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On equidistance graphs*

Anita Pasotti

Dipartimento di Matematica
Facoltà di Ingegneria
Università degli Studi di Brescia
Via Valotti, 9, I-25133 Brescia
Italy
anita.pasotti@ing.unibs.it

Abstract

Motivated by a problem concerning some combinatorial designs, we introduce the concept of an i-equidistance graph as a graph Γ with the property that its distance-i graph $\Gamma(i)$ has the same number of edges as it. In the special case where Γ is i-equidistance for every i not exceeding its diameter we say that Γ is an equidistance graph. We give several examples and properties and we propose some open questions.

1 Introduction

Let K_v be the complete graph on v vertices. We recall that a *decomposition* of K_v (see [4]) is a collection of graphs, called *blocks*, whose edges partition $E(K_v)$. In the case that all blocks are isomorphic to a given graph Γ one speaks of a (K_v,Γ)-*design*. We also recall that given a graph Γ and a positive integer i not exceeding its diameter, the *distance-i graph associated with* Γ is the graph $\Gamma(i)$ having the same vertices as Γ and in which two vertices are adjacent if and only if they have distance i in Γ (see [1]). It is obvious that $\Gamma(1)$ is Γ itself and that the set of all distance-i graphs associated with Γ is a decomposition of the complete graph with vertex-set $V(\Gamma)$.

Generalizing the well-known concept of an *i-perfect cycle system* (see, e.g., [3], [5] or [7]) we define a (K_v,Γ)-design $\mathcal{D}$ to be *i-perfect* if every two distinct vertices of K_v appear at distance i in exactly one block of $\mathcal{D}$ or, equivalently, if the distance-i graphs associated with all blocks of $\mathcal{D}$ form a $(K_v,\Gamma(i))$-design. Thus the number of blocks of an i-perfect (K_v,Γ)-design, that is $\frac{v(v-1)}{2|E(\Gamma)|}$, must be equal to the number of blocks of a $(K_v,\Gamma(i))$-design, that is $\frac{v(v-1)}{2|E(\Gamma(i))|}$. Obviously, this is equivalent to saying that Γ and $\Gamma(i)$ have the same number of edges.
This suggests the introduction of the following definition.

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Definition 1.1 *A graph* Γ *is* i-equidistance *if* $\Gamma(i)$ *has the same number of edges as* Γ. *Also,* Γ *is* I-equidistance *if it is* i-equidistance *for every positive integer* i *belonging to a given set* $I \subseteq \{1, 2, \dots, diam(\Gamma)\}$. *In particular,* Γ *is* equidistance *if it is i-equidistance for every positive integer i not exceeding its diameter, namely if it is* I*-equidistance with* $I = \{1, 2, \dots, diam(\Gamma)\}$.

In this way, the previous observation can be stated as follows.

Proposition 1.2 *If there exists an i-perfect* (K_v, Γ)*-design for some finite* $v > 1$*, then* Γ *is an* i*-equidistance graph.*

As a special case of the deep theory developed in [6], the above necessary condition is also sufficient in the sense that for any i-equidistance graph Γ there are (infinitely many) values of v for which there exists an i-perfect (K_v, Γ)-design. More detailed comments on this can be found in [2].

Proposition 1.3 *Let* Γ *be an equidistance graph and let* v*,* e *and* d *be its order, the number of its edges and its diameter, respectively. Then we have:*

$$v(v-1) = 2ed \tag{1}$$

Proof. As already observed $\{\Gamma(1), \Gamma(2), \dots, \Gamma(d)\}$ is a decomposition of K_v so that we have $\sum_{i=1}^{d} |E(\Gamma(i))| = |E(K_v)| = \frac{v(v-1)}{2}$. On the other hand we have, by hypothesis, $|E(\Gamma(i))| = e$ for every i so that $de = \frac{v(v-1)}{2}$ and the assertion follows.

