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Amicable complex orthogonal designs

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Abstract

In this paper, we define amicable complex orthogonal designs (ACOD) and propose two systematic methods to construct higher-order ACODs from lower-order ACODs. We found that the upper bound on the number of variables of an ACOD is the same as that of amicable orthogonal designs (AOD). We also show that certain types of AOD that were previously shown to be non-existent or undecided, such as AODs of order 8 with type (1, 1, 1, 1; 2, 2, 2, 2) and (1, 2, 2, 2; 1, 2, 2, 2), can be found from ACODs constructed using our proposed construction methods. Our proposed methods can also be used to systematically construct new AODs that are of the same type as, but not equivalent to, those previously found by Zhao, Wang and Seberry using computer search. An interesting finding arising from this study is that an AOD or ACOD can be constructed from a lower-order amicable family (AF) or amicable complex family (ACF). This implies that the component matrices for constructing a higher-order AOD/ACOD need not be disjoint.

1 Introduction

A series of combinatorial designs called orthogonal designs (OD) [3], amicable orthogonal designs (AOD) [8] and complex orthogonal designs (COD) [2] have been reported in the literature for many applications. For example, in [1, 5, 7, 9] and the references therein, an AOD is used to design space-time block codes for wireless communications with multiple transmit antennas. Although Ganesan and Stoica have discussed in [1] the complex version of AOD, or amicable complex orthogonal designs (ACOD), they did not try to find applications for ACOD as they did not expect ACOD to give any improvement in signal-to-noise ratio for space-time block codes. In this paper, we shall show that an ACOD does have useful applications, particularly in providing an approach to find new AODs not thought to be existent before.

The existence of AODs of order 8 in quite a number of types were thought to be non-existent or undecided in [6]. Although a few such AODs were found through computer search recently [10], not all cases were addressed.

In this paper, we shall discuss the maximum number of variables that exists in an ACOD and propose systematic methods to construct ACODs that achieve the maximum number of variables. We show that some AODs that were deemed to be non-existent or undecided in [6], including those found later in [10], can be systematically constructed with our proposed methods by allowing complex number as the entries of the code matrix.

Before we define an ACOD, we shall first review a COD and its related parameters.

Definition 1 [2, Proposition 3]: A *Complex Orthogonal Design* (COD) of *order* n and *type* $(w_1, \ldots, w_r)$ (w_j positive integers, for $1 \leq j \leq r$) on the real commuting *variables* $\{z_1, \ldots, z_r\}$ is an $n \times n$ matrix $\mathbf{Z}$ with entries from $\{0, \pm z_j$ or $\pm i z_j\}$, where $1 \leq j \leq r$, satisfying

$$\mathbf{Z}\mathbf{Z}^* = \left(\sum_{j=1}^{r} w_j z_j^2 \right) \mathbf{I}_n. \tag{1}$$

Here $\mathbf{Z}$ can be expressed as

$$\mathbf{Z} = \sum_{j=1}^{r} z_j \mathbf{C}_j \tag{2}$$

where $\mathbf{C}_j$ satisfies:

$$\begin{array}{llcll} (0) & \mathbf{C}_j * \mathbf{C}_k & = & 0 & 1 \leq j \neq k \leq r \\ (i) & \mathbf{C}_j \mathbf{C}_j^* & = & w_j \mathbf{I}_n & 1 \leq j \leq r \\ (ii) & \mathbf{C}_j \mathbf{C}_k^* + \mathbf{C}_k \mathbf{C}_j^* & = & 0 & 1 \leq j \neq k \leq r. \end{array} \tag{3}$$

The operator $*$ in (3)(0) represents Hadamard Product, and the superscript * represents complex conjugate transpose. The statement (3)(0) implies that the member matrices of this design are *pairwise disjoint*.

Definition 2 A *complex family* (CF) is a collection of matrices $\{\mathbf{C}_1, \ldots, \mathbf{C}_r\}$ satisfying (3)(i) and (3)(ii), but not (3)(0). This definition is analogous to the definition of *family* associated with AOD in [4, Definition 2.16].

Lemma 1 [2, Theorem 4] *Let $\tau(n)$ denote the maximum number of variables in a COD of order n. Then $\tau(n) = H(n)$, where $H(n) = 2a+2$ if $n = 2^a b$, b odd.*

Next we define an ACOD, using the approach presented in [8] for defining an AOD from a pair of OD.

