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Locating-domination, 2-domination and independence in trees

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Abstract

A set D of vertices in a graph G is 2-dominating if every vertex not in D has at least two neighbors in D and locating-dominating if for every two vertices u, v not in D, the sets $N(u) \cap D$ and $N(v) \cap D$ are non-empty and different. The minimum cardinality of a 2-dominating set (locating-dominating set) is denoted by $\gamma_2(G)$ ($\gamma_L(G)$). It is known that every tree T with $n \geq 2$ vertices, ℓ leaves, s support vertices and independence number $\beta(T)$, satisfies

$$\gamma_L(T) \leq (n + \ell - s)/2 \leq \beta(T) \leq \gamma_2(T).$$

We show that $\beta(T) + \gamma_L(T) \leq n + \ell - s$ and that $\gamma_2(T) = (n + \ell - s)/2$ if and only if $\gamma_L(T) = \gamma_2(T)$. Moreover, $\gamma_2(G) \leq 2\gamma_L(G)$ for every bipartite graph and $\gamma_2(G) \leq 2\gamma_L(G) - 1$ if G is a tree with $n \geq 3$.

1 Introduction

We consider graphs $G = (V, E)$ of order $|V| = n$. The graph is said *nontrivial* if $n > 1$. The *neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and its *degree* is $\deg_G(v) = |N(v)|$. A vertex of degree one is called a *pendant vertex* (or a *leaf*) and its neighbor is called a *support vertex*. We denote by $S(G)$ (respectively, $L(G)$) the set of support vertices (respectively, leaves) of T and by L_v the set of leaves adjacent to a support vertex v. A support vertex v is *strong (respectively, weak)* if $|L_v| \geq 2$ (respectively, $|L_v| = 1$). In this paper we call the *core* of G the subset $C(G) = V(G) \setminus (S(G) \cup L(G))$. For a graph G we denote by $n(G), \ell(G)$ and $s(G)$ the number of vertices, leaves and support vertices of G, respectively, (we use n, ℓ and s if there is no ambiguity). The *corona* $H \circ K_1$ of a graph H is obtained from H by adding a leaf at each of its vertices. The core of a corona is empty. More generally, we call a graph with an empty core a *pseudocorona*, and a *strong pseudocorona* is a pseudocorona having only strong support vertices.

A subset D of $V(G)$ is a 2-*dominating set* if every vertex not in S is adjacent to at least two vertices of D. The 2-*domination number* $\gamma_2(G)$ is the minimum cardinality of a 2-*dominating set* of G. The more general concept of k-domination was introduced by Fink and Jacobson [4]. A set $D \subseteq V$ is a *locating-dominating set* if it is dominating and every two vertices x, y of $V \setminus D$ satisfy $N(x) \cap D \neq N(y) \cap D$. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of a locating-dominating set. Locating domination was introduced by Slater [5, 6]. For any parameter $\mu(G)$ associated to a graph property $\mathcal{P}$, we refer to a set of vertices with Property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-set. The subgraph induced in G by a subset of vertices D is denoted $\langle D \rangle$ if there is no ambiguity on G.

Every tree with $n \geq 2$ vertices, s supports, ℓ leaves and independence number $\beta(T)$ satisfies the following equality chain. For each inequality, we indicate the reference of the paper in which it was established and the name given here to the family of extremal trees.

$$\gamma_L(T) \underset{[2]\ \mathcal{F}}{\leq} \frac{n+\ell-s}{2} \underset{[3]\ \mathcal{G}}{\leq} \beta(T) \underset{[1]\ \mathcal{H}}{\leq} \gamma_2(T) \qquad (1)$$

Our purpose in this paper is to complete the results summarized in (1) by showing that $\gamma_L(T) = (n+\ell-s)/2$ implies $\beta(T) = (n+\ell-s)/2$ and that $\gamma_2(T) = (n+\ell-s)/2$ implies $\gamma_L(T) = (n+\ell-s)/2$, but that the converses are false. For this we show that $\beta(T) + \gamma_L(T) \leq n + \ell - s$ and that $\gamma_2(T) = (n + \ell - s)/2$ if and only if $\gamma_2(T) = \gamma_L(T)$. We also show that $\gamma_2(G) \leq 2\gamma_L(G)$ for every nontrivial bipartite graph and $\gamma_2(G) \leq 2\gamma_L(G) - 1$ if G is a tree of order at least 3.