□

We will refer to Condition (1) as the *Steiner Condition*. We observe that this condition does not assure that Γ is an equidistance graph. For instance, although T_5, the prism on 10 vertices, satisfies that condition, it is not an equidistance graph since the number of edges of $T_5(2)$ and $T_5(3)$ is 20 and 10, respectively.

On the other hand one can easily check that every graph with diameter 2 satisfying the Steiner condition is certainly an equidistance graph.

Of course, more generally, in order to claim that a graph Γ satisfying (1) is an equidistance graph it suffices to check that $|E(\Gamma(i))| = e$ for all but one value of i in the set $\{1, 2, \dots, diam(\Gamma)\}$.

2 Some elementary i-equidistance graphs

We need the notion of a *Cayley graph*, whose definition is the following. Let G be an additive group and let Ω be a subset of $G - \{0\}$ such that $-\omega \in \Omega$ for every $\omega \in \Omega$. Then the *Cayley graph on* G *with connection set* Ω is the graph $Cay[G : \Omega]$ in which the vertices are the elements of G and where $[x, y]$ is an edge if and only if $x - y \in \Omega$. It is clear that this graph admits an automorphism group isomorphic to G acting sharply transitively on the vertices. It is well known [8] that the converse is also true, i.e., every graph admitting a sharply vertex transitive automorphism group G is, up to isomorphism, a Cayley graph on G.

In this section we establish the values of i for which some elementary graphs are i-equidistance. In some cases we will use the following obvious remark.

Remark 2.1 *If both* Γ *and* $\Gamma(i)$ *are regular graphs, then* Γ *is an* i*-equidistance graph if and only if* Γ *and* $\Gamma(i)$ *have the same degree.*

First of all, note that every complete graph trivially is an equidistance graph.

Proposition 2.2 *The* k*-cycle* C_k *is an equidistance graph if and only if* k *is odd while* C_{2h} *is* I*-equidistance with* $I = \{1, 2, \ldots, h-1\}$.

Proof. Observe that apart from the case where k is even and $i = k/2$, the distance-i graph of C_k has $g := \gcd(k, i)$ connected components each of which is a k/g-cycle so that we have $|E(C_k(i))| = g|E(C_{k/g})| = k = |E(C_k)|$. Also observe that if k is even we have that $C_k\left(\frac{k}{2}\right)$ is formed by $\frac{k}{2}$ disjoint edges. The assertion follows. □

Given an integer $k \geq 4$, the *wheel* on k vertices, denoted by W_k, is the "skeleton" of a pyramid whose base is a $(k-1)$-gon.

Proposition 2.3 *The wheel on* k *vertices is an equidistance graph if and only if* $k = 8$.

Proof. We have already observed that a graph of diameter two is an equidistance graph if and only if the Steiner Condition holds. So, in particular, the wheel W_k is an equidistance graph if and only if we have $k(k-1) = 2 \cdot 2(k-1) \cdot 2$. The assertion immediately follows. □

Proposition 2.4 *The complete bipartite graph* $K_{m,n}$ *is an equidistance graph if and only if* $(m, n) = \left(\binom{k}{2}, \binom{k+1}{2}\right)$ *for a suitable integer* $k > 1$.

Proof. Also here, having $diam(K_{m,n}) = 2$, we can say that $K_{m,n}$ is an equidistance graph if and only if Condition (1) holds. In this case this condition gives $(m+n)(m+n-1) = 4mn$, i.e., $m^2 - m(2n+1) + (n^2 - n) = 0$. This equation has positive integer solutions if and only if $8n+1$ is the square of an odd integer, say $8n+1 = (2k+1)^2$. This means $n = \binom{k+1}{2}$ and hence, consequently, $m = \binom{k}{2}$. □

Given an integer $n \geq 3$, the *prism* T_n is the graph corresponding to the "skeleton" of a prism with $2n$ vertices so that T_n has $3n$ edges.

Proposition 2.5 *The prism* T_n *is* i*-equidistance with* $i > 1$ *if and only if* n *is even and* $i = \frac{n}{2}$.

