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Results for the n-queens problem on the Möbius board

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Abstract

In this paper we consider the extension of the n-queens problem to the Möbius strip; that is, the problem of placing a maximum number of nonattacking queens on the $m \times n$ chessboard for which the left and right edges are twisted connected. We prove the existence of solutions for the $m \times n$ Möbius board for classes of m and n with density 25/48 in the set of all $m \times n$ Möbius boards, and show the impossibility of solutions for a set of m and n with density 1/16. We also have computed the total number of solutions for the $m \times m$ Möbius board for m from 1 to 16.

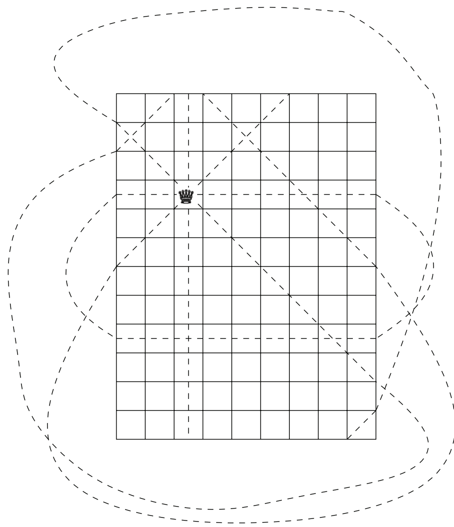
1 Introduction and definitions

The standard n-queens problem is to place n nonattacking queens on the $n \times n$ chessboard. It is well known that there exists a standard n-queens solution for all $n > 3$ [2, 3]. The n-queens problem has been generalized to the $m \times n$ board and higher dimensions, and also extended to the chessboard on the torus, i.e. the modular board; it is well known that modular n-queens solutions exist if and only if $\gcd(n, 6) = 1$ [4]. We give in [1] an exhaustive survey of the n-queens problem and generalizations and extensions of it.

In this paper we consider the extension of the n-queens problem to the Möbius strip. We impose an $m \times n$ chessboard on the Möbius strip to obtain the $m \times n$ *Möbius board*, for which the left and right edges are twisted connected.

Let us number the rows of an $m \times n$ board from 0 at the top to $m - 1$ at the bottom and the columns from 0 at the left to $n - 1$ at the right. We refer to the square on row i and column j by (i, j). A sum diagonal is a diagonal such that $i + j = c$ for all squares (i, j) on it for a certain c, and a difference diagonal is a diagonal such that $i - j = c$ for all squares (i, j) on it for a certain c.

Figure 1: A queen's attacks on the 12×9 Möbius board



It will be helpful to look at some examples of how a queen moves on the Möbius board. In Figure 1, we show the attacks of a queen in row 3 and column 2 on the 12×9 Möbius board, and in Figure 2 we show another view of the attacks of a queen in the same position on the 12×9 Möbius board. In Figure 3 we show the attacks of a queen in row 0 and column 0 on the 11×2 Möbius board. In Figure 4 we show the attacks of a queen in row 0 and column 1 on the 3×11 Möbius board. Finally in Figure 5 we show the attacks of a queen in row 0 and column 0 on the 6×2 Möbius board.

Since each queen on the $m \times n$ Möbius board attacks two distinct rows and one column, except a queen on the middle row for m odd which attacks one row and one column, it is clear that an upper bound to the maximum number of nonattacking queens that could be placed on the $m \times n$ Möbius board is $\min\{\lceil m/2 \rceil, n\}$. Thus we make the following definition.

Definition 1. *A placement of* $\min\{\lceil m/2 \rceil, n\}$ *nonattacking queens on the* $m \times n$ *Möbius board is defined to be a* solution for the $m \times n$ Möbius board.

This is equal to $\lceil m/2 \rceil$ when $m < 2n$, and n when $m \geq 2n$.