The following observation will be of use throughout the paper.

Observation 1 *For every connected graph G of order $n \geq 3$, the set $L(G)$ of all leaves is contained in every $\gamma_2(G)$-set and in some $\beta(G)$-set. The set $S(G)$ of all* *support vertices is contained in some $\gamma_L(G)$-set and for each $v \in S(G)$, every $\gamma_L(G)$-set contains at least $|L_v|$ vertices in $\{v\} \cup L_v$.*

2 The extremal families

In this section we give some precisions on the families $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and define a fourth one that will be encountered in Theorem 4.

Family $\mathcal{F}$ The family $\mathcal{F}$ of the nontrivial trees such that $\gamma_L(T) = (n + \ell - s)/2$ is described by Theorem 2.4 in [2] by the means of a recursive construction using five graph operations. We do not need to repeat this construction here.

Family $\mathcal{G}$ The family $\mathcal{G}$ consits of the nontrivial trees T such that $\beta(T) = (n + \ell - s)/2$. The inequality $\beta(G) \geq (n + \ell - s)/2$ is established by Theorem 8 in [3] for bipartite graphs but the extremal graphs are not determined. We give below a characterization of them.

Proposition 2 *A nontrivial connected bipartite graph G satisfies $\beta(G) = (n + \ell - s)/2$ if and only if its core $C(G) = V(G) \setminus (S(G) \cup L(G))$ induces a bipartite graph with a perfect matching.*

Proof. Every $\beta(G)$-set containing $L(G)$ contains a maximum independent set of $\langle C(G) \rangle$. Hence $\beta(G) = \ell + \beta(C(G))$. As a consequence of the König-Egerváry theorem, the maximum size of a matching of the bipartite graph $\langle C(G) \rangle$ is

$$\begin{aligned} |C(G)| - \beta(C(G)) &= (n - \ell - s) - (\beta(G) - \ell) \\ &= (n - \ell - s)/2 - (\beta(G) - (n + \ell - s)/2) \\ &= |C(G)|/2 - (\beta(G) - (n + \ell - s)/2)). \end{aligned}$$

Therefore $\langle C(G) \rangle$ has a perfect matching if and only if $\beta(G) = (n + \ell - s)/2$. ■

Family $\mathcal{H}$ The family $\mathcal{H}$ of the nontrivial trees T such that $\beta(T) = \gamma_2(T)$ is described by Theorem 6 in [1] by a recursive construction using two graph operations. We do not need this construction here and will only use the second characterization, namely T is in $\mathcal{H}$ if and only if it has a unique $\gamma_2(T)$-set which is also the unique $\beta(G)$-set.

Family $\mathcal{L}$ We denote by $\mathcal{L}$ the family of nontrivial trees which are strong pseudo-coronas.

3 Main results

Theorem 3 *Every tree T of order $n \geq 2$ satisfies $\beta(T) + \gamma_L(T) \leq n + \ell - s$. The bound is sharp.*

Proof. We make a proof by induction on n. The result is clearly true for $n = 2$ since P_2 has two leaves and two support vertices. Suppose that the property is true for every tree of order at most $n-1 \geq 2$ and let T be a tree of order n. If T is a star, then $\gamma_L(T) = \beta(T) = n-1$ and the result holds. We assume now that T has diameter at least 3 and we root T at a diametrical vertex r. Let u be a leaf at the last level and let v, w and t the father, grandfather and grandgrandfather of u.