Definition 3 Let the matrices $\mathbf{X} = x_1\mathbf{A}_1 + \cdots + x_s\mathbf{A}_s$ and $\mathbf{Y} = y_1\mathbf{B}_1 + \cdots + y_t\mathbf{B}_t$ be two CODs of the same order n, where $\mathbf{X}$ is of type $(u_1, \ldots, u_s)$ on the variables $\{x_1, \ldots, x_s\}$ and $\mathbf{Y}$ is of type $(v_1, \ldots, v_t)$ on the variables $\{y_1, \ldots, y_t\}$. Here $\mathbf{X}$ and $\mathbf{Y}$ are said to be an *Amicable Complex Orthogonal Design* (ACOD) denoted as $(u_1, \ldots, u_s; v_1, \ldots, v_t)$ if the family of matrices $\{\mathbf{A_1}, \ldots, \mathbf{A_s}; \mathbf{B_1}, \ldots, \mathbf{B_t}\}$ satisfies:

$$\begin{array}{llcll}
(0) & \mathbf{A}_j * \mathbf{A}_l & = & 0 & 1 \le j \ne l \le s \\
 & \mathbf{B}_k * \mathbf{B}_m & = & 0 & 1 \le k \ne m \le t \\
(i) & \mathbf{A}_j\mathbf{A}_j^* & = & u_j\mathbf{I}_n & 1 \le j \le s \\
 & \mathbf{B}_k\mathbf{B}_k^* & = & v_k\mathbf{I}_n & 1 \le k \le t \\
(ii) & \mathbf{A}_j\mathbf{A}_l^* + \mathbf{A}_l\mathbf{A}_j^* & = & 0 & 1 \le j \ne l \le s \\
 & \mathbf{B}_k\mathbf{B}_m^* + \mathbf{B}_m\mathbf{B}_k^* & = & 0 & 1 \le k \ne m \le t \\
(iii) & \mathbf{A}_j\mathbf{B}_k^* - \mathbf{B}_k\mathbf{A}_j^* & = & 0 & 1 \le j \le s,\ 1 \le k \le t
\end{array} \tag{4}$$

where the entries of $\mathbf{A}_j$ and $\mathbf{B}_k$ are 0, ± 1, or $\pm i$.

Definition 4 An *amicable complex family* (ACF) is a collection of matrices $\{\mathbf{A}_1, \ldots, \mathbf{A}_s; \mathbf{B}_1, \ldots, \mathbf{B}_t\}$ satisfying (4)(i–iii), but not (4)(0). This definition is analogous to the definition of an *amicable family* (AF) associated with AOD in [4, Definition 5.7].

Lemma 2 *Let $\mathbf{X}$ be a COD of type $(u_1, \ldots, u_s)$ on the variables $\{x_1, \ldots, x_s\}$, and $\mathbf{X} = x_1\mathbf{A}_1 + \cdots + x_s\mathbf{A}_s$; then*

(i) $\mathbf{X}' = x_1\mathbf{A}_1 + \cdots + x_{j-1}\mathbf{A}_{j-1} + x_j(\mathbf{A}_j + \mathbf{A}_k) + x_{j+1}\mathbf{A}_{j+1} + \cdots + x_{k-1}\mathbf{A}_{k-1} + x_{k+1}\mathbf{A}_{k+1} + \cdots + x_s\mathbf{A}_s$ *is a COD of type* $(u_1, \ldots, u_{j-1}, u_j + u_k, u_{j+1}, \ldots, u_{k-1}, u_{k+1}, \ldots, u_s)$ *on variables* $\{x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_s\}$;

(ii) $\mathbf{X}'' = x_1\mathbf{A}_1 + \cdots + x_{j-1}\mathbf{A}_{j-1} + x_{j+1}\mathbf{A}_{j+1} + \cdots + x_s\mathbf{A}_s$ *is an COD of type* $(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_s)$ *on variables* $\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_s\}$.

