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A note on the path cover number of regular graphs

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Abstract

Let G be a simple graph of order n. The path cover number $\mu(G)$ is defined to be the minimum number of vertex disjoint paths required to cover the vertices of G. Ore proved that in general $\mu(G) \leq \max\{1, n - \sigma_2(G)\}$. We conjecture that if G is k-regular, then $\mu(G) \leq \frac{n}{k+1}$ and we prove this for $k \leq 5$.

1 Introduction

All standard notation used in this note may be found in [3]. The *path cover number* $\mu(G)$ is defined to be the minimum number of vertex disjoint paths required to cover the vertices of G. The path cover number was first considered by Ore in [11] where he proved the following sharp bound on $\mu(G)$.

Theorem 1 *Given a graph G of order n, the path cover number satisfies*

$$\mu(G) \leq n - \sigma_2(G)$$

where $\sigma_2(G)$ is the minimum sum of degrees of two nonadjacent vertices.

Since then, many people (e.g. [1, 2, 4, 6, 8, 9]) have considered the problem of partitioning the vertices of a graph into paths. Others have generalized Ore's result but bounds on $\mu(G)$ have proven difficult to find. One such generalization is the following. Let $g(n, k)$ be the minimum integer such that every graph G on n vertices with at least $g(n, k)$ edges satisfies $\mu(G) \leq k$. Noorvash [10] proved the following generalization.

Theorem 2 *The function $g(n, k)$ satisfies*

$$\frac{1}{2}(n-k)(n-k-1) + 1 \leq g(n,k) \leq \frac{1}{2}(n-1)(n-k-1) + 1.$$

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Theorem 1 is sharp because of the following example. Consider the complete bipartite graph $H = K_{\delta,n-\delta}$. Notice $\delta(H) = \delta$ and $\mu(H) = n - 2\delta$.

Another strong bound on the path cover number is the following by I.B. Hartman [7].

Theorem 3 *Let G be a graph with connectivity k and $\alpha(G) = \alpha$. If $\alpha > k$, then $\mu(G) \le \alpha(G) - k$. Otherwise, we always have $\mu(G) \le \alpha$.*

The second part of this theorem actually follows from the trivial observation that for any minimum collection of paths, the end-vertices of these paths form an independent set. In spite of the above examples, we conjecture the following.

Conjecture 1 *Given a k-regular graph G of order n, the path cover number satisfies*

$$\mu(G) \le \frac{n}{k+1}.$$

First note this conjecture is clearly sharp because a collection of $\frac{n}{k+1}$ independent copies of K_{k+1} is k-regular and requires the use of $\frac{n}{k+1}$ paths in a cover. Also, since an $\lceil \frac{n}{2} \rceil$-regular graph is hamiltonian by Dirac's theorem [5], the conjecture is trivially true for $k \ge \frac{n}{2}$.

Another indication that the conjecture should be true is that almost all k-regular graphs are hamiltonian for $k \ge 3$ [13]. In a sense, this result leads to the following possible (although probably difficult) approach to proving our conjecture. It is known that, by a sequence of edge swaps (i.e. swapping two edges uv and wx for two other edges uw and vx), it is possible to go from a graph with degree sequence D to any other graph with the same degree sequence. If we define a measure of distance between two graphs G and H with degree sequence D to be the minimum number of edge swaps necessary to get from G to H, then our conjecture is similar to finding the distance between any k-regular graph and a hamiltonian graph.

If one was to restrict attention to connected graphs, the problem seems to become much more difficult. However, the following result of Reed [12] indicates that, in fact, a much stronger conjecture should hold in this case.

Theorem 4 *Any connected cubic graph of order n can be partitioned into $\lceil n/9 \rceil$ vertex disjoint paths.*

Furthermore, Reed conjectures in the same paper that if G is 2-connected, the above bound can be reduced to $\lceil n/10 \rceil$.

2 Results

In support of our conjecture, we prove the following result.

Theorem 5 *Given a k-regular graph G of order n with $0 \le k \le 5$, the path cover number satisfies*

$$\mu(G) \le \frac{n}{k+1}.$$

Proof: The proof is trivial for $k = 0, 1$ and 2. We start with a general claim about path covers of k-regular graphs for any integer $k > 0$.

Claim 1 *For $k > 0$, any k-regular graph G has a path cover with $\mu(G)$ paths such that no path is of order* 1.

Proof of Claim 1: Consider a minimum path cover with the smallest number of paths of order 1. Suppose there exists a path P_1 of order 1 and let $v \in P_1$. Certainly v is not adjacent to an endpoint of another path. First suppose v is adjacent to an internal vertex w of a path P of order at least 4. One may easily create two new paths from P_1 and P, each of order at least 2. Therefore v may be adjacent only to midpoints of paths of order 3.

