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On Böttcher's mysterious identity

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Abstract

In his work on Wiener-Hopf determinants, A. Böttcher came across what he termed a "mysterious" identity that evaluates a certain sum of a rational function of a primitive root of unity in terms of the Barnes G-function, which was later generalized by Basor and Forrester. We give a direct proof of Böttcher's identity and its many generalizations by using elements of the theory of symmetric functions.

1 Introduction

On his paper on the Wiener-Hopf determinants [2], Böttcher came across what he termed a "mysterious" sum that has the following evaluation:

$$\sum_{M} \frac{\left(\sum_{j \in \overline{M}} w^j \right)^{pq}}{\prod_{i \in M, j \in \overline{M}} (w^j - w^i)} = \frac{(pq)! G(p+1) G(q+1)}{G(p+q+1)} . \tag{1}$$

Here p and q are nonnegative integers, the summation is over all subsets $M \subseteq \{1, 2, \ldots, p+q\}$ with $|M| = p$, $\overline{M}$ is the complement of M, w is a primitive $(p+q)$-th root of unity, and G is the Barnes G-function. The Barnes G-function has values $G(0) = G(1) = G(2) = 1$ and

$$G(n) = 1!2! \cdots (n-2)!,$$

for $n > 2$.

The proof of (1) that appears in Böttcher is a consequence of the equality of the asymptotic evaluations of the Toeplitz determinant generated by the Laurent series expansion of a rational function in two essentially different ways: one computation with Day's formula [4], and the other as an application of a general asymptotic result of Böttcher and Silbermann [3]. The identity drops out once the two evaluations are equated to one another.

As an example, for $p = q = 2$, $w = i$ is the imaginary unit, and the left hand side of (1) evaluates to

$$\frac{4!\, G(3)\, G(3)}{G(5)} = 2.$$

Our starting point is an interesting property of this evaluation, that it is independent of w. For an indeterminate w, the left hand side of (1) can be expanded as

$$\begin{aligned}
&\frac{(w+w^2)^4}{(w-w^3)(w^2-w^3)(w-w^4)(w^2-w^4)} + \frac{(w+w^3)^4}{(w-w^2)(w^3-w^2)(w-w^4)(w^3-w^4)}\\
&+\frac{(w^2+w^3)^4}{(w^2-w)(w^3-w)(w^2-w^4)(w^3-w^4)} + \frac{(w+w^4)^4}{(w-w^2)(w-w^3)(w^4-w^2)(w^4-w^3)}\\
&+\frac{(w^2+w^4)^4}{(w^2-w)(w^2-w^3)(w^4-w)(w^4-w^3)} + \frac{(w^3+w^4)^4}{(w^3-w)(w^3-w^2)(w^4-w)(w^4-w^2)}
\end{aligned}$$

in which everything cancels and we are left with 2.

Even more appears to be true: in place of the monomials w^i in (1), consider the indeterminates $x_1, x_2, \ldots, x_{p+q}$ and replace w^i by x_i for all i. For $p = q = 2$, this sum now takes the form

$$\begin{aligned}
&\frac{(x_1+x_2)^4}{(x_1-x_3)(x_2-x_3)(x_1-x_4)(x_2-x_4)} + \frac{(x_1+x_3)^4}{(x_1-x_2)(x_3-x_2)(x_1-x_4)(x_3-x_4)}\\
&+\frac{(x_2+x_3)^4}{(x_2-x_1)(x_3-x_1)(x_2-x_4)(x_3-x_4)} + \frac{(x_1+x_4)^4}{(x_1-x_2)(x_1-x_3)(x_4-x_2)(x_4-x_3)}\\
&+\frac{(x_2+x_4)^4}{(x_2-x_1)(x_2-x_3)(x_4-x_1)(x_4-x_3)} + \frac{(x_3+x_4)^4}{(x_3-x_1)(x_3-x_2)(x_4-x_1)(x_4-x_2)}
\end{aligned}$$

which again simplifies to 2; an identity even more intriguing than Böttcher's original sum.

