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Constructing a large class of supertough graphs

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Abstract

For each odd $r \geq 3$ and each n of the form $2kb(r+2-2b)$ for $k \geq 1$ and $1 \leq b \leq (r-1)/2$, the first author has constructed an r-regular $r/2$-tough graph on n vertices. In this paper, we provide an alternate and more advantageous construction. First, both our new construction and its proof are simpler. Second, we use an extension of the notion of graph inflations, as used in a construction given by Chvátal in the case that $r = 3$. Third, we generalize a construction of our own used in the case that $r = 5$, that allowed us to further remove the restrictions on the number of vertices n.

1 Introduction

The toughness [1] of a non-complete graph $G = (V, E)$ is

$$\tau(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subseteq V \text{ and } \omega(G-S) > 1\},$$

where $\omega(G-S)$ is the number of components in the subgraph of G induced by $V \setminus S$. A graph G is said to be t-tough if $\tau(G) \geq t$. Since an r-regular graph G can have toughness at most $r/2$, we say that G is supertough if $\tau(G) = r/2$. The connectivity of a graph G is denoted $\kappa(G)$, and its edge connectivity is denoted $\lambda(G)$. All standard notation and terminology not presented here can be found in [7].

A $K_{1,3}$-center in a graph is a vertex with 3 non-adjacent neighbors. Graphs without $K_{1,3}$-centers are said to be $K_{1,3}$-free, and their toughness is strongly tied to their connectivity.

Theorem 1.1 ([6]). *If a graph G is $K_{1,3}$-free, then $\tau(G) = \frac{\kappa(G)}{2}$.*

Throughout this paper, r will represent an odd integer greater than or equal to 3. We shall provide a new proof of the following theorem of the first author.

Theorem 1.2 ([2]). *Let $r \geq 3$ be odd, $k \geq 1$, $1 \leq b \leq (r-1)/2$, and $n = 2kb(r+2-2b)$. Then there exists an r-regular $r/2$-tough graph $D_{k,b}(r)$ on n vertices.*

For our proof, we replace the construction $D_{k,b}(r)$ by a new construction $G_{k,b}(r)$. As we shall see, our new construction is more direct, rich in symmetries, and provides a natural generalization of constructions that have been used to obtain stronger results in the cases that $r = 3$ [1, 3] and $r = 5$ [4].

2 Harary Multigraphs and Inflations

For odd $q \geq 3$, let $H_q(2)$ be the graph on two vertices with q parallel edges. For even $p \geq 4$, the Harary multigraph $H_q(p)$ is constructed from the p-cycle C_p by replacing each single edge by $(q-1)/2$ parallel edges and further adding single edges between the antipodes. That is, $H_q(p)$ is the q-regular graph obtained from the simple cubic Harary graph [5] on p vertices by replacing each edge in the outer cycle by $(q-1)/2$ copies. The graph $H_5(4)$ is pictured on the left-hand side of Figure 1.

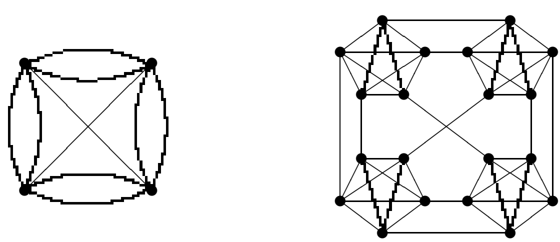

Figure 1: Harary Graph with its Inflation

The inflation of a graph H is the graph H^* whose vertices are all ordered pairs (v, e), where e is an edge of H and v is an endpoint of e, such that two vertices of H^* are adjacent if and only if they differ in exactly one coordinate. Note that each vertex v in H inflates to a clique K_d in H^* of order $d = \deg(v)$, and there is an injection of the edges of H into the edges of H^*. In the case of $H_q(p)^*$, each vertex in $H_q(p)$ is inflated to a clique K_q in $H_q(p)^*$, and $H_q(p)^*$ is also q-regular. We shall refer to the edges in $H_q(p)^*$ that correspond to the edges from $H_q(p)$ as fence edges. The graph $H_5(4)^*$ is pictured on the right-hand side of Figure 1. The utility of inflations for constructing tough graphs is noted by Chvátal [1].

Theorem 2.1 ([1]). *Let H be a graph without isolated vertices with $H \neq K_2$, and let H^* be its inflation. Then $\tau(H^*) = \lambda(H)/2$ and $\kappa(H^*) = \lambda(H^*) = \lambda(H)$.*

Since, for odd $q \geq 3$ and even $p \geq 2$, it is easy to see that $H_q(p)$ has edge-connectivity q, it follows from Theorem 2.1 that the q-regular graph $H_q(p)^*$ has toughness $q/2$ and connectivity q. The graphs $H_3(p)^*$ play a central role in [3].