In the next section we give the total number $M(m)$ of solutions for the $m \times m$ Möbius board for m from 1 to 16. Generally, we prove conditions that guarantee the existence of solutions for the $m \times n$ Möbius board for several large classes of m and n, with density at least 25/48 in the set of all $m \times n$ Möbius boards. We show that there is no solution for $m \geq 2n$ and $m \equiv n \equiv 0 \pmod 2$ or $m = 2n - 1$, $n \equiv 0 \pmod 2$, a set of density 1/16. We also show how to construct solutions for the

Figure 2: A different view of a queen's attacks on the 12×9 Möbius board

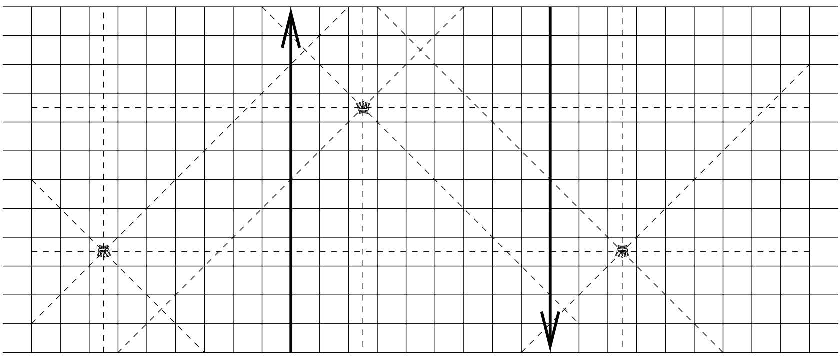


Figure 3: A queen's attacks on the 11×2 Möbius board

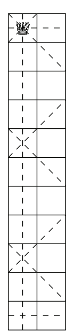


Figure 4: A queen's attacks on the 3×11 Möbius board

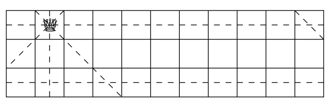


Figure 5: A queen's attacks on the 6×2 Möbius board

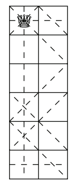


Table 1: Number of solutions $M(m)$ for $m \times m$ Möbius board for $m = 1, \dots, 16$

m	$M(m)$
1	1
2	4
3	0
4	16
5	40
6	192
7	560
8	3328
9	11,772
10	63,840
11	259,336
12	1,550,976
13	7,169,656
14	42,410,256
15	234,044,160
16	1,366,190,592

Möbius board using standard n-queens solutions. We conclude with open problems for the n-queens problem on the Möbius board.

For $\mathbb{N}$ the positive integers, $S \subseteq \mathbb{N} \times \mathbb{N}$ and $x \geq 1$, for convenience let us write $S_x = \{(m, n) \in S | m, n \leq x\}$. The set S has *density* α if

$$\lim_{x \to \infty} \frac{|S_x|}{x^2} = \alpha.$$

2 Results

We give in Table 1 the total number of solutions $M(m)$ that exist for the $m \times m$ Möbius board for $m = 1, \dots, 16$. These were calculated with a straight-forward backtracking algorithm and appear as sequence A137279 in the OEIS [5].

Theorem 2. *For all m and n such that $m = 2k \leq n$ and $n > 3$, $\lceil m/2 \rceil = k$ nonattacking queens can be placed on the $m \times n$ Möbius board.*