The purpose of this note is to give a direct proof of the following generalization of Böttcher's identity:

$$\sum_{M} \frac{\left(\sum_{j\in\overline{M}} x_j\right)^{pq}}{\prod_{i\in M, j\in\overline{M}} (x_j - x_i)} = \frac{(pq)!G(p+1)G(q+1)}{G(p+q+1)} \tag{2}$$

where as in (1), the summation is over all subsets $M \subseteq \{1, 2, \ldots, p+q\}$ with $|M| = p$.

This and other identities of similar flavor can be obtained by using elementary results from the theory of symmetric functions. We present a few of these in Section 4. Preliminaries are given in Section 2 and the proof of the main result in Section 3.

Most of what we recall about symmetric functions including tableaux, diagrams and dimension calculations are aimed at readers not familiar with this theory and can be skipped.

It should be mentioned that Basor and Forrester [1] also established general identities that arise from evaluations of Toeplitz determinants in two different ways, making use of Day's formula for one of the evaluations. The resulting identities of Basor and Forrester can be written in the form

$$\lim_{\epsilon\to 0}\sum_{M}\frac{1}{\epsilon^{pq}}\frac{\prod_{j\in\overline{M}}(1+\epsilon x_j)^{p+n}}{\prod_{i\in M, j\in\overline{M}}(x_j-x_i)} = \frac{G(n+2)G(p+q+n+1)G(q+1)G(p+1)}{n!G(p+n+1)G(q+n+1)G(p+q+1)} \qquad (3)$$

using the notation of this note.

2 Preliminaries

Let $[n] = \{1,2,\ldots,n\}$. For any $M = \{i_1 < i_2 < \cdots < i_p\} \subseteq [n]$ and a function $g(x_1,x_2,\ldots,x_p)$, define $g(M) = g(x_{i_1},x_{i_2},\ldots,x_{i_p})$. Let Λ^k denote the ring of symmetric functions in the x_i, homogeneous of degree k. Bases for Λ^k are indexed by partitions $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and $|\lambda| = \lambda_1+\lambda_2+\cdots+\lambda_n = k$. We denote this by $\lambda \vdash k$. The nonzero λ_i are the *parts* of λ. An alternate notation for λ is

$$\lambda = 1^{m_1}2^{m_2}\cdots n^{m_n}$$

indicating that λ has m_i parts equal to i. The *diagram* of λ is n left-justified rows of squares where the i-the row from the top consists of λ_i cells. The *conjugate* of λ is the partition λ' obtained by reflecting the diagram of λ across the main diagonal. $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ is the *hook-length* of the cell $(i,j) \in \lambda$. A *column-strict tableau* T of shape λ is obtained by placing a positive integer in each cell of λ so that the entries are weakly increasing along the rows and strictly increasing down the columns. The frequencies of the entries in T defines a partition μ of $|\lambda|$ called the *weight* or *content* of T. A *standard Young tableau* of shape λ is obtained by placing $1,2,\ldots,|\lambda|$ in λ such that the entries are increasing along the rows and down the columns. By the celebrated hook-length formula of Frame, Robinson, and Thrall [5], the number of standard Young tableaux of shape λ is given by

$$f_\lambda = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} h(i,j)} .$$

Given a partition λ with n parts, define the $n\times n$ alternant

$$a_\lambda = \det[x_i^{\lambda_j}] .$$

For the special selection $\delta = (n-1, n-2, \ldots, 0)$,

$$a_\delta = \det[x_i^{n-j}] = \prod_{i<j}(x_i - x_j) \tag{4}$$

is the Vandermonde determinant. We also use the notation $V([n])$ for the $n \times n$ Vandermonde determinant in (4). The *Schur* basis for Λ^k can be defined by the Cauchy's bi-alternants as

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta} \tag{5}$$

as λ runs through partitions of k into at most n parts, and addition of partitions is componentwise. The *elementary, homogeneous*, and *power* bases for Λ^k are

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}\ , \quad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}\ , \quad \psi_\lambda = \psi_{\lambda_1} \psi_{\lambda_2} \cdots \psi_{\lambda_n} \tag{6}$$

where e_r, h_r are the r-th elementary and the r-th homogeneous symmetric function of their arguments, and ψ_r is the r-th power sum. In particular

$$e_r = s_{1^r}\ , \quad h_r = s_r$$

and $e_1 = h_1 = \psi_1 = s_1$. By Pieri's rule, we have the expansion

$$s_1^k = \sum_{\lambda \vdash k} f_\lambda s_\lambda\ .$$

These and further properties of symmetric functions can be found in Macdonald [7].