Case 1: $d(w) > 2$ or $d(w) = 2$ and t is not a support vertex of T

The tree $T' = T - L_v \cup \{v\}$ has order at least 2 and satisfies $\beta(T) \leq \beta(T') + |L_v|$ and $\gamma_L(T) \leq \gamma_L(T') + |L_v|$ (since $D \cup L_v$, where D is any $\gamma_L(T')$-set, is a locating-dominating set of T). By the inductive hypothesis applied to T',

$$\beta(T) + \gamma_L(T) \leq \beta(T') + \gamma_L(T') + 2|L_v| \leq n(T') + \ell(T') - s(T') + 2|L_v|.$$

If $d(w) > 2$ then $n(T) = n(T') + 1 + |L_v|$, $\ell(T) = \ell(T') + |L_v|$ and $s(T) = s(T') + 1$. Therefore $n(T) + \ell(T) - s(T) = n(T') + \ell(T') - s(T') + 2|L_v|$ and $\beta(T) + \gamma_L(T) \leq n(T) + \ell(T) - s(T)$.

If $d(w) = 2$ and t is not a support vertex of T then $n(T) = n(T') + 1 + |L_v|$, $\ell(T) = \ell(T') + |L_v| - 1$ and $s(T) = s(T')$. Therefore $n(T) + \ell(T) - s(T) = n(T') + \ell(T') - s(T') + 2|L_v|$ and $\beta(T) + \gamma_L(T) \leq n(T) + \ell(T) - s(T)$.

Case 2: $d(w) = 2$ and t is a support vertex of T.

The tree $T' = T - L_v \cup \{v, w\}$ has order at least 2. Let I be a $\beta(T)$-set containing L_v. Then $w \in I$ for otherwise $t \in I$ and $(I \setminus \{t\}) \cup L_t \cup \{w\}$ contradicts the choice of I. Hence $\beta(T) \leq \beta(T') + |L_v| + 1$. If D is a $\gamma_L(T')$-set containing $S(T')$, and in particular t, and if L'_v is a subset of L_v of order $|L_v| - 1$, then $D \cup L'_v \cup \{v\}$ is a locating-dominating set of T. Hence $\gamma_L(T) \leq \gamma_L(T') + |L_v|$. By the inductive hypothesis,

$$\beta(T) + \gamma_L(T) \leq \beta(T') + \gamma_L(T') + 2|L_v| + 1 \leq n(T') + \ell(T') - s(T') + 2|L_v| + 1.$$

Now, $n(T) = n(T') + 2 + |L_v|$, $\ell(T) = \ell(T') + |L_v|$ and $s(T) = s(T') + 1$. Therefore $n(T) + \ell(T) - s(T) = n(T') + \ell(T') - s(T') + 2|L_v| + 1$ and thus $\beta(T) + \gamma_L(T) \leq n(T) + \ell(T) - s(T)$, which completes the proof.

We leave the reader check that every tree of order $n \geq 2$ and diameter at most 6 satisfies $\beta(T) + \gamma_L(T) = n(T) + \ell(T) - s(T)$. For every pseudocorona T of a tree, $\beta(T) = \gamma_L(T) = \ell(T) = (n(T) + \ell(T) - s(T))/2$. Therefore the pseudocoronas of trees are also extremal for Theorem 3. ∎

Corollary 1 *For every nontrivial tree T the property $\gamma_L(T) = (n + \ell - s)/2$ implies $\beta(T) = (n + \ell - s)/2$. The converse is false and there exist trees with $\beta(T) = (n + \ell - s)/2$ such that $(n + \ell - s)/2$ is arbitrarily larger than $\gamma_L(T)$.*

Proof. The implication comes from the fact that by Theorem 3 and (1), $0 \leq \beta(T) - (n + \ell - s)/2 \leq (n + \ell - s)/2 - \gamma_L(T)$ for every tree. The trees $T_1(k) = P_{10k}$ satisfy $\beta(T_1(k)) = 5k = (n + \ell - s)/2$ and $\gamma_L(T_1(k)) = 4k$. These trees are in $\mathcal{G}$ and not in $\mathcal{F}$. ∎

