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A note on orbit graphs of finite groups and colour-clique graphs of Cayley graphs

Jana Tomanová

Department of Algebra and Number Theory
Comenius University
Mlynská dolina, 842 15 Bratislava
Slovakia
tomanova@fmph.uniba.sk

Abstract

Let G be a (finite) group and let S be a non-empty subset of G. The vertex set of the *orbit graph* $O(G, S)$ is the union, over all $s \in S$, of orbits of left translations induced by s. If u and v are distinct vertices (each representing an orbit of some s and t from S), then for any $g \in G$ appearing in both orbits there is an edge coloured g in $O(G, S)$ joining u and v. Orbit graphs are an important special case of "G-graphs" introduced by Bretto and Faisant in Math. Slovaca 55 (2005). In this paper we establish a correspondence between simple orbit graphs $O(G, S)$ such that $1 \notin S$ and colour-clique graphs of certain Cayley graphs. This correspondence is used to answer questions about orbit graphs in terms of Cayley graphs.

1 Introduction

Let G be a (finite) group and let S be a non-empty subset of G. For each $s \in S$ let λ_s be the left translation $g \mapsto sg$ for all $g \in G$. The *orbit graph* $O(G, S)$ associated with the pair (G, S) is defined as follows. For each $s \in S$ and each orbit ω of λ_s there is a vertex corresponding to the pair (s, ω). For any unordered pair of distinct vertices (s, ω), (s', ω') and for any element $h \in G$ such that $h \in \omega \cap \omega'$ there is an edge with *colour* h that joins (s, ω) and (s', ω').

Motivated by potential application in group isomorphism testing, a more general version of orbit graphs were first introduced in [1] under the name "G-graphs". Vertices of G-graphs are defined as *cyclic orbits* of λ_s, that is, cyclic sequences $(g, sg, \ldots, s^{o(s)-1}g)$, $g \in G$, where $o(s)$ is the order of s (equivalently, cycles in the cyclic decomposition of λ_s regarded as a permutation of the set G). However, it is easy to see that the vertex set of a G-graph can be identified with the vertex set of our orbit graph. The only difference between G-graphs and orbit graphs is that, in

[1], the possibility $u = v$ was allowed in the description of adjacency by intersection, thus giving rise to loops at every vertex.

A number of properties of G-graphs were further studied in [2, 4, 5, 6, 7]. The interesting cases were, however, concerned with the restricted version we have introduced above. We strongly prefer the term "orbit graphs" in what follows, as well as keeping the graphs loopless.

Since no two vertices (s, ω) and (s, ω') representing distinct orbits of the *same* translation λ_s are joined by an edge, the graph $O(G, S)$ is $|S|$-partite. Observe that for any given $s \in S$, each of the $o(s)$ elements appearing in an orbit of λ_s is contained in some orbit of λ_t for all $t \in S$ such that $t \neq s$. It follows that the valency of a vertex corresponding to an orbit of λ_s is equal to $o(s)(|S| - 1)$. It is now easy to see that $O(G, S)$ is a simple graph (that is, a graph with no multiple edges) if and only if $\langle s \rangle \cap \langle t \rangle = \{1\}$ for any two distinct elements s, t in S, as was noted in [3]. Note that in this case for any two distinct elements $s, t \in S$ the set of all orbits of λ_s is disjoint from the set of all orbits of λ_t. We will therefore identify the vertex set of the *simple* orbit graph $O(G, S)$ with the union, over all $s \in S$, of orbits of left translations λ_s induced by s.

Once a class of graphs associated with groups has been defined, there are natural questions about relationships between the class and other known families of graphs associated with groups, in particular, Cayley graphs. The study of such connections in our case is also motivated by the following facts.

Generalizing an earlier observation of [4, 6], it was noted in [3] that if $|S| \geq 2$ and $g \in G$, then the subgraph K_g of $O(G, S)$ induced by all the edges coloured g is a complete graph on $|S|$ vertices. The *colour-clique graph* of the family $\mathcal{F}(G, S) = \{K_g;\ g \in G\}$ has vertex set $\mathcal{F}(G, S)$, with a pair of distinct vertices adjacent if the corresponding complete subgraphs share a vertex. Letting now $S^* = (\cup_{s \in S} \langle s \rangle) \setminus \{1\}$, it was proved in [3] that the mapping $K_g \mapsto g$ induces an isomorphism from the colour-clique graph of the family $\mathcal{F}(G, S)$ onto the Cayley graph $C(G, S^*)$. Briefly, the orbit graph $O(G, S)$ induces a certain colour-clique graph isomorphic to the Cayley graph $C(G, S^*)$.

Our main aim is to show a much more direct correspondence for *simple* orbit graphs: namely, that every simple orbit graph $O(G, S)$ such that $1 \notin S$ is isomorphic to a colour-clique graph of the Cayley graph $C(G, S^*)$ whose definition will be given in the next section. This correspondence will be used to answer questions (translated from the original "G-graphs" setting) regarding orbit graphs in terms of Cayley graphs.

