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On 3-choosability of plane graphs having no 3-, 6-, 7- and 8-cycles

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Abstract

A graph is k-choosable if it can be colored whenever every vertex has a list of available colors of size at least k. It is a generalization of graph coloring where all vertices do not have the same available colors. We show that every triangle-free plane graph without 6-, 7-, and 8-cycles is 3-choosable.

1 Introduction

List coloring is a generalization of graph coloring where every vertex v has its own list of colors $L(v)$. The coloring c assigns every vertex v a color from $L(v)$. Moreover, colors of vertices joined by an edge must be different. The concept of the list coloring was introduced independently by Vizing [7] and Erdős et al. [2].

We say that a graph G is *k-choosable* if it allows a list coloring for every list assignment where $|L(v)| \geq k$ for every vertex v. Observe that the graph coloring problem is a special case of the list coloring problem where all lists have the same content.

Thomassen [5] proved that every planar graph is 5-choosable. Voigt [8] showed that not all planar graphs are 4-choosable. Kratochvíl and Tuza [3] observed that every planar triangle-free graph is 4-choosable, and Voigt [9] exhibited an example of a non-3-choosable triangle-free planar graph.

There are several sufficient conditions for 3-choosability of planar triangle-free graphs. Alon and Tarsi [1] proved that every planar bipartite graph is 3-choosable. Thomassen [6] gave a proof showing that every planar graph without 3- and 4-cycles is 3-choosable. Lam, Shui and Song [4] proved that every planar graph without 3-, 5- and 6-cycles is 3-choosable. Zhang and Xu [11] proved that every planar graph without 3-, 6-, 7- and 9-cycles is 3-choosable. Zhang [10] proved that every planar graph without 3-, 5-, 8- and 9-cycles is 3-choosable.

We show that every planar graph without 3-, 6-, 7- and 8-cycles is 3-choosable. The result is derived as a corollary of Theorem 3 which restricts adjacency of small cycles in a graph.

The proof is done by the well known discharging method. We apply the method in the following manner. We consider a hypothetical smallest counterexample and identify some configurations which cannot occur in the smallest counterexample. Then faces and vertices are assigned charges such that sum of all charges is negative. Discharging rules redistribute charges between vertices and faces such that the resulting charge of every face and every vertex is non-negative. Hence we get a contradiction with assumption that the sum of the charges is negative.

We use the following notation. A cycle of length k is said to be a *k-cycle* and a vertex of degree k is said to be a *k-vertex*. Analogously a cycle of length at least k is said to be a *$(\geq k)$-cycle* and a vertex of degree at least k is said to be a *$(\geq k)$-vertex*. A similar notation is also used for faces.

2 Reducible configurations

A *configuration* R is (H, d) where H is a simple graph and d is a function $V(H) \to \mathbb{N}$. A graph G contains R if G contains a subgraph K such that there is an isomorphism $f : H \to K$ and for every vertex v from $V(H)$ holds that $\deg_G(f(v)) = d(v)$.

We say that a configuration R is *reducible* if removing R from any graph G does not affect the property of 3-choosability of G.

Note that removing any configuration R from a 3-choosable graph G does not make G non-3-choosable since we only delete edges and vertices. On the other hand removing R from a non-3-choosable graph G can turn G into a 3-choosable graph but this is not the case when the configuration R is reducible.

Observe that an isolated vertex v and a function $d(v) = 1$ or $d(v) = 2$ form a reducible configuration since adding a vertex of degree 1 or 2 to a graph G does not turn G into a non-3-choosable graph.

Next we describe a bigger reducible configuration. Let Z be a connected graph which is created as a union of 4-cycles $C_4^1, C_4^2, \ldots, C_4^n$ such that two 4-cycles are allowed to share at most one vertex and one vertex can be incident to at most two 4-cycles. Moreover, $C_4^1, C_4^2, \ldots, C_4^n$ are the only cycles in Z. Observe that the 4-cycles must form some kind of a tree structure. Refer to the black drawing in Figure 1.

We create H from Z by adding some edges joining vertices of degree two in distance more than two. Moreover, every vertex can be in at most one added edge. We call such graph H a *C_4-arbor*. We show that H with a suitable d is a reducible configuration. Note that we do not require C_4-arbors to be planar but we need only planar C_4-arbors in this paper. Refer to Figure 1 for an example of a planar C_4-arbor.