Theorem 4 *The following properties are equivalent for every nontrivial tree.*

(i) $\gamma_2(T) = (n + \ell - s)/2$

(ii) $\gamma_2(T) = \gamma_L(T)$

(iii) $T \in \mathcal{L}$, *i. e., T is a strong pseudocorona.*

Proof. (i) $\Rightarrow$ (iii) Let T be a graph satisfying $\gamma_2(T) = (n + \ell - s)/2$. By (1), $\gamma_2(T) = \beta(T)$. Hence $T \in \mathcal{H}$ and T has a unique $\beta(T)$-set. When the core $C(T) = V(T) \setminus (S(T) \cup L(T))$ of T is not empty, let $A \cup B$ be the bipartition of $\langle C(T)\rangle$. If $|A| = |B|$, then $A \cup L(T)$ and $B \cup L(T)$ are two different independent sets of order $(n - \ell - s)/2 + \ell = \beta(T)$, in contradiction to $T \in \mathcal{H}$. If, say, $|A| > |B|$ then $A \cup L(T)$ is an independent set of order $|A|+\ell > \ell+(n-\ell-s)/2 = (n+\ell-s)/2$, which contradicts $\beta(T) = \gamma_2(T)$. Hence $C(T) = \emptyset$ and T is a pseudocorona. Therefore $\beta(T) = \ell$. Since $L(T)$ is contained in every $\gamma_2(T)$-set and $\gamma_2(T) = \beta(T) = |L(T)|$, the set $L(T)$ is 2-dominating and each support vertex is strong. Hence T is a strong pseudocorona.

(iii) $\Rightarrow$ (ii) Let T be a strong pseudocorona of a tree H. The only $\gamma_2(T)$-set is the set $L(T)$ of all the leaves of T. Moreover $L(T)$ is a locating dominating set which is minimum by Observation 1. Hence $\gamma_2(T) = \gamma_L(T) = \ell(T)$.

(ii) $\Rightarrow$ (i) Obvious from (1). ∎

Corollary 2 *For every nontrivial tree T the property $\gamma_2(T) = (n + \ell - s)/2$ implies $\gamma_L(T) = (n + \ell - s)/2$. The converse is false and there exist trees with $\gamma_L(T) = (n + \ell - s)/2$ such that $\gamma_2(T)$ is arbitrarily larger than $(n + \ell - s)/2$.*

Proof. The implication is an immediate consequence of (1) and Theorem 4. The coronas $T_2(k) = P_{3k} \circ K_1$ satisfy $\gamma_L(T_2(k)) = 3k = (n + \ell - s)/2$ ($= \beta(T_2(k))$ by Proposition 1) but $\gamma_2(T_2(k)) = 4k$. Note that $T_2(k)$, a corona but not a strong pseudocorona, is in $\mathcal{F}$ and not in $\mathcal{H}$. ∎

Remark By analogy between Corollary 1 and Corollary 2, one can wonder whether $(n + \ell - s)/2 - \gamma_L(T)$ is not always smaller than $\gamma_2(T) - (n + \ell - s)/2$. i. e., whether $\gamma_2(T) + \gamma_L(T)$ is not at least $n + \ell - s$ for every tree T. This is not the case as shown by the following example. The tree H_x represented in Figure 1 satisfies $n(H_x) = 21$, $\gamma_2(H_x) = 12$ and $\gamma_L(H_x) = 8$. For the tree $T_3(k)$ constructed from k disjoint trees $H_{x_1}, \cdots, H_{x_k}$ by adding the $k-1$ edges $x_1x_2, x_2x_3, \cdots, x_{k-1}x_k$, $\beta(k) = \gamma_2(T_3(k)) = 12k$, $\gamma_L(T_3(k)) = 8k$ and the sum $\gamma_2(T) + \gamma_L(T)$ is arbitrarily smaller than $n + \ell - s = 21k$. Note that $T_3(k)$ is in $\mathcal{H}$ and not in $\mathcal{G}$.