Let now $n = p+q$ and assume $M \subseteq [p+q]$ with $|M| = p$. Define

$$i(\overline{M}, M) = |\{(j,i) \mid j \in \overline{M}, i \in M \text{ and } j > i\}| \tag{7}$$

This is the number of *inverted* pairs in $\overline{M} \times M$. We record the following properties of $i(\overline{M}, M)$.

Lemma 1 *Suppose $M \subseteq [p+q]$ with $|M| = p$ and $i(\overline{M}, M)$ is as defined in (7). Then*

(a) $i(\overline{M}, M) + i(M, \overline{M}) = pq$,

(b) If $(j,i) \in \overline{M} \times M$ with $j > i$ and $N = M \setminus \{i\} \cup \{j\}$, then $i(\overline{N}, N) = i(\overline{M}, M) + i - j$,

(c) $i(\overline{M}, M) = pq + \dfrac{p(p+1)}{2} - \sum_{i \in M} i$.

Proof The proof of (a) is immediate since there are a total of pq pairs and the two sets are disjoint. For the proof of (b), let m_1, m_2, m_3 (resp. $\overline{m}_1, \overline{m}_2, \overline{m}_3$) denote the number of elements of M (resp. $\overline{M}$) in the intervals $[1,i)$, (i,j), $(j, p+q]$, respectively. In exchanging i and j, we are replacing the pairs

$$\overline{M} \times \{i\} \ \cup \{j\} \times M$$

which contribute $\overline{m}_2 + \overline{m}_3 + m_1 + m_2 + 1$ inversions to $i(\overline{M}, M)$ by the pairs

$$\overline{M} \times \{j\} \ \cup \{i\} \times M$$

which contribute $\overline{m}_3 + m_1$ inversions. All other pairs have the same contribution to both $i(\overline{M}, M)$ and $i(\overline{N}, N)$. Therefore $i(\overline{N}, N)$ has $m_2 + \overline{m}_2 + 1 = (j-i-1)+1 = j-i$ fewer inversions.

To prove (c), suppose $M = \{i_1 < i_2 < \cdots < i_p\}$. Then $p+q-i_p$ elements of $\overline{M}$ larger than i_p contribute $p(p+q-i_p)$ inversions, the $i_p - i_{p-1} - 1$ elements of $\overline{M}$ between i_{p-1} and i_p contribute $(p-1)(i_p - i_{p-1} - 1)$ inversions, etc. Therefore

$$\begin{aligned} i(\overline{M}, M) &= p(p+q-i_p) + (p-1)(i_p - i_{p-1} - 1) + \cdots + (i_2 - i_1 - 1) \\ &= p(p+q) - pi_p - (p-1)(i_p - i_{p-1}) - \cdots - (i_2 - i_1) - \frac{p(p-1)}{2} \\ &= pq + \frac{p(p+1)}{2} - \sum_{i \in M} i \ . \end{aligned}$$

□

Recall that if $M = \{i_1 < i_2 < \cdots < i_p\}$ then

$$V(M) = \det \begin{bmatrix} x_{i_1}^{p-1} & \ldots & x_{i_1} & 1 \\ x_{i_2}^{p-1} & \ldots & x_{i_2} & 1 \\ \vdots & \ldots & \vdots & \vdots \\ x_{i_p}^{p-1} & \ldots & x_{i_p} & 1 \end{bmatrix} = \prod_{r<s} (x_{i_r} - x_{i_s})$$

The following identity expresses the difference-product in the denominator of (2) as a ratio of Vandermonde determinants:

Lemma 2 *Suppose $M \subseteq [p+q]$ with $|M| = p$. Then*

$$V([p+q]) = (-1)^{i(\overline{M}, M)} V(M) V(\overline{M}) \prod_{i \in M, j \in \overline{M}} (x_j - x_i) \ . \tag{8}$$

Proof Both sides of (8) have the same factors $x_i - x_j$ up to sign. Each pair $j > i$ in the product

$$\prod_{i \in M, j \in \overline{M}} (x_j - x_i)$$

contributes a -1 when we flip it around, so that

$$\prod_{i \in M, j \in \overline{M}} (x_j - x_i) = (-1)^{i(\overline{M}, M)} \prod_{\substack{i \in M, j \in \overline{M} \\ i < j}} (x_i - x_j) \ .$$

□

In the algebra of symmetric functions in $x_1, x_2, \ldots, x_q$, define the operator $c_{p,q}$ on Λ^{pq} by setting for $f \in \Lambda^{pq}$

$$c_{p,q}(f) = \sum_{\substack{M \subseteq [p+q] \\ |M| = p}} (-1)^{i(\overline{M},M)} \frac{V(M)V(\overline{M})}{V([p+q])} f(\overline{M}) \ . \tag{9}$$

3 The main result

Rewriting the denominator in (2) in terms of Vandermonde determinants by using (8) and with the notation of (9), identity (2) can be restated as

Theorem 1

$$c_{p,q}(s_1^{pq}) \ = \ \frac{(pq)!G(p+1)G(q+1)}{G(p+q+1)} \ . \tag{10}$$

Proof Writing (10) in the form

$$\sum_M (-1)^{i(\overline{M},M)} V(M)V(\overline{M}) s_1(\overline{M})^{pq} \ = \ \frac{(pq)!G(p+1)G(q+1)}{G(p+q+1)} V([p+q]), \tag{11}$$

it is clear that both sides are homogeneous polynomials of total degree $\binom{p+q}{2}$.

Consider a difference $x_i - x_j$ with $i < j$ that appears as a factor in $V([p+q])$. $x_i - x_j$ divides all $\binom{p+q-2}{p-2}$ terms in the sum on the left for which $i, j \in M$, and the $\binom{p+q-2}{p}$ terms for which $i, j \in \overline{M}$, since in these cases $V(M)$ and $V(\overline{M})$ are divisible by $x_i - x_j$, respectively. This leaves $2\binom{p+q-2}{p-1}$ terms on the left to account for. We will pair these up. We can assume that $i \in M$ and $j \in \overline{M}$ and show that the sum of the two terms corresponding to the pair M and $N = M \setminus \{i\} \cup \{j\}$ in (11) is divisible by $x_i - x_j$.

This sum is of the following form:

$$(-1)^{i(\overline{M},M)} V(M)V(\overline{M})(S + x_j)^{pq} + (-1)^{i(\overline{N},N)} V(N)V(\overline{N})(S + x_i)^{pq} \ . \tag{12}$$

In the Vandermonde determinants above, only the terms that involve i and j are relevant. These terms can be written as

$$\pm \prod_{r \in M, r \neq i} (x_i - x_r) \prod_{s \in \overline{M}, s \neq j} (x_j - x_s) \tag{13}$$

for $V(M)V(\overline{M})$ and

$$\pm \prod_{r \in N, r \neq j} (x_j - x_r) \prod_{s \in \overline{N}, s \neq i} (x_i - x_s) \tag{14}$$

for $V(N)V(\overline{N})$. As in the proof of Lemma 1 (b), the total number of switches needed to write x_i and x_j as the first element in each factor in (13) is $m_1 + \overline{m}_1 + \overline{m}_2$. This number is $m_1 + m_2 + \overline{m}_1$ for (14). The difference $m_2 - \overline{m}_2$ has the same parity as