Lemma 1. *Let H be a C_4-arbor and let $d : V(H) \to \mathbb{N}$ be defined as follows:*

$$d(v) = \begin{cases} \deg_H(v) & \text{if } \deg_H(v) \geq 3; \\ 3 & \text{if } \deg_H(v) = 2. \end{cases}$$

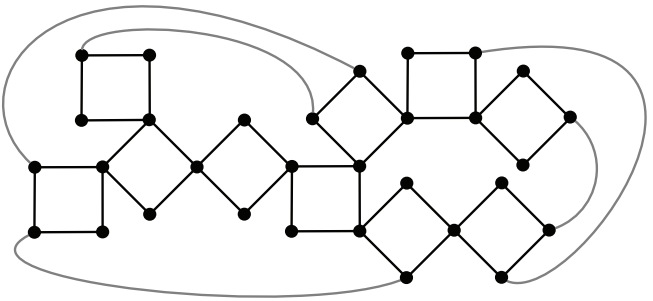

Figure 1: C_4-arbor

Then $R = (H, d)$ is a reducible configuration.

Proof. Let G be a graph and let H be a C_4-arbor subgraph with degrees according to d. Assume that $G - H$ is 3-choosable. Our goal is to show that G is also 3-choosable. More precisely we want to extend a list coloring c of $G - H$ to a list coloring c' of G. We assume that every vertex v of still uncolored H has $L(v)$ of size 3.

The proof proceeds by induction on n. Let H be a C_4-arbor formed by a union of 4-cycles $Z = \{C_4^1, C_4^2, \ldots, C_4^n\}$ and some additional edges. Observe that every vertex v in H with $\deg_H(v) = 2$ has $\deg_G(v) = 3$ and the other vertices of H have the same degree in G as in H. Since v of $\deg_H(v) = 2$ has exactly one neighbor u in $G - H$, it has a forbidden color $c(u)$. We alter $L(v)$ to $L(v) \setminus \{c(u)\}$ to avoid conflict during extending c.

$$|L(v)| = \begin{cases} 2 & \text{if } \deg_H(v) = 2; \\ 3 & \text{otherwise.} \end{cases}$$

First we consider case $Z = \{C_4^1\}$ for starting the induction. Then $H = C_4$ since we are not allowed to add any new edge and $d(v) = 2$ for every vertex v from $V(H)$. The discussion why C_4 can be colored is analogous to the discussion for the induction step or we could use the fact that C_4 is 2-choosable.

If all 4-cycles have at least two shared vertices, then we can find a long cycle which is forbidden by the definition of C_4-arbor. Thus there is a 4-cycle C with exactly one shared vertex x.

We color the non-shared vertices u, v, w of C such that there will remain two possible colors for x. Then we remove u, v and w from H. This decreases the number of 4-cycles and the shared vertex x becomes a vertex of degree 2 with $L(x)$ of size 2. Hence we can apply the induction hypothesis.

Let the shared vertex x be adjacent to vertices u and v; refer to Figure 2. We distinguish two cases.

If a is a common color of $L(u)$ and $L(v)$ then we color u and v by a. Since u and v have the same color a, there are still at least two colors left in $L(x)$ and at least

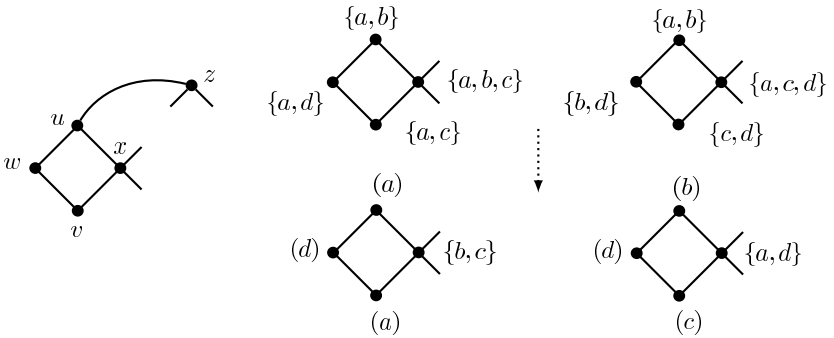

Figure 2: The 4-face in the induction step.

one color is left in $L(w)$. Thus we can assign such remaining color to w and preserve a color list of size 2 for x.