Lemma 3 *Let $\mathbf{X}$ and $\mathbf{Y}$ be CODs with the same order, and let $\mathbf{X}'$ and $\mathbf{X}''$ be CODs obtained from Lemma 2. If $\mathbf{X}$, $\mathbf{Y}$ are amicable, so are $\mathbf{X}'$ and $\mathbf{Y}$, and $\mathbf{X}''$ and $\mathbf{Y}$.*

PROOF of Lemmas 2 and 3:
Lemma 2 and Lemma 3 follow directly from Lemma 4.1 and Theorem 4.1 of [10], except that we apply them on COD/ACOD instead of on OD/AOD. □

2 Upper bound on number of variables in an ACOD

In this section, we shall derive an upper bound on the number of variables that exist in an ACOD.

Theorem 1 *The maximum number of variables in an ACOD of order n is bounded by $H(n) = 2a + 2$ if $n = 2^a b$, b odd.*

PROOF: If $\{\mathbf{A}_j, 1 \le j \le s; \mathbf{B}_k, 1 \le k \le t\}$ is an ACOD, then $\{\mathbf{A}_j, 1 \le j \le s, i\mathbf{B}_k, 1 \le k \le t\}$ is a CF with $s+t$ variables; the result is immediate from Lemma 1. □

We note from our results above that both an ACOD and an AOD have the same bound on the maximum number of variables. Since ACOD includes AOD as a special case, and an AOD that achieves its maximum number of variables exists [4, Corollary 5.32], we can conclude that an ACOD that achieves the maximum number of variables also exists. Despite this similarity in property, we will demonstrate subsequently in this paper that some AODs thought to be non-existent or undecided in [6] can actually be found via ACOD.

3 Construction of an ACOD

In this section, we propose two systematic ways to construct an ACOD; one is to construct an ACOD of order $4n$ from an ACOD of order n, and another is to construct an ACOD of order $2n$ from an ACOD of order n. Since an ACOD is a generalization of an AOD, the following discussions and proposed construction methods also apply to AOD.

Construction 1: If $\{\mathbf{A}_j, 1 \le j \le s; \mathbf{B}_k, 1 \le k \le t\}$ is an ACOD of order n and type $(u_1, \dots, u_s; v_1, \dots, v_t)$ on the $s+t$ variables $\{x_1, \dots, x_s; y_1, \dots, y_t\}$, then

$$\begin{aligned}\{&\mathbf{B}_1 \otimes \mathbf{M}_1, \mathbf{B}_1 \otimes \mathbf{M}_2, \mathbf{B}_1 \otimes \mathbf{M}_3, \mathbf{A}_j \otimes \mathbf{I}_4, \quad 2 \le j \le s;\\ &\mathbf{A}_1 \otimes \mathbf{N}_1, \mathbf{A}_1 \otimes \mathbf{N}_2, \mathbf{A}_1 \otimes \mathbf{N}_3, \mathbf{B}_k \otimes \mathbf{I}_4, \quad 2 \le k \le t\}\end{aligned}$$

is an ACOD of order $4n$ and type $(v_1, v_1, v_1, u_2, \dots, u_s; u_1, u_1, u_1, v_2, \dots, v_t)$ on the $s+t+4$ variables $\{x_1, \dots, x_s, x_{s+1}, x_{s+2}; y_1, \dots, y_t, y_{t+1}, y_{t+2}\}$, where $\otimes$ represents Kronecker Product and

$$\mathbf{M}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \mathbf{M}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \mathbf{M}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{N}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{N}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \mathbf{N}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Construction 2: If $\{\mathbf{A}_j, 1 \le j \le s; \mathbf{B}_k, 1 \le k \le t\}$ is an ACOD of order n and type $(u_1, \dots, u_s; v_1, \dots, v_t)$ on the $s+t$ variables $\{x_1, \dots, x_s; y_1, \dots, y_t\}$, then

$$\{\mathbf{B}_1 \otimes \mathbf{N}_1, \mathbf{A}_j \otimes \mathbf{I}_2, 1 \le j \le s; \mathbf{B}_1 \otimes \mathbf{N}_2, \mathbf{B}_1 \otimes \mathbf{N}_3, \mathbf{B}_k \otimes \mathbf{I}_2, 2 \le k \le t\}$$

is an ACOD of order $2n$ and type $(v_1, u_1, u_2, \dots, u_s; v_1, v_1, v_2, \dots, v_t)$ on the $s+t+2$ variables $\{x_1, \dots, x_s, x_{s+1}; y_1, \dots, y_t, y_{t+1}\}$, where

$$\mathbf{N}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{N}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{N}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Theorem 2 *The ACOD of order* $4n$ *constructed by* Construction 1 *and the ACOD of order* $2n$ *constructed by* Construction 2 *have the maximum number of variables if each is constructed from an ACOD of order* n *having the maximum number of variables.*