Note that when studying simple orbit graphs we may restrict ourselves to the case $1 \notin S$. Indeed, if $1 \in S$ then the vertex class of $O(G, S)$ corresponding to λ_1 has $|G|$ vertices, each one corresponding to an element of G. If in addition $|S| \geq 2$ then the graph $O(G, S)$ can be easily recovered from $O(G, S \setminus \{1\})$ as follows. We extend the vertex set of $O(G, S \setminus \{1\})$ by adding all elements of G, and for each $g \in G$ and every vertex v that represents an orbit of λ_s for $s \in S \setminus \{1\}$ the element g belongs to, we add the edge joining g and v.

2 The correspondence theorem

We recall that if X is a subset of G closed under taking inverses and not containing 1, the *Cayley graph* $C(G, X)$ has vertex set G and vertices $g, h \in G$ are adjacent if $gh^{-1} \in X$, which is equivalent to $hg^{-1} \in X$. We do not require X to be a generating set for G and therefore we allow also disconnected Cayley graphs. If $gh^{-1} = x \in X$, we consider the set $\{x, x^{-1}\}$ to be the *colour* of the edge joining g and h.

We aim at showing that every *simple* orbit graph $O(G, S)$ with $1 \notin S$ can be derived from a Cayley graph on the group G. To see this, we introduce the set $S^* = (\cup_{s\in S}\langle s\rangle) \setminus \{1\}$ and consider the Cayley graph $C(G, S^*)$. Since vertices of $C(G, S^*)$ are the elements of G, for any given $s \in S$ the set of all orbits of translation λ_s forms a partition of the vertex set of $C(G, S^*)$. By the definition of adjacency relation in a Cayley graph, any two distinct vertices appearing in the same orbit of λ_s are joined by an edge coloured $\{x, x^{-1}\}$ for some $x \in \langle s\rangle \setminus \{1\}$. Conversely, the pair of vertices incident with any edge coloured $\{x, x^{-1}\}$ where $x \in \langle s\rangle \setminus \{1\}$ belongs to the same orbit of λ_s. We conclude that the set of all orbits of translation λ_s induces a spanning subgraph F_s of the Cayley graph $C(G, S^*)$ that consists of vertex disjoint complete subgraphs and the edge set of F_s coincides with the set of all edges coloured $\{x, x^{-1}\}$, $x \in \langle s\rangle \setminus \{1\}$.

Now we define the *colour-clique graph* $KC(G, S^*)$ of the Cayley graph $C(G, S^*)$ as follows. The vertex set of $KC(G, S^*)$ is the set of all complete graphs in F_s for all $s \in S$. Two distinct vertices u, v in this set are adjacent if the complete graphs they represent share a vertex.

We shall show now that any simple orbit graph $O(G, S)$ with the property $1 \notin S$ is isomorphic to the colour-clique graph $KC(G, S^*)$. Given $s \in S$, a vertex $u \in F_s$ represents a complete graph induced by all the vertices that belong to an orbit u' of λ_s. Thus for each $s \in S$ there is a 1-1 correspondence between the vertex class F_s and the set of all orbits of λ_s given by $u \mapsto u'$ for all $u \in F_s$. Further, the assumption $\langle s\rangle \cap \langle t\rangle = \{1\}$ for any two distinct s, t in S guarantees that any pair of vertices $u \in F_s$ and $v \in F_t$, $s \neq t$, share at most one vertex, i.e. the vertex class F_s is disjoint from F_t. Therefore the natural extension of the mapping we have defined above is a bijection. Now, two vertices $u, v \in KC(G, S^*)$ are adjacent if and only if the complete graphs they represent have a (unique) vertex $g \in G$ in common, that is, if and only if g appears in both orbits u' and v', which is equivalent to the existence of an edge coloured g that joins u' and v'.

Thus we have proved the following result:

Theorem 1. *Every simple orbit graph $O(G, S)$ such that $1 \notin S$ is isomorphic to the colour-clique graph $KC(G, S^*)$.*

3 Application

In this section we present examples of application of Theorem 1. First, let us examine connectivity of simple orbit graphs. It was observed in [1] that an orbit graph $O(G, S)$ is connected if and only if S is a generating set for G. In what follows we show that every connected, *regular*, simple orbit graph $O(G, S)$ is 2-connected unless it is one-vertex graph.

Observe first that by our regularity assumption, the order $o(s)$ of every $s \in S$ is constant, say, $o(s) = m \geq 2$. Clearly, if $O(G, S)$ has at least two vertices then $|S| \geq 2$; also, since $O(G, S)$ is a simple graph, it cannot be complete. Therefore to prove 2-connectedness it is sufficient to show that $O(G, S)$ has no cut-vertex.

If $|S| = 2$, then $O(G, S)$ is a connected, m regular bipartite graph and the assertion follows directly from Hall's Theorem on 1-factorability of regular bipartite graphs.

In the case when $|S| \geq 3$ we will make use of the following two results.