In the other case the color lists of u and v are disjoint. Their color lists have 4 different colors together while $L(x)$ contains only 3 colors. Hence there is a color b which is not in $L(x)$. Without loss of generality assume that b is in $L(u)$. We color u by the color b. Then we color w by a color distinct from b. Finally we color v by a color different from the color of w. The color of v may be present in $L(x)$, but there are still at least 2 different colors left in $L(x)$.

We also need to deal with vertices of H joined to vertices u, v, and w by edges, if there are any. Let z be a vertex connected to vertex u. The vertex z has a color list of size 3. So we can remove the color of u from $L(z)$ to avoid conflict. The vertex z is then treated as a vertex of degree two. □

3 Initial charges

In this section we define the initial charges for faces and vertices. We start with defining that for a plane graph G the *degree* of a face f is the number of incident edge sides. We denote it by $\deg(f)$. Observe that one edge can raise $\deg(f)$ by two if both sides of the edge are incident with f.

Recall that for every plane graph G holds that $\sum_{v\in V}\deg(v) = 2|E|$ and observe that also holds that $\sum_{f\in F}\deg(f) = 2|E|$ since every edge is counted twice.

The *initial charge* of a face f and the *initial charge* of a vertex v are defined by

$$\mathrm{ch}(f) := \deg(f) - 6 \text{ and } \mathrm{ch}(v) := 2\deg(v) - 6.$$

Refer to Table 1 for the initial charges of faces and vertices of small degrees.

Lemma 2. *If $G = (V, E)$ is a plane connected graph, then the sum of all initial charges is negative.*

deg	3	4	5	6	7
ch(f)	-3	-2	-1	0	1
ch(v)	0	2	4	6	8

Table 1: Initial charges of a face f and a vertex v depending on their degree.

Proof. The idea of the proof is based on counting with Euler's formula, which says that $|E| - |V| - |F| = -2$ where F is the set of faces. Counting with the charges gives:

$$\begin{aligned}\sum_{f\in F}\mathrm{ch}(f) + \sum_{v\in V}\mathrm{ch}(v) &= \sum_{f\in F}(\deg(f)-6) + \sum_{v\in V}(2\deg(v)-6)\\ &= (2|E| - 6|F|) + (4|E| - 6|V|)\\ &= -12.\end{aligned}$$

□

The goal is to show that the sum of all charges is non-negative in the assumed counterexample and hence it violates the Euler's formula. In Table 1 we observe that 3-, 4- and 5-faces have negative initial charge hence we need to deal with these negative charges. We satisfy 3-faces by the condition that our graph is triangle-free. Hence we have to deal only with 4- and 5-faces. Observe that forbidding C_3, C_4 and C_5 is a sufficient condition for a planar graph to be 3-choosable.

4 Discharging 4-faces and 5-faces

We denote by $C_{x|y}$ a graph constructed from a cycle on $x + y - 2$ vertices by adding a single edge to form two cycles of lengths x and y. We say that cycles are *touching* if they share exactly one vertex. We call two cycles *adjacent* if they form $C_{x|y}$.

A 4-vertex v is called *non-shared* if it is incident to only one 4-face with negative charge. We say that a 4-vertex v is *shared* if it is incident to two 4-faces with negative charges. It may seem superfluous to require that the 4-faces have a negative charge since they all have it at the beginning. But during the discharging process a shared vertex may become non-shared if one of its incident 4-faces receives charge from somewhere else.

We say that a 5-face is *isolated* if it does not share any vertex with another 5-face.

Theorem 3. *Every planar graph without C_3, $C_{4|4}$, $C_{4|5}$, $C_{5|5}$, and $C_{5|6}$ is 3-choosable.*

Proof. Assume for a contradiction that G is a minimal counterexample. By minimality G does not contain vertices of degree two or configurations from Lemma 1. Moreover, G is also a connected graph.

We claim that there are no two 4-faces, or a 4-face and a 5-face, or two 5-faces or a 5-face and a 6-face that would share two edges because G does not contain any vertices of degree two or triangles; refer to Figure 7.

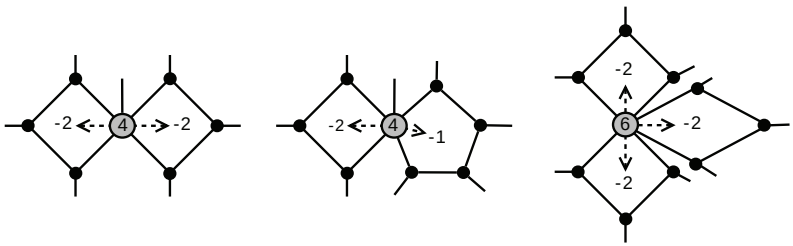

Figure 3: Illustration of Rule 1. The grey vertices have a sufficient charge to eliminate the negative charges of the incident 4-faces and 5-faces.