So, suppose v is adjacent to a vertex w which is an internal vertex of a path uwx of order 3. As above, we may rearrange paths to form the paths xwv and u or uwv and x. This implies u and x must also be adjacent only to midpoints of paths of order 3.

Let T be the set of all vertices which can be made into paths of order 1 by repeated application of the above rearrangement process and recall that we are assuming $|T| \geq 1$. Notice first that T is an independent set. Let M be the neighborhood of T. Every vertex of M must be a midpoint of a path of order 3, each endpoint of which is in T. This implies $|M| \leq \frac{|T|}{2}$. Conversely, each vertex of T has k adjacencies and all such adjacencies must be in M. Each vertex in M also has only k adjacencies so, since $k > 0$, we get $|M| \geq |T|$, which is a contradiction. Therefore we may assume there are no paths of order 1 in a minimum path cover. $\square_{Claim\ 1}$

Although the proofs are similar in essence, we use the following cases to demonstrate the increasing complexity of the problem as k increases.

Case 1 *Suppose $k = 3$.*

Choose a minimum path cover of G. Let B be the set of end-vertices of these paths and let C be the set of internal vertices. Every vertex of B must have at least one non-path edge to a vertex of C. Since every vertex of C has exactly one non-path edge, $|B| \leq |C|$. The sets B and C partition the graph so $|B| + |C| = n$ which means the number of paths is equal to $\frac{|B|}{2} \leq \frac{n}{4}$ proving the desired result for $k = 3$. This case also follows easily from Theorem 4 of Reed [12].

Case 2 *Suppose $k = 4$.*

Let $\mathscr{P} = \{P_i\}$ be a minimum path cover of G. Let A be the set of all vertices in paths of the form $(v_1, \ldots, v_k)$ with $k \geq 3$ and $v_1 v_k \in E(G)$, let B be the set of endpoints of paths which do not use vertices of A and let $C = G \setminus (A \cup B)$. Let A_3 be the set of vertices in A which are in paths of order 3, let A_4 be the set of vertices in A which are in paths of order 4 and let $A_{\geq 5} = A \setminus (A_3 \cup A_4)$.

For a given vertex v, let $d'(v)$ be the number of edges incident to v which are not in any path of $\mathscr{P}$. Suppose $v \in A$ and let P be the path containing v. Notice

v cannot be adjacent to any vertices in $B \cup A \setminus V(P)$. Therefore, for every vertex $v \in A_4$, $d'_C(v) \geq 1$ and for every vertex $u \in A_3$, $d'_C(u) = 2$. Also, since the vertices of B are endpoints of paths, for any vertex $v \in B$, $d'_{A\cup B}(v) = 0$ so $d'_C(v) = 3$. Finally, note that $d'(v) = 2$ for all $v \in C$. This means

$$\begin{aligned} 3|B| + 2|A_3| + |A_4| &\leq 2|C| \\ &= 2(n - |B| - |A_3| - |A_4| - |A_{\geq 5}|), \end{aligned}$$

which implies

$$\tfrac{|B|}{2} + \tfrac{2|A_3|}{5} + \tfrac{3|A_4|}{10} + \tfrac{|A_{\geq 5}|}{5} \leq \tfrac{n}{5}.$$

Therefore since

$$|\mathscr{P}| \leq \frac{|B|}{2} + \frac{|A_3|}{3} + \frac{|A_4|}{4} + \frac{|A_{\geq 5}|}{5} \leq \frac{n}{5}$$

we get the desired result for $k = 4$.

Case 3 *Suppose* $k = 5$.

Again consider a minimum path cover $\mathscr{P}$. Let c_i be the number of paths in $\mathscr{P}$ of the form $(v_1, v_2, \ldots, v_i)$ with the edge $v_1 v_i \in E(G)$ for $i \geq 3$ (we abuse notation by calling such paths *cycles*) and let t_i be the number of paths of order i not counted above. We would like to show that the average order of paths in $\mathscr{P}$ is at least 6. In order to show this, it suffices to prove

$$\frac{\sum i(t_i + c_i)}{\sum (t_i + c_i)} \geq 6. \tag{1}$$

If the average order is at least 6, we know $|\mathscr{P}| \leq \frac{n}{6}$.