Proof. Note that the prism T_n can be represented as the Cayley graph $T_n = Cay[\mathbb{Z}_n \times \mathbb{Z}_2 : \{(1,0), (-1,0), (0,1)\}]$. It is easy to see (see also Figure 1) that the set $N_{(0,0)}(i)$ of *neighbours* of $(0,0)$ in $T_n(i)$, namely the set of vertices having distance i from $(0,0)$ in T_n, is given by:

$$N_{(0,0)}(i) = \begin{cases} \{(i,0), (-i,0), (i-1,1), (1-i,1)\} & \text{for } 2 \leq i < \frac{n}{2}; \\ \{(\frac{n}{2},0), (\frac{n}{2}-1,1), (1-\frac{n}{2},1)\} & \text{for } n \text{ even and } i = \frac{n}{2}; \\ \{(\frac{n-1}{2},1), (\frac{1-n}{2},1)\} & \text{for } n \text{ odd and } i = \frac{n+1}{2}. \end{cases}$$

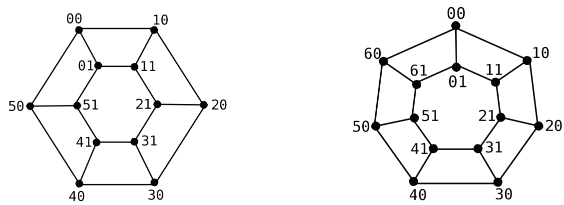


Figure 1: The prisms T_6 and T_7

Now consider that the degree of a vertex (x, y) in $T_n(i)$ does not depend on the particular choice of (x, y) since T_n is sharply vertex transitive. Hence $T_n(i)$ is regular of degree $|N_{(0,0)}(i)|$, namely we have:

$$deg(T_n(i)) = \begin{cases} 4 & \text{for } 2 \le i < \frac{n}{2}; \\ 3 & \text{for } n \text{ even and } i = \frac{n}{2}; \\ 2 & \text{for } n \text{ odd and } i = \frac{n+1}{2}. \end{cases}$$

The assertion immediately follows in view of Remark 2.1 considering that every prism is 3-regular. □

Proposition 2.6 *The only $i > 1$ for which the hypercube Q_t is i-equidistance is $t-1$.*

Proof. Recall that the t-dimensional hypercube, denoted by Q_t, is the Cayley graph $Cay[\mathbb{Z}_2^t : \Omega]$ where Ω is the canonical basis of $\mathbb{Z}_2^t$. From the definition it is clear that two vertices are adjacent in $Q_t(i)$ if and only if they differ in exactly i coordinates. In this way we see that $Q_t(i)$ is regular of degree $\binom{t}{i}$. Hence, by Remark 2.1 and considering that Q_t is regular of degree t, Q_t is i-equidistance if and only if $t = \binom{t}{i}$ which gives $i = 1$ or $i = t - 1$. □

Given $t > 1$ integers $n_1, \ldots, n_t$ greater than 1, the *Hamming graph* $H_{n_1,\ldots,n_t}$ can be seen as the Cayley graph $Cay[\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_t} : \Omega]$ where Ω is the set of all t-uples of $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_t}$ having exactly one non-zero coordinate. Note, in particular, that if $n_1 = \ldots = n_t = 2$, then $H_{n_1,\ldots,n_t}$ is the hypercube Q_t.

Proposition 2.7 *For $i > 1$, a Hamming graph H is i-equidistance if and only if $H = Q_t$ and $i = t - 1$, or $H = H_{3,3}$ and $i = 2$.*

Proof. Let $H = H_{n_1,n_2,\ldots,n_t}$ and assume that $n_1 \ge n_2 \ge \ldots \ge n_t$. We can also assume that $n_1 \ge 3$ since in the opposite case we would have $n_1 = n_2 = \ldots = n_t = 2$ in which case the assertion is true by Proposition 2.6.