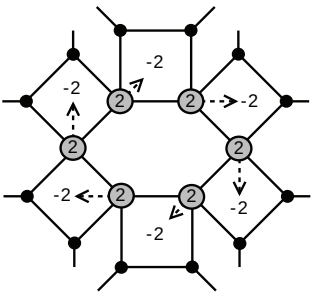

Figure 4: Illustration of Rule 2. A cycle of touching 4-faces. The arrows illustrate sending charges along the cycle.

Note that two 5-faces or a 5-face and a 6-face can share an edge and one extra vertex. Let v be a vertex. A 5-face f is *proper* with respect to v if either f does not share an edge with another 5-face or it shares an edge e with another 5-face but v is not incident to e; refer to Figure 8. Observe that a 5-face can share no edge with two 5-faces.

4.1 The discharging rules

We claim that we are able to raise charge of all 4-faces and 5-faces in G to a non-negative value by subsequent application of the following rules:

Rule 1. *A* (≥ 5)*-vertex* v *sends charge* 2 *to all incident* 4*-faces and charge* 1 *to all incident proper* 5*-faces; refer to Figure 3.*

Rule 2. *On a cycle of touching* 4*-faces with negative charges the shared vertices send charge* 2 *along the cycle to the* 4*-faces; refer to Figure 4.*

Rule 3. *A non-shared* 4*-vertex sends charge* 2 *to its* 4*-face with negative charge; refer to Figure 5.*

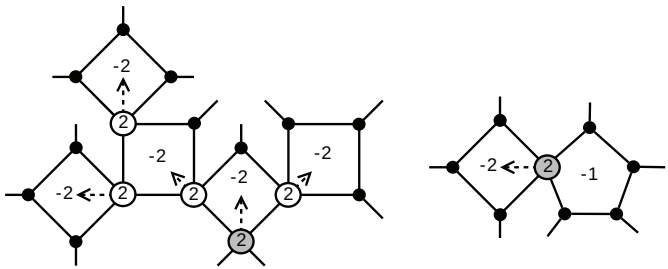

Figure 5: Illustration of Rule 3. The grey vertex eliminates the negative charge of the incident 4-face. The other white shared vertices then become non-shared. Note that the negative charge of the 5-face is not discharged.

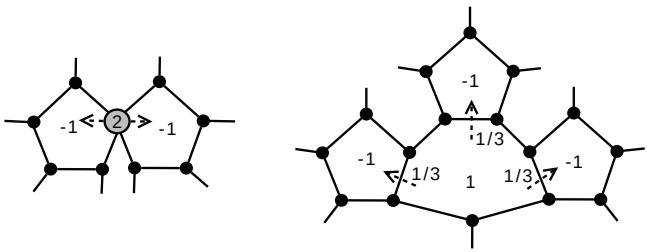

Figure 6: Illustration of Rules 4 and 5. Discharging touching 5-faces and distributing charge 1 from a 7-face to adjacent 5-faces.

Rule 4. *A* 4*-vertex which was not used in Rule 3 sends charge* 1 *to every incident proper* 5*-face; refer to Figure 6.*

Rule 5. *A* (≥ 7)*-face sends charge* $\frac{1}{5}$ *through every edge which is shared with an isolated* 5*-face; refer to Figure 6.*

The rules are applied sequentially from Rule 1 to Rule 5. Note that in Rule 5 a big face can send charge to the same 5-face through several edges.

Next we check that the final charge of all faces and vertices is non-negative.

Let v be a k-vertex. Regarding its degree we consider following cases:

$k \geq 5$**:** Only Rule 1 can apply on v. Charge transfered from v in is at most $2\lfloor \frac{k}{2} \rfloor$. Hence we compute $\mathrm{ch}(v) - 2\lfloor \frac{k}{2} \rfloor = 2k - 2\lfloor \frac{k}{2} \rfloor - 6 \geq 0$.

$k = 4$**:** Only one of the Rules 2,3 or 4 can apply on v. Charge at most two is transferred from v hence the resulting charge is non-negative.

$k = 3$**:** The initial zero charge is not changed.