$j-i-1$ since $m_2+\overline{m}_2 = j-i-1$. Now multiply through by ± 1 as necessary to make the sign of (14) equal to 1, then the sign of (13) becomes $(-1)^{j-i-1}$. Expanding the powers in (12) by the binomial theorem, and using Lemma 1 (b), it suffices to show that each of the expressions

$$x_j^k \prod_{r\in M, r\neq i} (x_i - x_r) \prod_{s\in \overline{M}, s\neq j} (x_j - x_s) \;-\; x_i^k \prod_{r\in N, r\neq j} (x_j - x_r) \prod_{s\in \overline{N}, s\neq i} (x_i - x_s)$$

is divisible by $x_i - x_j$. For any $t \notin \{i,j\}$, exactly one of $x_i - x_t$ or $x_j - x_t$ is a factor of the first two products above, and and the other a factor of last two products. Therefore this expression vanishes for $x_j = x_i$ and the left hand side is divisible by $x_i - x_j$.

The same argument above actually shows that for any $g(x_1, x_2, \ldots, x_q) \in \Lambda^{pq}$, $c_{p,q}(g)$ is a scalar.

Let $a_{(\delta,\lambda)}$ denote the determinant of the $(p+q)\times(p+q)$ matrix

$$\left[\begin{cases} x_i^{p-j} & \text{if } 1 \le j \le p \\ x_i^{p+q-j+\lambda_{q+p-j+1}} & \text{if } p < j \le p+q \end{cases}\right]_{1\le i,j\le p+q} .$$

Thus

$$a_{(\delta,\lambda)} = \det \begin{bmatrix} x_1^{p-1} & \cdots & x_1 & 1 & x_1^{q-1+\lambda_1} & x_1^{q-2+\lambda_2} & \cdots & x_1^{\lambda_q} \\ x_2^{p-1} & \cdots & x_2 & 1 & x_2^{q-1+\lambda_1} & x_2^{q-2+\lambda_2} & \cdots & x_2^{\lambda_q} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ x_{p+q}^{p-1} & \cdots & x_{p+q} & 1 & x_{p+q}^{q-1+\lambda_1} & x_{p+q}^{q-2+\lambda_2} & \cdots & x_{p+q}^{\lambda_q} \end{bmatrix} . \tag{15}$$

Expand $a_{(\delta,\lambda)}$ by the first p columns of (15) by Laplace's rule. In this expansion we form the minor corresponding to a selection M of p row indices and multiply this by the complementary minor. The sign of the pair is given by the parity of the sum of the row and column indices picked, in other words the sign is

$$(-1)^{\sum_{r=1}^{p} r + \sum_{i\in M} i} .$$

The minor corresponding to M is $V(M)$, and the complementary minor is $s_\lambda(\overline{M})V([\overline{M}])$ by (5). By Lemma 1 (c), we can rewrite the sign as

$$(-1)^{\frac{p(p+1)}{2} + \sum_{i\in M} i} = (-1)^{i(\overline{M},M)+pq}$$

and so

$$a_{(\delta,\lambda)} = (-1)^{pq} \sum_{\substack{M\subseteq [p+q] \\ |M|=p}} (-1)^{i(\overline{M},M)} V(M) V(\overline{M}) s_\lambda(\overline{M}) .$$

Dividing through by $V([p+q])$,

$$\frac{a_{(\delta,\lambda)}}{V([p+q])} = (-1)^{pq} c_{p,q}(s_\lambda) \ .$$

But unless $\lambda_q \geq p$, $a_{(\delta,\lambda)}$ has two identical columns and therefore vanishes. Since $\lambda \vdash pq$, when $\lambda_q \geq p$, we are forced to have $\lambda_1 = \lambda_2 = \cdots = \lambda_q = p$, and $\lambda = p^q$ is a rectangle. Reordering the columns in (15) with pq column switches,

$$a_{(\delta,p^q)} = (-1)^{pq} V([p+q])$$

so that

$$c_{p,q}(s_\lambda) = \begin{cases} 1 & \text{if } \lambda = p^q \\ 0 & \text{otherwise} \end{cases} \ .$$