From the definition it is easy to see that H has diameter t and that two vertices are adjacent in $H(i)$ if and only if they differ in exactly i coordinates so that $H(i)$ is a regular graph whose degree is given by:

$$deg(H(i)) = \sum_{1 \le k_1 < k_2 < \ldots < k_i \le t} (n_{k_1} - 1)(n_{k_2} - 1) \ldots (n_{k_i} - 1).$$

Then, setting $n_j - 1 = m_j$ for each j, by Remark 2.1 we can say that H is i-equidistance if and only if the following relation holds:

$$\sum_{1\le k_1<k_2<\ldots<k_i\le t} m_{k_1}m_{k_2}\ldots m_{k_i} = m_1+m_2+\ldots+m_t \tag{2}$$

In fact the right-hand side of the above equality represents the degree of H.

For $i>1$, among the summands of the left-hand side of (2) we have all possible products between two distinct m_j's. Thus, if (2) holds for some $i>1$ we would have

$$m_1+m_2+\ldots+m_t \ge m_1m_2+m_1m_3+\ldots+m_1m_t$$

and hence

$$2\le m_1\le \frac{m_2+\ldots+m_t}{m_2+\ldots+m_t-1}$$

which gives $m_2+\ldots+m_t\le 2$. This would be possible only for $t=3$ and $m_2=m_3=1$ or for $t=2$ and $m_2=2$ or, finally, for $t=2$ and $m_2=1$.
If $(t,m_2,m_3)=(3,1,1)$, then condition (2) has no solutions for $i=3$ and the unacceptable solution $m_1=1$ for $i=2$.
Analogously, if $(t,m_2)=(2,1)$ we have no solutions for $i=2$.
Finally, if $(t,m_2)=(2,2)$, then for $i=2$ condition (2) gives $m_1=2$. The assertion immediately follows. □

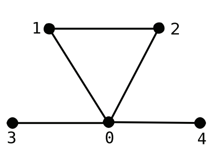
From the above proposition we have that $H_{3,3}$ is an equidistance graph.

3 Some examples of small equidistance graphs

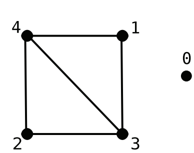
The complete graphs, the odd cycles and the Hamming graph $H_{3,3}$ are the only equidistance graphs that we have found among the classical graphs considered in the previous section.

Here we want to exhibit some less trivial examples. In the following, by speaking of a (v,e,d)-*graph* we mean a graph of order v with e edges and diameter d.

Example 3.1 *The graph* Γ *below is the only example of an equidistance graph of order* 5 *besides* K_5 *and* C_5. *In fact its diameter is* 2 *and* $\Gamma(2)$ *has* 5 *edges as* Γ.