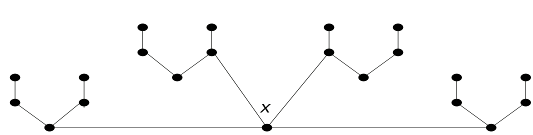

Figure 1: $H(x)$

We recapitulate in Figure 2 the relative positions of the four classes $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{L}$. By Corollary 1, $\mathcal{F} \subseteq \mathcal{G}$. By Theorem 4, $\mathcal{H} \cap \mathcal{G} \cap \mathcal{F} = \mathcal{H} \cap \mathcal{G} = \mathcal{L}$, which reduces to $\mathcal{H} \cap \mathcal{F} = \mathcal{H} \cap \mathcal{G} = \mathcal{L}$ since $\mathcal{F} \subseteq \mathcal{G}$. Moreover, the trees $T_1(k), T_2(k)$ and $T_3(k)$ show that the differences $\mathcal{G} \setminus \mathcal{F}$, $\mathcal{F} \setminus \mathcal{L}$ and $\mathcal{H} \setminus \mathcal{G}$ are not empty. To show that the difference $\mathcal{H} \setminus \mathcal{L}$ is not empty, consider the tree $T_4(k)$ obtained by subdividing twice each edge of a star $K_{1,k}$ with $k \geq 2$. Then $\beta(T_4(k)) = \gamma_2(T_4(k)) = 2k$ and $(n + \ell - s)/2 = (3k + 1)/2$. Hence $T_4(k)$ is in $\mathcal{H}$ but not in $\mathcal{G}$. Note that the core $C(T_4(k))$, a star $K_{1,k}$, has no perfect matching, in accordance to $T_4(k) \notin \mathcal{G}$.

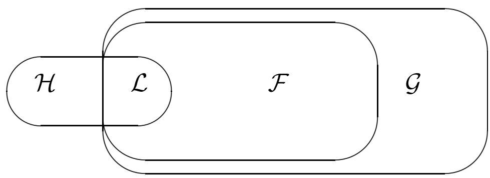

Figure 2

Let us denote by $\mathcal{J}$ the family of trees such that $\beta(T) + \gamma_L(T) = n + \ell - s$. By Corollary 1, every tree in $\mathcal{F}$ is in $\mathcal{J}$. Since moreover for the trees of $\mathcal{J}$, the equality $\beta(T) = (n+\ell-s)/2$ is equivalent to $\gamma_L(T) = (n+\ell-s)/2$, $\mathcal{J} \cap \mathcal{G} = \mathcal{J} \cap \mathcal{F}$. Therefore $\mathcal{J} \cap \mathcal{G} = \mathcal{F} \supseteq \mathcal{H} \cap \mathcal{G}$. But $\mathcal{H} \setminus \mathcal{G}$ and $\mathcal{J} \setminus \mathcal{G}$ are not comparable as shown by the following examples. The tree $T_3(k)$ is in $\mathcal{H} \setminus \mathcal{F}$ and not in $\mathcal{J}$. Let $T_5(k)$ be obtained from a path $x_1x_2x_3x_4x_5x_6$ by attaching $k \geq 2$ leaves at x_3. Then $\gamma_L(T_5(k)) = k + 2$, $\beta(T_5(k)) = k + 3$ and $\gamma_2(T_5(k)) = k + 4$, and $n + \ell - s = 2k + 5$. The tree $T_5(k)$ is in $\mathcal{J} \setminus \mathcal{G}$ and not in $\mathcal{H}$.

The last result of the paper is related to the problem of finding upper bounds on the ratios $\gamma_2(T)/\beta(T)$, $\beta(T)/\gamma_L(T)$ and $\gamma_2(T)/\gamma_L(T)$.

The bound $\gamma_2(T)/\beta(T) \leq 3/2$ for every tree is already established by Theorem 12 in [1] where the class of extremal trees is constructed from $T_1 = P_4$ by recursively adding a new path $P_{i+1} = P_4$ attached by an edge joining an internal vertex of P_{i+1} to a nonpendant vertex of T_i of degree at least 3 from the second step (this last condition on the degree was forgotten in [1]). Equivalently, such a tree can be obtained from any tree H and $n(H)$ disjoint P_4's by identifying each vertex of H with an internal vertex of a P_4.

The next theorem gives a bound on the ratio $\gamma_2(T)/\gamma_L(T)$.

Theorem 5 *1. Every graph G satisfies $\gamma_2(T) \leq 2\gamma_L(T)$ and the bound is sharp.*
2. Every tree T of order $n \geq 3$ satisfies $\beta(T) \leq \gamma_2(T) \leq 2\gamma_L(T) - 1$ and the inequality $\beta(T) \leq 2\gamma_L(T) - 1$ is sharp.