In order to prove Inequality (1), we prove the Inequality (2) below. The left hand side of Inequality (2) can be thought of as the average order of paths and small cycles (of order at most 5). Certainly Inequality (2) implies Inequality (1) despite the fact that $\frac{\sum i(t_i+c_i)}{\sum(t_i+c_i)} \geq \frac{\sum it_i + \sum_{i\leq 5} ic_i}{\sum t_i + \sum_{i\leq 5} c_i}$ is not necessarily true.

$$\frac{\sum it_i + \sum\limits_{i\leq 5} ic_i}{\sum t_i + \sum\limits_{i\leq 5} c_i} \geq 6. \tag{2}$$

Let B be the collection of end-vertices of paths which are not cycles. Let D be the collection of edges e satisfying the following conditions:

- e is not an edge of a path.
- e is not incident to two vertices of a single cycle.
- e is incident to either an end-vertex of a path or a vertex of a cycle.

For ease of notation, direct the edges of D away from end-vertices of paths or cycle vertices.

One may easily see that the number of edges in D starting in a cycle of order 5 is at least 5, in a cycle of order 4 is at least 8 and in a cycle of order 3 is equal to 9.

Claim 2 *The number of edges in D going into a path of order i minus the number of edges of D going out of the same path is at most $b(i)$ where:*

$$b(i) = \begin{cases} \frac{3i-19}{2} & \text{if } i \text{ is odd} \\ \frac{3i-22}{2} & \text{if } i \text{ is even.} \end{cases}$$

Proof of Claim 2:

Note, by Claim 1, we may assume there are no paths of order 1 in $\mathscr{P}$. For a path of order 2, each end-vertex sends out 4 edges of D and there are no internal vertices so $b(2) = -8$. For a path of order 3, each end-vertex again sends out 4 edges of D but the internal vertex can take in at most 3 edges of D. This means $b(3) = -5$.

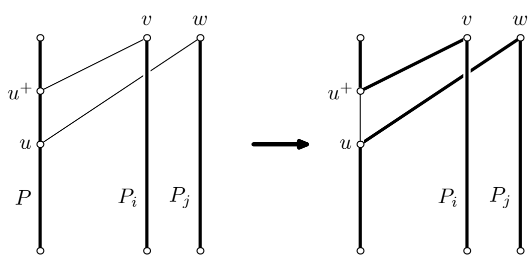

Figure 1: Eliminating the path P.

Suppose we have a path P of order $p \geq 4$. Consider two consecutive internal vertices $u, u^+ \in P$. If there exist two edges in D which end at u and u^+ and begin at distinct vertices v and w, we may reduce the number of paths in $\mathscr{P}$ by using these edges to break the path P as seen in Figure 1. Since $\mathscr{P}$ is minimum, one of the following must hold.

1. Each of u and u^+ is the endpoint of an edge of D starting at a single vertex.

2. Only one of u or u^+ is the endpoint of an edge in D.

Hence, for every pair of consecutive internal vertices of a path in $\mathscr{P}$, these vertices may be the end-vertices of at most three edges of D. Therefore, if p is even, the internal vertices can take in at most $\frac{3(p-2)}{2} = \frac{3p-6}{2}$ edges of D and the endpoints of P each send out 4 edges of D so $b(p) = \frac{3p-22}{2}$ if p is even. A similar argument shows $b(p) = \frac{3p-19}{2}$ if p is odd. $\square_{Claim\ 2}$

Since each edge of D must end at some vertex, we may say that the sum of the differences (bounded by $b(i)$) noted in Claim 2 minus the edges coming from cycles is not negative. More precisely we have

$$\begin{aligned} 0 &\leq \sum(t_i b(i)) - 5c_5 - 8c_4 - 9c_3 \\ &\leq \sum\left(t_i \frac{3i-19}{2}\right) - 5c_5 - 8c_4 - 9c_3. \end{aligned}$$

We may rearrange the above terms to get:

$$\begin{aligned} \sum(it_i) + \sum_{i\leq 5}(ic_i) &\geq \frac{19}{3}\sum(t_i) + \frac{25}{3}c_5 + \frac{28}{3}c_4 + 9c_3 \\ &\geq 6\left(\sum(t_i) + \sum_{i\leq 5}(c_i)\right). \end{aligned}$$

This leads to the following inequality:

$$\frac{\sum it_i + \sum_{i\leq 5} ic_i}{\sum t_i + \sum_{i\leq 5} c_i} \geq 6$$

which proves Inequality (2) and completes the proof of the theorem. □

3 Conclusion

Regarding future work, the goal is of course to prove (or disprove) the conjecture. One may easily prove an analogue of Claim 2 for any k but one would have to be more careful in the details to apply the above argument when $k \geq 6$.

It would also be of interest to consider a corresponding conjecture for connected regular graphs or stronger results for connected graphs in general but, as mentioned above, this seems very difficult.

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