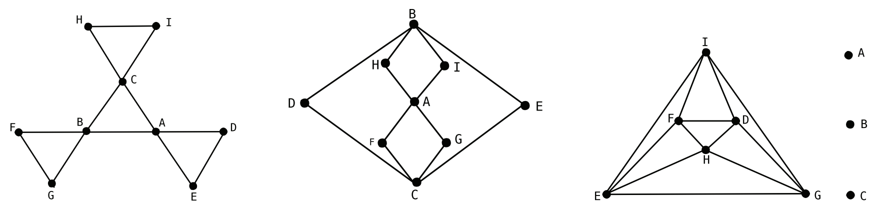


Γ

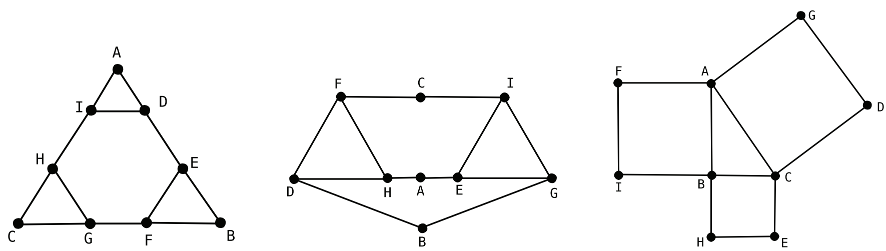


$\Gamma(2)$

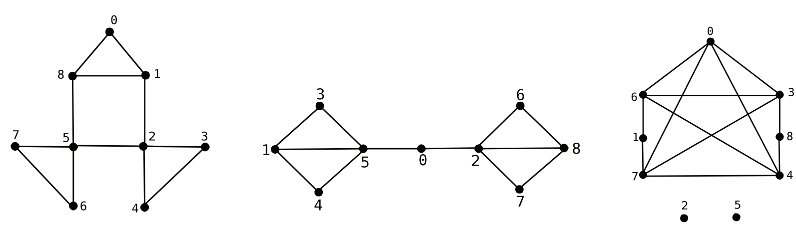
Example 3.2 *Here we have three non-isomorphic equidistance* $(9, 12, 3)$*-graphs.*



Γ $\Gamma(2)$ $\Gamma(3)$



Γ $\Gamma(2)$ $\Gamma(3)$



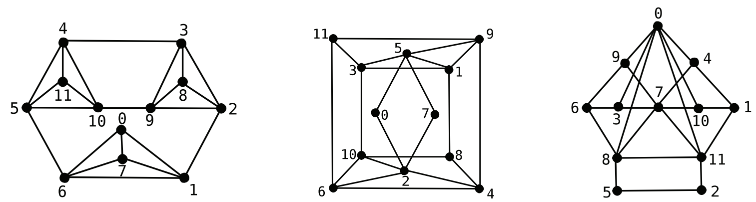
Γ $\Gamma(2)$ $\Gamma(3)$

Example 3.3 *An example of an equidistance* $(12, 22, 3)$*-graph (overleaf).*

Example 3.4 *The graphs of Figure 2 and Figure 3 overleaf are the only cubic equidistance* $(10, 15, 3)$*-graphs.*

Note that the two non isomorphic equidistance $(10, 15, 3)$-graphs given overleaf admit isomorphic distance-3 graphs.

Now, more generally, given a connected graph Γ and a set of positive integers I, let us define the *distance-I* graph of Γ as the graph $\Gamma(I) = \bigcup_{i \in I} \Gamma(i)$. In particular, by writing $\Gamma(m, \ldots, n)$, we mean the distance-I graph of Γ where I is the set of



Γ   Γ(2)   Γ(3)

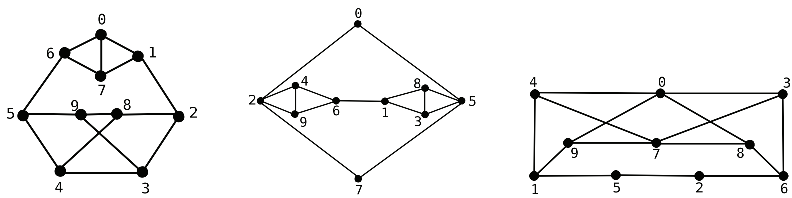


Figure 2: An equidistance $(10, 15, 3)$-graph and its associated distance-i graphs

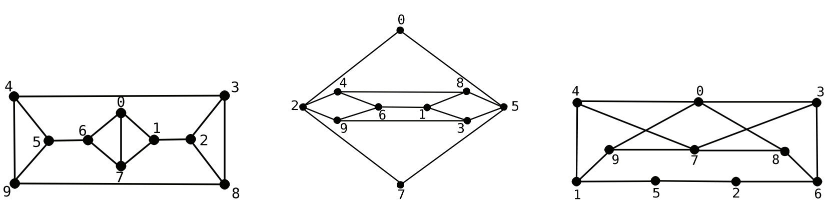


Figure 3: Another equidistance $(10, 15, 3)$-graph and its associated distance-i graphs

consecutive integers $\{m, m+1, \ldots, n\}$. In other words, $\Gamma(m, \ldots, n)$ is the graph with vertex-set $V(\Gamma)$ and where $[x, y]$ is an edge if and only if $m \leq d_\Gamma(x, y) \leq n$ where $d_\Gamma(x, y)$ denotes the distance between x and y in the graph Γ.