Going back to the expansion of s_1^{pq} in terms of Schur functions

$$s_1^{pq} = \sum_{\lambda \vdash pq} f_\lambda s_\lambda \tag{16}$$

and applying $c_{p,q}$ to (16), all the terms on the right vanish except for $\lambda = p^q$. Therefore

$$c_{p,q}(s_1^{pq}) = f_{p^q}$$

But the hook-lengths for $\lambda = p^q$ are

$$h(i,j) = p - j + q - i + 1$$

for $1 \leq i \leq q$ and $1 \leq j \leq p$. Therefore

$$\begin{aligned} \prod_{(i,j)\in p^q} h(i,j) &= \prod_{i=1}^{p}\prod_{j=1}^{q}(p-j+q-i+1) \\ &= \prod_{i=1}^{p} \frac{(p+q-i)!}{(p-i)!} = \frac{G(p+q+1)}{G(p+1)G(q+1)} \end{aligned}$$

and consequently by the hook-length formula

$$f_{p^q} = \frac{(pq)!G(p+1)G(q+1)}{G(p+q+1)} \ .$$

□

What we have actually proved can be summarized as the following:

Theorem 2 *Suppose $g(x_1, x_2, \ldots, x_q)$ is a symmetric function, homogeneous of degree pq with the expansion*

$$g = \sum_{\lambda \vdash pq} c_\lambda s_\lambda$$

in the Schur basis. Then

$$\sum_{\substack{M \subseteq [p+q] \\ |M| = p}} (-1)^{i(\overline{M},M)} \frac{V(M)V(\overline{M})}{V([p+q])} g(\overline{M}) = c_{p^q} \ .$$

4 Further identities

We give a few easy specializations with the the corresponding evaluations in the form of the original generalization of Böttcher's identity in (2). In (17)-(20) below, the summations are over all subsets $M \subseteq \{1, 2, \ldots, p+q\}$ with $|M| = p$.

1. When $g = e_q^p$, $c_{p^q} = 1$. Therefore

$$\sum_M \frac{\prod_{j \in \overline{M}} x_j^p}{\prod_{i \in M, j \in \overline{M}} (x_j - x_i)} = 1 \ . \tag{17}$$

2. When $g = \psi_{pq}$, $c_{p^q} = 0$ by the Frobenius formula for the characters of the symmetric group and the Murnaghan-Nakayama rule [6]. Therefore

$$\sum_M \frac{\sum_{j \in \overline{M}} x_j^{pq}}{\prod_{i \in M, j \in \overline{M}} (x_j - x_i)} = 0 \ . \tag{18}$$

3. Using Pieri's rule for the expansion of the product of h_r and s_λ, we obtain

$$\sum_M \frac{h_r(\overline{M}) \left(\sum_{j \in \overline{M}} x_j \right)^{pq-r}}{\prod_{i \in M, j \in \overline{M}} (x_j - x_i)} = \frac{\binom{p}{r}\binom{q+r-1}{r}}{\binom{pq}{r}} \frac{(pq)! G(p+1) G(q+1)}{G(p+q+1)} \ . \tag{19}$$

4. Dually,

$$\sum_M \frac{e_r(\overline{M}) \left(\sum_{j \in \overline{M}} x_j \right)^{pq-r}}{\prod_{i \in M, j \in \overline{M}} (x_j - x_i)} = \frac{\binom{q}{r}\binom{p+r-1}{r}}{\binom{pq}{r}} \frac{(pq)! G(p+1) G(q+1)}{G(p+q+1)} \ . \tag{20}$$

The transition matrices that express e_μ and h_μ in (6) in terms of Schur basis can be all written in terms of the Kostka matrix K where $K_{\lambda,\mu}$ is the number of column-strict tableaux of shape λ and weight μ [7]. Therefore $c_{p,q}(h_\mu)$ is the number of column-strict tableaux of shape p^q and weight μ, and dually $c_{p,q}(e_\mu)$ is the number of column-strict tableaux of shape q^p and weight μ. In the case of ψ_μ, $c_{p,q}(\psi_\mu)$ is the value of the irreducible character of the symmetric group corresponding to $\lambda = p^q$ at the conjugacy class μ.

It is clear that other specializations similar to Böttcher's identity (2) and (17)-(20) can be constructed at will by picking particular partitions μ for which these values have nice closed form expressions.

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