Theorem A. (Mader, [9]) *Let Γ be a connected vertex-transitive graph of valency k, then Γ is $\lfloor \frac{2}{3}(k+1) \rfloor$ connected and k edge-connected.*

Theorem B. (Chartrand and Stewart, [8]) *Let Γ be a k edge-connected graph, then the line graph $L(\Gamma)$ is k connected and $2k-2$ edge-connected.*

Consider the colour-clique graph $KC(G, S^*)$. Since S generates G, S^* is a generating set for G as well, and therefore the Cayley graph $C(G, S^*)$ is connected. If S consists of involutions (that is, if $m = 2$) then $S = S^*$ and thus for each $s \in S$ the set of all cliques in F_s constitutes a perfect matching of $C(G, S^*)$. It follows that $KC(G, S^*)$ is the line graph of $C(G, S^*)$. Combining this fact together with the above theorems we obtain that $O(G, S)$ is, in fact, $|S|$-connected.

It remains to analyze the case when $m \geq 3$. First we show that if $KC(G, S^*)$ has a cut-vertex w, then the graph $C(G, S^*) \setminus w$ obtained from $C(G, S^*)$ by deleting all the vertices of a complete graph that corresponds to w is disconnected. Suppose the contrary and let u and v be vertices that belong to different components of $KC(G, S^*) \setminus \{w\}$ and let $C(G, S^*) \setminus w$ be connected. Then the complete graphs represented by u and v must be vertex disjoint and each of them can share at most one vertex with w. Consequently, there are two distinct vertices $u', v' \in C(G, S^*) \setminus w$ such that $u' \in u$ and $v' \in v$. Let W be an (oriented) walk from u' to v' in $C(G, S^*) \setminus w$. Recall that every arc on W is coloured x for some $x \in S^*$ and therefore it belongs to a (unique) complete graph in F_s for some $s \in S$. Now, any two consecutive arcs on W either belong to the same clique or they belong to different cliques (which, clearly, share a vertex). Since F_s is disjoint from F_t for all $s \neq t$, any complete graph an arc on W belongs to is vertex disjoint from the clique represented by w. We conclude that W induces a walk from u to v in $KC(G, S^*) \setminus \{w\}$, which contradicts our assumption.

Finally, assume that $KC(G, S^*)$ is not 2-connected and let w be a cut-vertex of

$KC(G,S^*)$. It follows that $C(G,S^*)\setminus w$ is disconnected. But $m < \lfloor \frac{2}{3}(k+1) \rfloor$ where $k = |S|(m-1)$ is the valency of $C(G,S^*)$, contradicting Theorem A. We summarize these facts in

Proposition 1. *Every connected, regular simple orbit graph on at least two vertices is* 2*-connected.*

Next, we derive a condition for the automorphism group $Aut(G)$ of a group G that ensures a simple orbit graph $O(G,S)$ will be vertex-transitive. Generalizing an earlier observation of [4, 6], it was noted in [3] that the group G acts (as a subgroup of automorphisms) transitively on each vertex class of the vertex set of $O(G,S)$ by right multiplication. Let $Aut(G,S)$ denote the subgroup of $Aut(G)$ consisting of all automorphisms that fix S setwise. It was shown in [1] that every automorphism in $Aut(G,S)$ induces an automorphism of $O(G,S)$. From this the following result follows immediately.

Proposition 2. *Let $O(G,S)$ be a simple orbit graph. If $Aut(G,S)$ has a subgroup acting transitively on S, then $O(G,S)$ is vertex-transitive.*

For simple orbit graphs with the property $1 \notin S$ this can be alternatively proved as follows. Let us consider the Cayley graph $C(G,S^*)$.

• The group G acts regularly (as a subgroup of automorphisms) on the vertex set of any Cayley graph on the group G by right multiplication. It is easily seen that for any given $h \in G$ and each $s \in S$ this action induces an action of G on the set of all orbits of λ_s given by $\{g, sg, \ldots, s^{o(s)-1}g\} \mapsto \{gh, sgh, \ldots, s^{o(s)-1}gh\}$ for $g \in G$, which is transitive but not regular any longer in general. It follows that the right translation of G by h induces a permutation on the vertex set of $KC(G,S^*)$ that fixes each vertex class F_s setwise and preserves adjacencies.

• Given a Cayley graph $C(G,X)$, then each $\pi \in Aut(G,X)$ induces an automorphism of $C(G,X)$. Applied to $C(G,S^*)$ one sees that each $\pi \in Aut(G,S) \subseteq Aut(G,S^*)$ induces the action on the set of all orbits of λ_s for $s \in S$, defined by $\{g, sg, \ldots, s^{o(s)-1}g\} \mapsto \{g\pi, (sg)\pi, \ldots, (s^{o(s)-1}g)\pi\}$ for $g \in G$. Now, two vertices $u, v \in KC(G,S^*)$ are adjacent if and only if the complete graphs they represent share a vertex $h \in G$ or equivalently, if and only if h appears in both orbits on which the complete graphs are defined. It follows that each $\pi \in Aut(G,S)$ induces an automorphism of $KC(G,S^*)$. This completes the alternative proof of Proposition 2.

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