Proof. 1. Let A be a $\gamma_L(G)$-set and A' the set of the external A-private neighbors of the vertices of A (a vertex a' of $V \setminus A$ is an external private neighbor of a vertex

a of A if $N_A(a') = \{a\}$). Since the dominating set A is locating, each vertex of A has at most one external A-private neighbor. Therefore $|A'| \le |A|$. Every vertex of $B = V \setminus (A \cup A')$ has at least two neighbors in A. Hence $A \cup A'$ is a 2-dominating set of G and $\gamma_2(G) \le |A| + |A'| \le 2\gamma_L(G)$.

Let G be obtained from an even cycle $x_1x_2\cdots x_{2k}x_1$ with $k \ge 3$ by attaching a leaf at each vertex x_i with i odd. Then $\gamma_L(G) = k$ ($\{x_1, x_3, \cdots, x_{2k-1}\}$ is a $\gamma_L(G)$-set) and $\gamma_2(G) = 2k$. Hence G is an example of extremal bipartite graph.

2. To prove that the previous inequality is strict when G is a tree T, we start from a $\gamma_L(T)$-set A such that the number of edges incident with two vertices of $V \setminus A$ is minimum. Since every vertex of B has at least two neighbors in A and T is a tree, the number $m(A,B)$ of edges of T between A and B is such that $2|B| \le m(A,B) \le |A| + |B| - 1$. Therefore $|B| \le |A| - 1$. If $|A'| < |A|$, then $\gamma_2(T) \le |A| + |A'| \le 2\gamma_L(T) - 1$. Assume now that every vertex of A has exactly one external A-private neighbor. If a vertex v of A has no neighbor in B then, since T is connected and $n \ge 3$, either $d(v) = 1$, the external A-private neighbor v' of v has a neighbor in $V \setminus A$ and $(A \setminus \{v\}) \cup \{v'\}$ is a γ_L-set contradicting the choice of A, or v has a neighbor in A and $A' \cup (A \setminus \{v\})$ is a 2-dominating set of T of order $2\gamma_L(T) - 1$. If every vertex of A has at least one neighbor in B, then $A' \cup B$ is a 2-dominating set of order $|A| + |B|$ and $\gamma_2(T) \le |A| + |B| \le 2|A| - 1 = 2\gamma_L(T) - 1$. The proof is complete.

We give some examples of extremal trees. The caterpillar $T_6(k)$ obtained from a path $x_1x_2\cdots x_{2k+1}$ by attaching a leaf at each vertex x_i with i odd or the tree $T_7(k)$ obtained from a star $K_{1,k+1}$ with $k \ge 1$ by subdividing k edges twice satisfy $\gamma_L(T) = k+1$ and $\beta(T) = \gamma_2(T) = 2k+1 = 2\gamma_L(T) - 1$. The tree $T_8(k)$ obtained from a star $K_{1,k+2}$ with $k \ge 1$ by subdividing one edge once and k edges twice is such that $\gamma_L(T) = k+2$, $\gamma_2(T) = 2k+3 = 2\gamma_L(T) - 1$ but $\beta(T) = 2k+2 < 2\gamma_L(T) - 1$.

■

The trees such that $\beta(T) = 2\gamma_L(T) - 1$ are clearly in $\mathcal{H} \setminus \mathcal{F}$. The following proposition gives the exact value of γ_L for these trees.

Proposition 6 *If a nontrivial tree T satisfies $\beta(T) = 2\gamma_L(T) - 1$ then $\gamma_L(T) = (n + \ell - s + 1)/3$.*

Proof. Let T be a nontrivial tree such that $\beta(T) = 2\gamma_L(T) - 1$. By Theorem 3, $3\gamma_L(T) - 1 \le n + \ell - s$. It is proved in [2] that $\gamma_L(T) \ge (n + \ell - s + 1)/3$ for every tree with $n \ge 3$, and this results hold for P_2 with $\ell = s = 2$. Therefore $\gamma_L(T) = (n + \ell - s + 1)/3$. ■

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