3 Constructing r-regular $r/2$-tough Graphs

Let $r \geq 3$ be odd. For each $k \geq 1$ and $1 \leq b \leq (r-1)/2$, we construct an r-regular simple graph $G_{k,b}(r)$ on $2kb(r+2-2b)$ vertices from the product graph $H_{r+2-2b}(2k)^* \times K_b$ by adding more edges. First, regard $K_b = \langle 1, \ldots, b \rangle$ and, for each $1 \leq i \leq b$, call $H_{r+2-2b}(2k)^* \times \{i\}$ the i^{th} level of $G_{k,b}(r)$. For each fence edge in the first level, joining some vertices $(x, 1)$ and $(y, 1)$ in $H_{r+2-2b}(2k)^* \times K_b$, add edges so that, for all $1 \leq i, j \leq b$, vertices (x, i) and (y, j) become joined in $G_{k,b}(r)$. These new edges shall also be called fence edges. In sum, we say that $G_{k,b}(r)$ is the r-regular graph obtained from $H_{r+2-2b}(2k)$ by inflating each vertex of $H_{r+2-2b}(2k)$ to a product $K_{r+2-2b} \times K_b$ in $G_{k,b}(r)$ and inflating each edge in $H_{r+2-2b}(2k)$ to a clique K_{2b} in $G_{k,b}(r)$. A portion of $G_{2,3}(9)$ is shown in Figure 2.

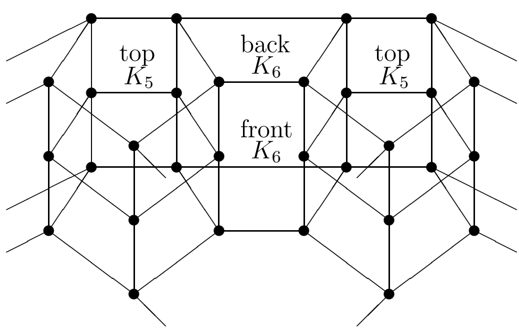

Figure 2: The Structure of $G_{2,3}(9)$

The main result of this paper is the following theorem. Its proof amounts to a simple listing of paths. The only difficulty is carefully denoting these otherwise easy-to-describe paths.

Theorem 3.1. *Let r be odd, $r \geq 3$, $k \geq 1$, and $1 \leq b \leq (r-1)/2$. Then, $G_{k,b}(r)$ is an r-regular $r/2$-tough graph on $n = 2kb(r+2-2b)$ vertices.*

Proof. Since the neighborhood of each vertex in $G_{k,b}(r)$ is a union of two cliques K_{r+1-2b} and K_{2b-1}, we immediately see that $G_{k,b}(r)$ is $K_{1,3}$-free. By Theorem 1.1, it therefore suffices to show that $G_{k,b}(r)$ is r-connected. Our approach uses the characterization of connectivity given by Menger's theorem. First, note that each level of $G_{k,b}(r)$ is the $(r+2-2b)$-connected graph $H_{r+2-2b}(2k)^*$. This is sufficient when $b = 1$. So we assume that $b \geq 2$. In general, given distinct vertices u and v in $H_{r+2-2b}(2k)^*$, there must be $r+2-2b$ internally disjoint uv-paths $P^1, P^2, \ldots, P^{r+2-2b}$ in $H_{r+2-2b}(2k)^*$. We assume that the indexing is such that P^1 leaves u via the fence edge and one of P^2 or P^1 enters v via the fence edge. To specify indices, we say that P^f enters v via the fence edge and $\{P^1, P^2\} = \{P^f, P^g\}$. For each $1 \leq i \leq b$, we shall make use of the $r+2-2b$ internally disjoint paths $P^1 \times \{i\}, P^2 \times \{i\}, \ldots, P^{r+2-2b} \times \{i\}$ in $G_{k,b}(r)$.

To prove that $G_{k,b}(r)$ is r-connected, suppose that (u,i) and (v,l) are distinct non-adjacent vertices in $G_{k,b}(r)$. We shall specify r internally disjoint paths between them.

