconnected graph with at least three edges. A P-shape γ of G is drawable if and only if the embedding induced by γ is a planar embedding and the P-shape of each face is drawable.*

Let G be a biconnected graph with at least three edges and let γ be a C-shape of G. The proof of necessity in Theorem 3 is true for γ. In fact, if γ is drawable then the embedding induced by γ is a planar embedding and the C-shape of each face is drawable. Also, if the C-shapes of all faces of G are drawable and the rotation in every face is 4 other than one whose rotation is -4, then by Theorem 3 and Theorem 1, γ is drawable and G has a type-1 ortho-radial drawing. The following theorem formally states the necessary condition for C-shape γ to be drawable.

Theorem 4 *If γ is a drawable C-shape, $F_1, F_2, ..., F_k$ are the faces of G and $\sigma_i, 1 \leq i \leq k$, is the C-shape cycle of F_i then one of the following happens:*

1. *For some face $F_j, rot(\sigma_j) = -4$ and $rot(\sigma_i) = 4$ for the other faces. In this case γ has a type-1 drawing.*

2. *$rot(\sigma_j) = rot(\sigma_l) = 0$, for some $1 \leq j \neq l \leq k$, $rot(\sigma_i) = 4$ for other faces, one of σ_j and σ_l is an I-shape cycle and the other is an E-shape cycle. In this case γ has a type-2 drawing.*

The example in Figure 3(a) shows that Theorem 3 does not hold in the case of C-shapes. It is easy to check that the C-shapes of all its faces are drawable and it has a cycle whose C-shape is not drawable. In the example in Figure 3(b), even the C-shapes of all the cycles of the graph are drawable, but by Theorem 5 we can prove that the C-shape of the graph itself is not drawable.

For a path $p = v_1, \ldots, v_k$ denote by $\bar{p}$ the reverse path $v_k, \ldots, v_1$. By *removing* p, we mean removing all the edges of p and all the vertices of p other than end-vertices.

Let T be a theta graph with three paths p_1, p_2 and p_3 connecting vertex p to vertex q. Let σ be a C-shape of T which induces a planar embedding of T, and let σ_i be the C-shape path of p_i induced by σ, $i = 1, 2, 3$. We denote by τ_1, τ_2 and τ_3 the C-shape cycles $\sigma_1\bar{\sigma}_3$, $\sigma_2\bar{\sigma}_1$ and $\sigma_3\bar{\sigma}_2$, respectively. The next theorem presents necessary and sufficient conditions for σ to have a type-2 ortho-radial drawing.

Theorem 5 *Considering the above definitions, T has a type-2 ortho-radial drawing with C-shape σ if and only if the C-shape cycles τ_1, τ_2 and τ_3 are drawable, the rotation of one of them, say τ_2, is 4, one of them, say τ_3, is an I-shape cycle and the other one, τ_1, is an E-shape cycle, and one of the following cases happens.*

1. *All the labels of τ_3 are A.*
2. *All the labels of τ_1 are C.*
3. *τ_1 has a C-shape path UCD such that at least one of the darts of the subpath labeled by C is on p_3.*
4. *τ_3 has a C-shape path DAU such that at least one of the darts of the subpath labeled by A is on $\bar{p}_3$.*
5. *σ_2 has a C-shape path DCU and $\bar{\sigma}_1$ has a C-shape path UAD such that τ_2 is $\tau_{21}DCU\tau_{22}UAD\tau_{23}$ for some C-shape paths τ_{21}, τ_{22} and τ_{23}, and $rot(D\tau_{23}\tau_{21}D) = rot(U\tau_{22}U) = 0$.*

Proof: Suppose that T has a type-2 ortho-radial drawing Γ. Let τ_1, τ_2 and τ_3 be the C-shape of the external face, the internal face not containing point S and the internal face containing point S, respectively. Thus, τ_1 is an E-shape cycle, $rot(\tau_2) = 4$ and τ_3 is an I-shape cycle. Suppose none of the cases 1, 2, 3 and 4 happens. Let r and r' be the nearest and farthest segments of Γ from S. Then r is a subpath of cycle $p_3\bar{p}_2$ labeled by A and the edges before and after r are labeled by D and U respectively, and r' is a subpath of cycle $p_1\bar{p}_3$ labeled by C with the edges before and after r' are labeled by U and D respectively. Since cases 3 and 4 do not happen, r and r' are respectively on p_2 and p_1 which are therefore the nearest and farthest segments from S on cycle $p_2\bar{p}_1$ in Γ . Thus, there are some C-shape paths τ_{21}, τ_{22} and τ_{23} such that $\tau_2 = \tau_{21}DCU\tau_{22}UAD\tau_{23}$ and by considering the corresponding orthogonal drawing of $p_2\bar{p}_1$, Lemma 2 and Theorem 1, we conclude $rot(D\tau_{23}\tau_{21}D) = rot(U\tau_{22}U) = 0$. This proves the necessity.