Theorem 3.5 *Given any graph* Γ *and any positive integer* $r \leq diam(\Gamma)$*, we have* $\Gamma(1, \ldots, r)(i) = \Gamma((i-1)r+1, \ldots, ir)$*.*

Proof. Take two distinct vertices a, b of Γ and let j be their distance in $\Gamma(1, \ldots, r)$. We have $d_\Gamma(a, b) = (i-1)r + s$ for suitable positive integers i, s with $s \leq r$. The assertion will be proved if we show that $j = i$.

Let $(a = x_0, x_1, \ldots, x_j = b)$ be a *geodesic* (i.e., a path of minimun length) between a and b in $\Gamma(1, \ldots, r)$. By definition of $\Gamma(1, \ldots, r)$ we have $d_\Gamma(x_{h-1}, x_h) \leq r$ for

$h = 1, \dots, j$. Thus, using the *Schwartz inequality* we have

$$jr \geq \sum_{h=1}^{j} d_\Gamma(x_{h-1}, x_h) \geq d_\Gamma(a, b) = (i-1)r + s > (i-1)r$$

which implies $j \geq i$.
On the other hand, if $(a = y_0, y_1, \dots, y_{(i-1)r+s} = b)$ is a geodesic in Γ from a to b, we have $d_\Gamma(y_h, y_k) = k - h$ for $0 \leq h \leq k \leq (i-1)r + s$ so that we can say that $P = (y_0, y_r, y_{2r}, \dots, y_{(i-1)r}, y_{(i-1)r+s})$ is a path of $\Gamma(1, \dots, r)$ from a to b. Since P has length i we have $j \leq i$ and we can conclude that $j = i$. □

Corollary 3.6 *The diameter of* $\Gamma(1, \dots, r)$ *is given by* $\left\lceil \frac{diam(\Gamma)}{r} \right\rceil$.

Proof. The diameter of $\Gamma(1, \dots, r)$ is the maximum i for which $\Gamma(1, \dots, r)(i)$ is not empty. Hence, by Theorem 3.5, it is the maximum i for which $(i-1)r+1 \leq diam(\Gamma)$. The assertion easily follows. □

Corollary 3.7 *Let* Γ *be an equidistance graph of diameter* rd. *Then* $\Gamma(1, \dots, r)$ *is an equidistance graph of diameter* d.

Proof. The diameter of $\Gamma(1, \dots, r)$ is d by Corollary 3.6. Now, by Theorem 3.5, for $i = 1, \dots, d$ we have

$$\Gamma(1, \dots, r)(i) = \bigcup_{h=1}^{r} \Gamma((i-1)r + h).$$

Thus, since Γ is an equidistance graph, we have $|E(\Gamma(1, \dots, r)(i))| = r|E(\Gamma)|$ for any i. The assertion follows. □

The last corollary allows us to construct a $2r$-regular equidistance graph of diameter d for any pair of positive integers (r, d).

Theorem 3.8 *Given any pair of positive integers* (r, d), *the graph* $C_{2rd+1}(1, \dots, r)$ *is a* $2r$*-regular equidistance graph of diameter* d.

Proof. Considering that any odd cycle is equidistance, it suffices to apply Corollary 3.7 with $\Gamma = C_{2rd+1}$. □

Note that $C_{2rd+1}(1, \dots, r)$ can be viewed as the Cayley graph $Cay[\mathbb{Z}_{2rd+1} : \{\pm 1, \dots, \pm r\}]$.

Example 3.9 *In* Figure 4 *we have* $C_9(1, 2) \simeq Cay[\mathbb{Z}_9 : \{\pm 1, \pm 2\}]$ *and its distance*-2 *graph* $C_9(3, 4) \simeq Cay[\mathbb{Z}_9 : \{\pm 3, \pm 4\}]$.