Proof. Let us place queens on the squares $\{(2j, j) \mid j = 0, \dots, \frac{m}{2} - 1\}$. It is clear that there are no column attacks between these $m/2$ queens. Each queen is placed on a distinct row, so there are no attacks on the original rows of the queens. Since a row i carries over to a row $m - i - 1$, for a queen $(2j, j)$ to carry over to another row and attack a distinct queen $(2l, l)$ would imply $m - 2j - 1 = 2l$, so $2k - 2j - 2l = 1$ and $2(k - j - l) = 1$, a contradiction. Suppose that a queen $(2j, j)$ attacked another queen $(2l, l)$ along the sum diagonal it is on. Then $2j + j = 2l + l$ so $j = l$, a contradiction. If its sum diagonal hits the left edge of the board (excluding the last row), then it will carry over to the difference diagonal $-(2j + j) - 1 - (n - m)$; if the queen on $(2j, j)$ attacks a queen on $(2l, l)$ with such a carry-over diagonal attack then $2l - l = -(2j + j) - 1 - (n - m)$, hence $l = -3j - 1 - (n - m) < 0$ since $m \leq n$, rendering an attack a contradiction. If its sum diagonal hits the right edge of the board (excluding the top row), then it will carry over to the difference diagonal $2l - l = 2m - (2j + j) - 1 + (n - m)$, thus $l = 2m - 3j - 1 + n - m$ and $3j + l = m + n - 1$. The maximum value of the left hand side of this equation is given by $j = \frac{m}{2} - 1$ and $l = \frac{m}{2} - 2$, giving $3j + l \leq 2m - 5 < m + n - 1$, showing that no attack is possible. Suppose that a queen $(2j, j)$ attacked another queen $(2l, l)$ along the difference diagonal it is on. Then $2j - j = 2l - l$ and $j = l$, a contradiction. Thus for $m = 2k \leq n$, $\{(2j, j) \mid j = 0, \dots, \frac{m}{2} - 1\}$ places $m/2 = \lceil m/2 \rceil$ nonattacking queens on the $m \times n$ Möbius board. □

Here let $S = \{(m, n) \mid m \equiv 0 \pmod 2, m \leq n, n > 3\}$, the set of (m, n) covered by the above solution. Then

$$\begin{aligned}
|S_x| &= -1 + \sum_{j=1}^{\lfloor \frac{x}{2} \rfloor} (x - 2j) \\
&= -1 + x \left\lfloor \frac{x}{2} \right\rfloor - 2 \sum_{j=1}^{\lfloor \frac{x}{2} \rfloor} j \\
&= -1 + x \left\lfloor \frac{x}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor \left(\left\lfloor \frac{x}{2} \right\rfloor + 1 \right) \\
&= \frac{x^2}{4} + O(x).
\end{aligned}$$

It follows that $\lim_{x \to \infty} \frac{|S_x|}{x^2} = \frac{1}{4}$. Thus the density of the (m, n) covered by Theorem 2 is $1/4$.

Theorem 3. *For all m and n such that $m = 2k + 1 \leq n$ and $n > 3$, $\lceil m/2 \rceil = k + 1$ nonattacking queens can be placed on the $m \times n$ Möbius board.*

Proof. We will divide this into two cases of m modulo 4. We first consider $m \equiv 1 \pmod 4$. We place the first $\frac{m-1}{4}$ queens on the squares $A = \{(2j, j) \mid j = 0, \dots, \frac{m-1}{4} -$