PROOF: A new ACOD of order $4n$ or $2n$ constructed by *Construction 1* or *Construction 2* has respectively 4 or 2 more variables than the lower order ACOD used to construct it. Hence Theorem 2 is a result of Theorem 1. □

Example 1: An AOD of order 8 and type (2, 2, 2, 2; 2, 2, 2, 2)

From the AF $\left\{ \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right\}$ of order 2 and type (2, 2; 2, 2), the following AOD of order 8 and type (2, 2, 2, 2; 2, 2, 2, 2) is obtained by *Construction 1*:

$$\mathbf{X} = \begin{bmatrix} x_4 & x_4 & x_1 & -x_1 & x_2 & -x_2 & x_3 & -x_3 \\ x_4 & -x_4 & x_1 & x_1 & x_2 & x_2 & x_3 & x_3 \\ -x_1 & x_1 & x_4 & x_4 & x_3 & -x_3 & -x_2 & x_2 \\ -x_1 & -x_1 & x_4 & -x_4 & x_3 & x_3 & -x_2 & -x_2 \\ -x_2 & x_2 & -x_3 & x_3 & x_4 & x_4 & x_1 & -x_1 \\ -x_2 & -x_2 & -x_3 & -x_3 & x_4 & -x_4 & x_1 & x_1 \\ -x_3 & x_3 & x_2 & -x_2 & -x_1 & x_1 & x_4 & x_4 \\ -x_3 & -x_3 & x_2 & x_2 & -x_1 & -x_1 & x_4 & -x_4 \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} y_4 & y_4 & -y_1 & y_1 & -y_2 & y_2 & -y_3 & y_3 \\ -y_4 & y_4 & y_1 & y_1 & y_2 & y_2 & y_3 & y_3 \\ y_1 & -y_1 & y_4 & y_4 & y_3 & -y_3 & -y_2 & y_2 \\ -y_1 & -y_1 & -y_4 & y_4 & -y_3 & -y_3 & y_2 & y_2 \\ y_2 & -y_2 & -y_3 & y_3 & y_4 & y_4 & y_1 & -y_1 \\ -y_2 & -y_2 & y_3 & y_3 & -y_4 & y_4 & -y_1 & -y_1 \\ y_3 & -y_3 & y_2 & -y_2 & -y_1 & y_1 & y_4 & y_4 \\ -y_3 & -y_3 & -y_2 & -y_2 & y_1 & y_1 & -y_4 & y_4 \end{bmatrix} \quad (5)$$

Remarks:

- It should be noted that (5) is not equivalent to the AOD of order 8 and type (2, 2, 2, 2; 2, 2, 2, 2) obtained in [10]. Two designs are *equivalent* if one can be obtained from the other by the following sequence of operations [10]:
  - Multiply one row (one column) by -1.
  - Swap two rows (columns).
  - Rename or negate a variable throughout the design.
- Since (5) contains no zero entries, it has practical importance as it provides power-balanced transmit diversity in wireless communication systems [7, 9].
- By applying Lemmas 2 and 3, we can use (5) to construct new AODs that were undecided in [6], such as the AOD of type (2,2,2,2; 2,2,2).
- The above example demonstrates that, by using our proposed *Construction 1*, we can construct an AOD from an AF, instead of from an AOD.
- Clearly, *Construction 1* can also be used to construct an ACOD from a smaller order ACF.

Example 2: An AOD of order 8 and type (1, 2, 2, 2; 1, 1, 1, 2)