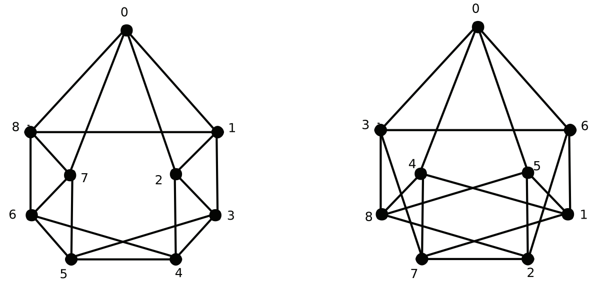


Figure 4: $C_9(1,2)$ and its distance-2 graph.

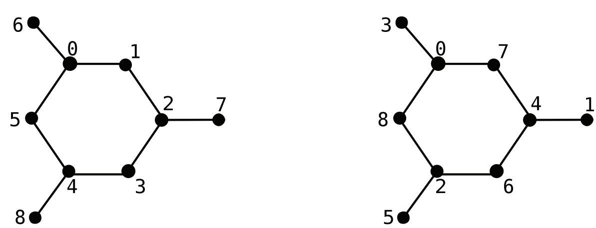
4 Isodistance graphs

Considering equidistance graphs naturally leads one to consider a more particular class of graphs that we call *isodistance graphs*. Given a graph Γ and a positive integer i not exceeding its diameter we say that Γ is an *i-isodistance graph* if Γ is isomorphic to its distance-i graph.

In the special case that all distance-i graphs associated with a graph Γ are isomorphic to Γ itself, we say that Γ is an *isodistance graph*. It is obvious that every isodistance graph is, in particular, an equidistance graph but the converse is not true in general. Observations made in the proof of Proposition 2.2 allows us to state:

Proposition 4.1 *Every p-cycle with p a prime is an isodistance graph.*

Example 4.2 *The graph Γ below is an example of a 3-isodistance graph.*



Γ $\qquad\qquad$ $\Gamma(3)$

We recall that the *complement* $\overline{\Gamma}$ of a graph Γ has the same vertex-set as Γ and that two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ. A *self-complementary graph* is a graph Γ isomorphic to its complement. In this case it is clear that Γ and its complement form a (K_v,Γ)-design where v is the order of Γ. We also recall that a self-complementary graph has order $\equiv 0$ or 1 (mod 4) and that its diameter is 2 or 3 (see, e.g., [9]).

Proposition 4.3 *The isodistance graphs of diameter two are precisely the self-complementary graphs of diameter two.*

Proof. It is sufficient to observe that $diam(\Gamma) = 2$ if and only if $\overline{\Gamma} = \Gamma(2)$. □

We point out that there are equidistance graphs of diameter two that are not self-complementary as, for instance, the graph Γ of Example 3.1. It is easy to see that an infinite class of graphs with this property is given by the Cayley graphs of the form $Cay[\mathbb{Z}_{4n+1} : \{\pm 1, \pm 2, \ldots, \pm n\}]$ with $n \geq 2$.

Now we present two infinite classes of self-complementary graphs of diameter two and hence of isodistance graphs. The graphs belonging to the first class have order divisible by 4 while the graphs of the second class have order a prime power $q \equiv 1 \pmod 4$.

For every $n > 1$, let Γ_n be the graph with vertex-set $\mathbb{Z}_{4n}$ and edge-set $E(\Gamma_n) = E_0 \cup E_1 \cup E_2$ where E_0, E_1, E_2 are defined by

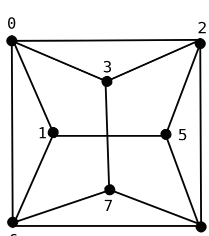
$$[x, y] \in E_0 \Longleftrightarrow x \equiv y \equiv 0 \pmod 2 \text{ and } x - y \in \left\{\pm 2i \mid 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\right\}$$