Initially, consider the case in which (u,i) and (v,l) are on the same level. So, it suffices to assume that $i = l = 1$. For each $1 \leq j \leq r+2-2b$, let $P_1^j = P^j \times \{1\}$. The paths $P_1^1, P_1^2, \ldots, P_1^{r+2-2b}$ provide $r+2-2b$ internally disjoint paths from $(u,1)$ to $(v,1)$ that are completely contained within the first level. For each remaining level i, we shall specify two paths from $(u,1)$ to $(v,1)$ that are internally disjoint from each other and whose interior vertices are entirely contained in level i. This will bring the number of internally disjoint paths from $(u,1)$ to $(v,1)$ to a total of $(r+2-2b) + 2(b-1) = r$, as desired. The paths we now describe are pictured in Figure 3. For each $2 \leq i \leq b$, form paths P_i^1 and P_i^2 from paths $P^1 \times \{i\}$ and

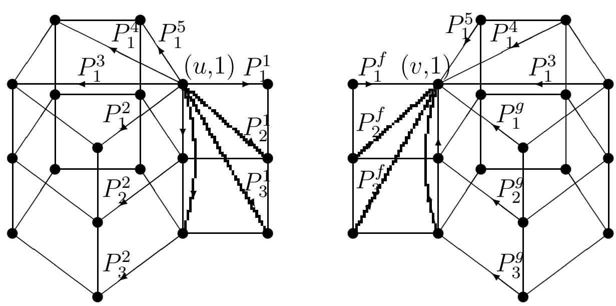

Figure 3: Linking Vertices on the Same Level

$P^2 \times \{i\}$, respectively, as follows. Replace the initial edge of $P^1 \times \{i\}$, which joins (u,i) to some (w,i), by the edge joining $(u,1)$ to (w,i). To the beginning of $P^2 \times \{i\}$ add the edge joining $(u,1)$ to (u,i). Replace the final edge of $P^f \times \{i\}$, which joins some (z,i) to (v,i), by the edge joining (z,i) to $(v,1)$. To the end of $P^g \times \{i\}$ add the edge joining (v,i) to $(v,1)$.

Finally, consider the case in which (u,i) and (v,l) are on different levels. So, it suffices to assume that $i = 2$ and $l = 1$. The paths we choose are essentially the same as those used in the previous case, except that we need to make new choices for the first few edges of our paths in levels one and two. Since the remainder of each path, ending at $(v,1)$, is the same as that used in the previous case, we do not redescribe that end here. The paths are pictured in Figure 4. Let P_1^1 be the path $P^1 \times \{1\}$ with its initial edge joining $(u,1)$ to $(w,1)$ replaced by the edge joining $(u,2)$ to $(w,1)$.

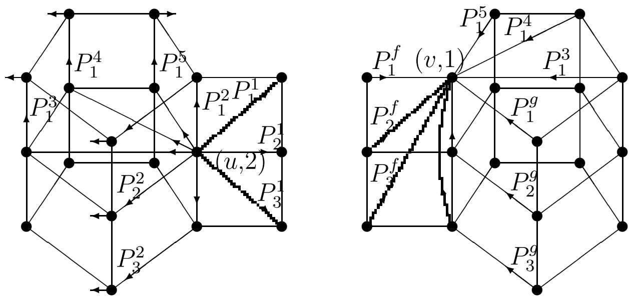

Figure 4: Linking Vertices on the Different Levels

Form P_1^2 by adding to the beginning of $P^2 \times \{1\}$ the edge joining $(u, 2)$ to $(u, 1)$. For $3 \leq j \leq r + 2 - 2b$, form P_1^j by replacing the initial edge of $P^j \times \{1\}$, which joins $(u, 1)$ to some $(x, 1)$, by the edge from $(u, 2)$ to $(x, 2)$ followed by the edge from $(x, 2)$ to $(x, 1)$. For each $3 \leq i \leq b$, the paths P_i^1 and P_i^2 are constructed as in the previous case, except that they start at $(u, 2)$ instead of $(u, 1)$. □

4 Conclusions

The graphs $G_{k,1}(3) = H_3(2k)^*$ are the inflations used by Chvátal [1] to construct 3/2-tough cubic graphs on $6k$ vertices. Moreover, our constructions [3] of 3/2-tough graphs with degree sequence either $3, 3, \ldots, 3$ or $4, 3, \ldots, 3$ also agree with these graphs when the number of vertices is divisible by 6.

The $\frac{5}{2}$-tough 5-regular graphs on $12k$ vertices constructed in [4] coincide with the special case of $G_{k,b}(r)$ when $r = 5$ and $b = 2$. In that case, we showed that each subgraph $K_{r+2-2b} \times K_b$ on $t = b(r+2-2b)$ vertices can be replaced by an alternative subgraph on $t + 1$ or $t + 2$ vertices, while retaining toughness 5/2. By using those substitutes, we further constructed, for almost every number n, a graph on n vertices with toughness 5/2 and degree sequence either $5, 5, \ldots, 5$ or $6, 5, \ldots, 5$.

For our general construction $G_{k,b}(r)$, we have been unable to find appropriate substitution subgraphs allowing us to construct $r/2$-tough r-regular graphs on n vertices when n is not divisible by $2b(r + 2 - 2b)$ for some $1 \leq b \leq (r - 1)/2$. However, we conjecture that such substitutes exist and can be used to exploit and extend the construction $G_{k,b}(r)$ introduced here.

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