From the AF $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right\}$ of order 2 and type (1, 1; 2, 2), the following AOD of order 8 and type (1, 2, 2, 2; 1, 1, 1, 2) is obtained by *Construction 1*:

$$\mathbf{X} = \begin{bmatrix} 0 & x_4 & x_1 & x_1 & x_2 & x_2 & x_3 & x_3 \\ x_4 & 0 & -x_1 & x_1 & -x_2 & x_2 & -x_3 & x_3 \\ -x_1 & -x_1 & 0 & x_4 & x_3 & x_3 & -x_2 & -x_2 \\ x_1 & -x_1 & x_4 & 0 & -x_3 & x_3 & x_2 & -x_2 \\ -x_2 & -x_2 & -x_3 & -x_3 & 0 & x_4 & x_1 & x_1 \\ x_2 & -x_2 & x_3 & -x_3 & x_4 & 0 & -x_1 & x_1 \\ -x_3 & -x_3 & x_2 & x_2 & -x_1 & -x_1 & 0 & x_4 \\ x_3 & -x_3 & -x_2 & x_2 & x_1 & -x_1 & x_4 & 0 \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} y_4 & -y_4 & y_1 & 0 & y_2 & 0 & y_3 & 0 \\ y_4 & y_4 & 0 & -y_1 & 0 & -y_2 & 0 & -y_3 \\ -y_1 & 0 & y_4 & -y_4 & -y_3 & 0 & y_2 & 0 \\ 0 & y_1 & y_4 & y_4 & 0 & y_3 & 0 & -y_2 \\ -y_2 & 0 & y_3 & 0 & y_4 & -y_4 & -y_1 & 0 \\ 0 & y_2 & 0 & -y_3 & y_4 & y_4 & 0 & y_1 \\ -y_3 & 0 & -y_2 & 0 & y_1 & 0 & y_4 & -y_4 \\ 0 & y_3 & 0 & y_2 & 0 & -y_1 & y_4 & y_4 \end{bmatrix} \qquad (6)$$

Although (6) is equivalent to the AOD obtained in [10], Examples 1 and 2 show that our proposed *Construction 1* is general and can be used to construct many AODs systematically. Furthermore, we can apply Lemmas 2 and 3 on (6) to construct new AODs thought to be non-existent or which were undecided in [6], such as AODs of type (1,1,1,2; 1,2,2,2), (1,1,1,2; 1,2,4), (1,1,1,2; 2,2,2), (1,1,1,2; 2,2,3), (1,1,1,2; 2,4), (1,1,1,2; 3,4), (1,1,1,2; 1,6), (1,1,1,2; 2,5), (1,1,1,2; 6), (1,1,1,2; 7), (1,1,1; 2,2,2), (1,1,1; 2,2,3), (1,1,2; 1,2,4), (1,1,2; 2,2,3), (1,1,2; 2,5), (1,1; 2,2,3), (1,2; 2,2,3), (1; 1,2,2,2), (2; 1,2,2,2), (3; 1,2,2,2), (4; 1,2,2,2), (5; 1,2,2,2).

Example 3: An AOD of order 8 and type (1, 1, 2, 2; 1, 1, 2, 2).

$$\text{From the AOD}\left\{\begin{array}{l}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix},\begin{bmatrix}0&0&1&1\\0&0&1&-1\\1&1&0&0\\1&-1&0&0\end{bmatrix};\\[2ex]\begin{bmatrix}0&0&1&1\\0&0&1&-1\\-1&-1&0&0\\-1&1&0&0\end{bmatrix},\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\end{array}\right\}$$

of order 4 and type (1, 1, 2; 2, 1, 1), the following AOD of order 8 and type (1, 1, 2, 2; 1, 1, 2, 2) is obtained by *Construction 2*:

$$\begin{aligned}\mathbf{X}&=\begin{bmatrix}x_2&x_3&x_4&x_4&0&0&x_1&x_1\\-x_3&x_2&x_4&-x_4&0&0&x_1&-x_1\\x_4&x_4&-x_2&-x_3&-x_1&-x_1&0&0\\x_4&-x_4&x_3&-x_2&-x_1&x_1&0&0\\0&0&-x_1&-x_1&x_2&x_3&x_4&x_4\\0&0&-x_1&x_1&-x_3&x_2&x_4&-x_4\\x_1&x_1&0&0&x_4&x_4&-x_2&-x_3\\x_1&-x_1&0&0&x_4&-x_4&x_3&-x_2\end{bmatrix}\\ \mathbf{Y}&=\begin{bmatrix}y_3&y_4&y_2&y_2&0&0&y_1&y_1\\-y_4&-y_3&y_2&-y_2&0&0&y_1&-y_1\\-y_2&-y_2&y_4&y_3&-y_1&-y_1&0&0\\-y_2&y_2&y_3&-y_4&-y_1&y_1&0&0\\0&0&y_1&y_1&y_3&y_4&-y_2&-y_2\\0&0&y_1&-y_1&-y_4&-y_3&-y_2&y_2\\-y_1&-y_1&0&0&y_2&y_2&y_4&y_3\\-y_1&y_1&0&0&y_2&-y_2&y_3&-y_4\end{bmatrix}\end{aligned}\tag{7}$$

Similarly, we can apply Lemmas 2 and 3 on (7) to construct new AODs that were undecided in [6], such as that of type (1,1,2,2; 1,1,4).