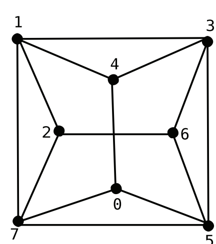
$$[x, y] \in E_1 \Longleftrightarrow x \equiv y \equiv 1 \pmod 2 \text{ and } x - y \in \left\{\pm 2i \mid \left\lfloor \frac{n}{2} \right\rfloor < i \leq n\right\}$$

$$E_2 = \{[2z, 2z + 2i + 1] \mid 0 \leq z \leq 2n - 1; 0 \leq i \leq n - 1\}.$$

It is a quite easy matter to see that $diam(\Gamma_n) = 2$ and that the map $f : x \in \mathbb{Z}_{4n} \longrightarrow x + 1 \in \mathbb{Z}_{4n}$ is an isomorphism between Γ_n and its complement.

Example 4.4 *Here are Γ_2 and its complement.*





Definition 4.5 *Given a prime power $q = 4t + 1$, the* Paley graph *of order q is the Cayley graph $\mathbb{P}_q = Cay[\mathbb{F}_q : \mathbb{F}_q^{\square}]$, where $\mathbb{F}_q$ and $\mathbb{F}_q^{\square}$ are the additive group of the finite field of order q and the set of its non-zero squares, respectively.*

If x and 0 are not adjacent in $\mathbb{P}_q$, then, by definition of a Cayley graph, x is a non-square. In this case note that 0 and x have at least one common neighbour that is x^2 if $x-1$ is a non-square and $x-1$ itself in the opposite case. Hence, two is the maximum distance of a vertex of $\mathbb{P}_q$ from 0. This, taking into account that $\mathbb{P}_q$ is sharply vertex transitive, implies that $diam(\mathbb{P}_q) = 2$.

Of course the complement of $\mathbb{P}_q$ is $\overline{\mathbb{P}_q} = Cay[\mathbb{F}_q : \mathbb{F}_q^{\boxslash}]$, where $\mathbb{F}_q^{\boxslash}$ is the set of non-squares of $\mathbb{F}_q$. Also, for fixed $y \in \mathbb{F}_q^{\boxslash}$, it is obvious that the map $x \in \mathbb{F}_q \longrightarrow xy \in \mathbb{F}_q$ is an isomorphism between $\mathbb{P}_q$ and its complement. This observation allow us to state:

Proposition 4.6 *For every prime power $q = 4t+1$, the Paley graph $\mathbb{P}_q$ is a self-complementary graph of diameter 2 and hence an isodistance graph.*

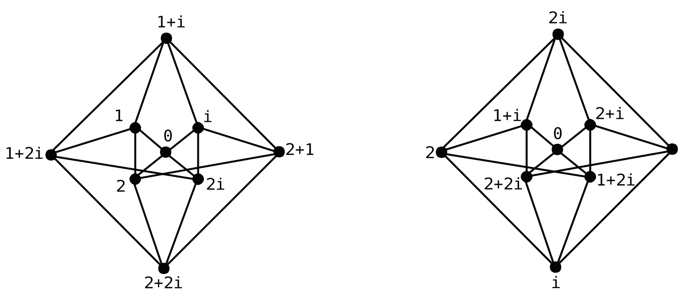


Figure 5: $\mathbb{P}_9$ and its complement. Here the elements of $\mathbb{F}_9$ are represented in the form $a+ib$ with $a, b \in \mathbb{Z}_3$ and $i^2 = 2$

Note that the Hamming graph $H_{3,3}$ is isomorphic to $\mathbb{P}_9$.

Open questions. An interesting question on equidistance graphs naturally arises from Theorem 3.8: "Do there exist $(2r+1)$-regular equidistance graphs of diameter d for any pair of positive integers (r, d)?"

At the moment the more crucial question on isodistance graphs is: "Do there exist isodistance graphs of diameter $d > 2$ which are not cycles?"

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