Now we prove the sufficiency for each case. In the first case, it is easy to see that the first and last dart of $\bar{p}_1$ are respectively labeled by U and D. Thus, by considering the corresponding orthogonal drawing of cycle $p_2\bar{p}_1$, Lemma 1 and Theorem 1, there is a drawing of cycle $p_2\bar{p}_1$ with C-shape cycle τ_2 such that vertices p and q are drawn on the smallest cycle of the ortho-radial grid. Without loss of generality, we can suppose that there are no other vertices on that cycle except the vertices of path p_2. By inserting an anticlockwise circular segment from p to q, we obtain a drawing of T. For the second case, we can use a similar approach to obtain a drawing of T.

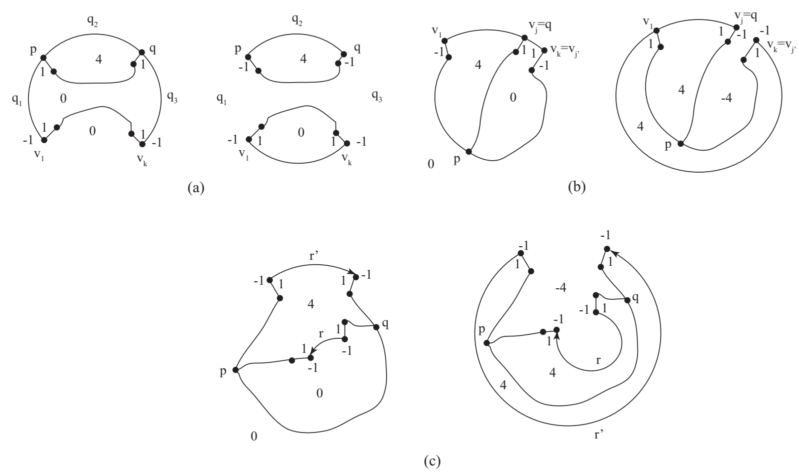

Figure 10: The number written in each face is the rotation of the C-shape cycle corresponding to the face

Suppose τ_1 has a C-shape path UCD such that at least one of the darts of the subpath $r = v_1, ..., v_k$, labeled by C is on $\bar{p}_3$. If for some i_p and i_q, $1 < i_p < i_q < k$, $p = v_{i_p}$ and $q = v_{i_q}$ (see Figure 10(a)), then the first and last darts of p_2 are labeled by D and U respectively. Vertices p and q divide r into three subpaths, $q_1 = v_1, ..., v_{i_p}$, $q_2 = v_{i_p}, ..., v_{i_q}$ and $q_3 = v_{i_q}, ..., v_k$. Remove q_1 and q_2, add edge $v_k v_1$ and label dart (v_k, v_1) by C. The resulting graph consists of two disjoint cycles which have type-1 ortho-radial drawings. By considering corresponding orthogonal drawings of the cycles, Lemma 1 and Theorem 1 show that there are coincidence drawings of the cycles such that $v_1 v_k$ and q_2 are drawn as the farthest segments from S. The drawing can be easily converted to a drawing of T with C-shape σ. Otherwise, let v_j be the first vertex of r which is on $\bar{p}_3$ and let $v_{j'}$ be the last vertex of r on $\bar{p}_3$ such that all the darts of r which are between v_j and $v_{j'}$ are on $\bar{p}_3$. Remove the subpath of r which is between v_j and $v_{j'}$, then add edge $v_k v_1$ and label dart (v_k, v_1) by C (see Figure 10(b)). The resulting graph T' is a theta graph. Let σ' be the C-shape of T'. Computing the rotation of C-shape cycles corresponding to the faces proves that T' has a type-1 ortho-radial drawing. Let r'' be the path in T' consisting of $v_k v_1$ and the remaining part of r. The P-shape corresponding to σ' satisfies the conditions of Lemma 1 and so by Theorem 1, T' has a type-1 drawing such that r'' is drawn as the farthest segment from S in the drawing. This is easily converted into a type-2 ortho-radial drawing of graph T with C-shape σ. A similar approach works for case 4.

In order to prove the sufficiency in case 5, let r and r' be the subpaths labeled with C and A, respectively. By replacing the labels of the darts of r and r' with A and C respectively, we obtain a drawable C-shape σ' of T which has a type-1 ortho-radial drawing (see Figure 10(c)). By considering the corresponding orthogonal drawing of T, Lemma 2 and Theorem 1 shows that T has an ortho-radial drawing with C-shape σ' such that r and r' are the nearest and farthest paths from S in the drawing and this drawing can be easily transformed to a type-2 ortho-radial drawing of T with C-shape σ. □

4 Concluding remarks

The problem of determining C-shapes that have an ortho-radial drawing is still open. The solution of this problem is a suitable base for studying ortho-radial drawings. Since there are some graphs that have ortho-radial drawings with fewer bends than orthogonal drawings, the problem of computing ortho-radial drawings of graphs with a minimum number of bends is an interesting problem.

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