Example 4: An ACOD of order 8 and type (1, 1, 1, 1; 2, 2, 2, 2).

From the ACF $\left\{\begin{bmatrix} i & -i \\ i & i \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right\}$ of order 2 and type (2, 1; 1, 2) (an AF of this type could not be found using exhaustive computer search), the following ACOD of order 8 and type (1, 1, 1, 1; 2, 2, 2, 2) is obtained by *Construction 1*:

$$\mathbf{X} = \begin{bmatrix} x_4 & 0 & 0 & ix_1 & 0 & ix_2 & 0 & ix_3 \\ 0 & -x_4 & ix_1 & 0 & ix_2 & 0 & ix_3 & 0 \\ 0 & -ix_1 & x_4 & 0 & 0 & ix_3 & 0 & -ix_2 \\ -ix_1 & 0 & 0 & -x_4 & ix_3 & 0 & -ix_2 & 0 \\ 0 & -ix_2 & 0 & -ix_3 & x_4 & 0 & 0 & ix_1 \\ -ix_2 & 0 & -ix_3 & 0 & 0 & -x_4 & ix_1 & 0 \\ 0 & -ix_3 & 0 & ix_2 & 0 & -ix_1 & x_4 & 0 \\ -ix_3 & 0 & ix_2 & 0 & -ix_1 & 0 & 0 & -x_4 \end{bmatrix}$$
$$\mathbf{Y} = \begin{bmatrix} y_4 & y_4 & iy_1 & -iy_1 & iy_2 & -iy_2 & iy_3 & -iy_3 \\ -y_4 & y_4 & iy_1 & iy_1 & iy_2 & iy_2 & iy_3 & iy_3 \\ -iy_1 & iy_1 & y_4 & y_4 & -iy_3 & iy_3 & iy_2 & -iy_2 \\ -iy_1 & -iy_1 & -y_4 & y_4 & -iy_3 & -iy_3 & iy_2 & iy_2 \\ -iy_2 & iy_2 & iy_3 & -iy_3 & y_4 & y_4 & -iy_1 & iy_1 \\ -iy_2 & -iy_2 & iy_3 & iy_3 & -y_4 & y_4 & -iy_1 & -iy_1 \\ -iy_3 & iy_3 & -iy_2 & iy_2 & iy_1 & -iy_1 & y_4 & y_4 \\ -iy_3 & -iy_3 & -iy_2 & -iy_2 & iy_1 & iy_1 & -y_4 & y_4 \end{bmatrix} \quad (8)$$

To our knowledge, (8) is the first ACOD of type (1, 1, 1, 1; 2, 2, 2, 2) ever reported. Furthermore, by applying Lemmas 2 and 3 on (8), we can now construct new ACODs whose corresponding AODs of the same types were thought to be non-existent or were undecided in [6], as shown in Table 1. This again demonstrates that by allowing complex entries in the code matrix, more types of AODs become possible.

Example 5: An ACOD of order 8 and type (1, 2, 2, 2; 1, 2, 2, 2)

From the ACF $\left\{\begin{bmatrix} i & -i \\ i & i \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\right\}$ of order 2 and type (2, 1; 2, 1), the following ACOD of order 8 and type (1, 2, 2, 2; 1, 2, 2, 2) is obtained by *Construction 1*:

$$\mathbf{X} = \begin{bmatrix} x_4 & 0 & x_1 & x_1 & x_2 & x_2 & x_3 & x_3 \\ 0 & -x_4 & -x_1 & x_1 & -x_2 & x_2 & -x_3 & x_3 \\ -x_1 & -x_1 & x_4 & 0 & x_3 & x_3 & -x_2 & -x_2 \\ x_1 & -x_1 & 0 & -x_4 & -x_3 & x_3 & x_2 & -x_2 \\ -x_2 & -x_2 & -x_3 & -x_3 & x_4 & 0 & x_1 & x_1 \\ x_2 & -x_2 & x_3 & -x_3 & 0 & -x_4 & -x_1 & x_1 \\ -x_3 & -x_3 & x_2 & x_2 & -x_1 & -x_1 & x_4 & 0 \\ x_3 & -x_3 & -x_2 & x_2 & x_1 & -x_1 & 0 & -x_4 \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} 0 & iy_4 & iy_1 & -iy_1 & iy_2 & -iy_2 & iy_3 & -iy_3 \\ iy_4 & 0 & iy_1 & iy_1 & iy_2 & iy_2 & iy_3 & iy_3 \\ -iy_1 & iy_1 & 0 & iy_4 & -iy_3 & iy_3 & iy_2 & -iy_2 \\ -iy_1 & -iy_1 & iy_4 & 0 & -iy_3 & -iy_3 & iy_2 & iy_2 \\ -iy_2 & iy_2 & iy_3 & -iy_3 & 0 & iy_4 & -iy_1 & iy_1 \\ -iy_2 & -iy_2 & iy_3 & iy_3 & iy_4 & 0 & -iy_1 & -iy_1 \\ -iy_3 & iy_3 & -iy_2 & iy_2 & iy_1 & -iy_1 & 0 & iy_4 \\ -iy_3 & -iy_3 & -iy_2 & -iy_2 & iy_1 & iy_1 & iy_4 & 0 \end{bmatrix} \quad (9)$$

To our knowledge, an ACOD of this type also has not been reported before. By applying Lemmas 2 and 3 on (9), new ACODs whose corresponding AODs of the same types were thought to be undecided in [6] are found, as shown in Table 2.

We wish to point out an interesting observation that although the ACFs of order 2 used in *Example 5* and *Example 4* are the same because they differ only in the order of variables, they can be used to generate two totally different higher-order ACODs.

The above examples demonstrate the generality of our proposed construction method, and some basic differences between ACOD and AOD even though they have the same maximum number of variables. New applications of high-order AODs or ACODs, such as those with order greater than 8, are interesting areas to investigate.

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Legend for Tables 1 and 2

- ? Undecided
- × Non-existent
- √ Existent

Table 1: ACOD of order 8 based on *Example 4*

Order 8		AOD	ACOD
Type	Type	[6]	This paper
1 1 1 1	2 2 2 2	×	√ *Example 4* in (10)
1 1 1 1	2 2 4	×	√
1 1 1 1	2 2 2	?	√
1 1 1 1	4 4	×	√
1 1 1 1	2 4	?	√
1 1 1 1	2 6	×	√
1 1 1 1	6	?	√
1 1 1 1	8	×	√
1 1 1	2 2 2 2	×	√
1 1 1	2 2 2	?	√
1 1 1	2 2 4	×	√
1 1 1	8	×	√
1 1 2	2 2 2 2	×	√
1 1	2 2 2 2	×	√
1 1	4 4	×	√
1 1	2 6	×	√
1 2	2 2 2 2	×	√
1	2 2 2 2	×	√
3	2 2 2 2	?	√

Table 2: ACOD of order 8 based on *Example 5*

Order 8		AOD	ACOD
Type	Type	[6]	This paper
1 2 2 2	1 2 2 2	?	√ *Example 5* in (11)
1 2 2 2	1 2 2	?	√
1 2 2 2	1 2 4	?	√
1 2 2 2	2 2 2	?	√
1 2 2 2	2 2 3	?	√
1 2 2 2	1 2	?	√
1 2 2 2	2 2	?	√
1 2 2 2	3 4	?	√
1 2 2 2	1 6	?	√
1 2 2 2	2 5	?	√
1 2 2 2	2 3	?	√
1 2 2 2	1	?	√
1 2 2 2	2	?	√
1 2 2 2	3	?	√
1 2 2 2	4	?	√
1 2 2 2	5	?	√
1 2 2 2	6	?	√
1 2 2 2	7	?	√
1 2 2	2 2 2	?	√
1 2 4	2 2 2	?	√
2 2 2	2 2 3	?	√
1 2 2	3 4	?	√
1 2 2	1 6	?	√
1 2 2	2 5	?	√
2 2 2	3 4	?	√
2 2 2	1 6	?	√
2 2 2	2 5	?	√
2 2 2	7	?	√

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