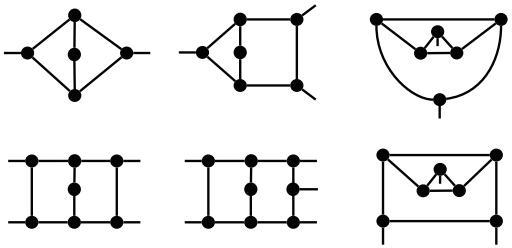

Figure 7: Two small cycles drawn as faces that share two edges require a vertex of degree two or triangles.

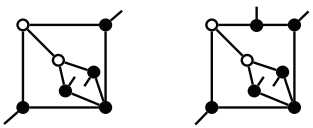

Figure 8: Two 5-faces and a 5-face with a 6-face sharing an edge and one more vertex. The 5-faces are considered proper with respect to the black vertices.

Let f be a k-face. We consider cases regarding k:

$k \geq 7$: Only Rule 5 applies on f. Note that the number of edges shared with isolated 5-faces is at most $\lfloor \frac{k}{2} \rfloor$ since these 5-faces share no vertices; refer to Figure 9. Hence the resulting charge of f is $k - 6 - \frac{1}{5}\lfloor \frac{k}{2} \rfloor$ which is non-negative.

$k = 6$: The initial zero charge is not changed.

$k = 5$: If Rule 1 or Rule 4 applies then the resulting charge of f is non-negative. Hence assume that they do not apply.

If f shares an edge with another 5-face or a 6-face f' then f also shares a (≥ 4)-vertex v with f'. Observe that f is proper with respect to v. If v is a 4-vertex then it is not incident to any 4-face. Hence Rule 4 applies on f. If v is a (≥ 5)-vertex then Rule 1 applies. Therefore we can assume that f is proper with respect to all vertices of G.

If f shares a vertex v with another 5-face then Rule 1 or Rule 4 applies on v and f. Hence f is isolated.

Every isolated 5-face is adjacent to five (≥ 7)-faces. Hence Rule 5 applies five times on f and the incoming charge is $\frac{5}{5}$. Thus the resulting charge is non-negative.

$k = 4$: If any of the Rules $1, 2$ or 3 applies on f then the incoming charge is 2 and the resulting charge is 0. Hence assume that none of the Rules applies.

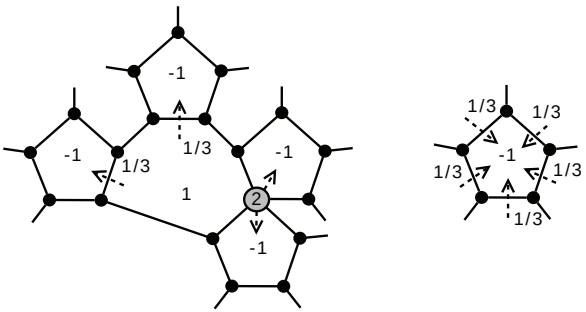

Figure 9: A big face is sending charges only to isolated 5-faces and incoming charges to an isolated 5-face.

Since G is a minimal counterexample, f must be incident to at least one 4-vertex v. We create a bipartite graph $B = (V_B = V_B^1 \cup V_B^2, E_B)$ where vertices of V_B^1 correspond to 4-faces of G with negative charge and the vertices of V_B^2 correspond to 4-vertices incident with the 4-faces. An edge uv where $u \in V_B^1$ and $v \in V_B^2$ is present if the vertex of G corresponding to v is incident to the 4-face of G corresponding to u. If B is not connected we process each connected component separately.

If B contains a cycle, then Rule 2 applies. Hence B is a forest. If $v \in V_B^2$ is a leaf, then the corresponding 4-vertex in G is non-shared and Rule 3 applies. Hence the faces corresponding to V_B^1 induce a C_4-arbor and Lemma 1 applies. It is a contradiction with the assumption that G is minimal.

Therefore after redistribution of charges every face and every vertex has a non-negative charge and we have a contradiction with Lemma 2. □

Observe that graphs without C_6 do not contain $C_{4|4}$. We can also use absence of $C_{4|5}$ in graphs without C_7 and absence of $C_{5|5}$ in graphs without C_8. Hence Theorem 3 implies the result of Lam et al. [4] that planar graphs without C_3, C_5 and C_6 are 3-choosable, as well as the following two corollaries.

Corollary 4. *Every planar graph without C_3, C_6, C_7 and C_8 is 3-choosable.*

Corollary 5. *Every planar graph without C_3, C_5 and $C_{4|4}$ is 3-choosable.*

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