$1\}$ and the next $\frac{m-1}{4}$ queens on the squares $B = \{(\frac{m-1}{2} + 1 + 2j, \frac{m-1}{4} + j) \mid j = 0, \ldots, \frac{m-1}{4} - 1\}$, leaving one queen unplaced. Let us denote $C = \{(\frac{m-1}{2}, j) \mid j = \frac{m+1}{2}, \ldots, n-1\}$. Clearly there are no column attacks, and no queen in A or B attacks a column in C. None of the queens share a row. Recall that a row i carries over to a row $m-i-1$. Suppose a queen $(2j, j)$ in A carried over to another row to hit a queen $(\frac{m-1}{2} + 1 + 2l, \frac{m-1}{4} + l)$ in B. Then $m - 2j - 1 = \frac{m-1}{2} + 1 + 2l$, so $2k+1-2j-1 = \frac{m-1}{2}+1+2l$ and $2k-2j-2l = \frac{m-1}{2}+1$, for which the left hand side is even and the right hand side is odd, since $m \equiv 1 \pmod 4$, a contradiction. Also, none of the queens are on the row $\frac{m-1}{2}$ of C or carry over to that row. Clearly within A and B there are no sum diagonal attacks. Consider a queen $(2j, j)$ in A. If its sum diagonal hits the left edge of the board (excluding the bottom row), it will carry over to the difference diagonal $-(2j+j)-1-(n-m) = -3j-1-(n-m)$ that hits the right edge of the board (and not at the bottom row); however, none of the queens are on difference diagonals that hit the right edge of the board except possibly for the queen at $(0, 0)$, but it would hit the right edge of the board at the bottom row. This carry over attack will intersect C at column l for $\frac{m-1}{2} - l = -3j - 1 - (n-m)$, thus $l = -\frac{m+1}{2} + n + 1 + 3j$, hence these carry over attacks will hit exactly every third square in C starting at column $-\frac{m+1}{2} + n + 1$. None of the queens in A are on sum diagonals that hit the right side of the board. Now consider a queen $(\frac{m-1}{2} + 1 + 2j, \frac{m-1}{4} + j)$ in B. If it is on a sum diagonal that hits the left side of the board, recall as for above that no queens are placed on the difference diagonal that it carries over onto. Also, the difference diagonal that it carries over onto will not intersect C. If it is on a sum diagonal that hits the right side of the board (aside from at the top row), it will intersect C along this very sum diagonal at column l for $\frac{m-1}{2} + l = \frac{m-1}{2} + 1 + 2j + \frac{m-1}{4} + j$ and $l = \frac{m-1}{4} + 1 + 3j$; yet C only includes columns after and including column $\frac{m+1}{2}$, so the queens in B on sum diagonals that hit the right edge of the board will hit C exactly at every third square starting at column $\frac{m+1}{2} + 1$. These queens will carry over on difference diagonals from the left side of the board that do not intersect C, since they will intersect row $\frac{m-1}{2}$ at columns before C. Hence the squares in C with column numbers incongruent to both $\frac{m+1}{2}+1$ and $-\frac{m+1}{2} + n + 1$ modulo 3 will be unattacked. In any case, there will be at least one square in C that is unattacked since C always has at least three squares except possibly when $m = 5$, but for $m = 5$ the sum diagonal of the queen in B does not hit C, and therefore we can place the last queen there, giving us $\frac{m+1}{2} = \lceil m/2 \rceil$ nonattacking queens on the $m \times n$ Möbius board for $m = 2k + 1 \equiv 1 \pmod 4$.

We now consider $m \equiv 3 \pmod 4$. Let us place the first $\frac{m+1}{4}$ queens on the squares $A = \{(2j, j) \mid j = 0, \ldots, \frac{m+1}{4} - 1\}$, and the next $\frac{m-3}{4}$ queens on the squares $B = \{(\frac{m-1}{2} + 2 + 2j, \frac{m+1}{4} + j) \mid j = 0, \ldots, \frac{m-3}{4} - 1\}$, leaving one queen unplaced. Let us denote $C = \{(\frac{m-1}{2}, j) \mid j = \frac{m+1}{2}, \ldots, n-1\}$. Clearly there are no column attacks, and none of the squares in C have queens in their columns. There are no row attacks within A and B, and as in the case of $m \equiv 1 \pmod 4$, the rows of the queens in A and the queens in B are of different parity, showing that there can be no carry over row attacks. As well, row $\frac{m-1}{2}$ of C is not attacked directly or indirectly, as it carries over to itself. Clearly there are no sum diagonal attacks. Similarly to the previous case, the squares in C with column numbers incongruent to $-\frac{m-1}{2}+n+1$ and $\frac{m+1}{4}+2$ modulo 3 will be unattacked. Indeed in any case, for $m \geq 7$, there will be at least three squares in C, and thus at least one unattacked square in which we can put the last queen. In the case of $m=3$, when $n=3$, there are no placements of two nonattacking queens on the Möbius board (cf. Table 1). When $m=3$ and $n>3$, C will have at least three squares, and thus at least one unattacked square in which we can put the last queen. This gives us $\frac{m+1}{2}=\lceil m/2 \rceil$ nonattacking queens on the $m \times n$ Möbius board for $m=2k+1 \equiv 3 \pmod 4$ for $n>3$. This completes the proof. □

The density of the (m,n) covered by Theorem 3 is $1/4$.

On the $m \times n$ Möbius board, we find that a queen on the cell (i,j), on the sum diagonal $s=i+j$ and the difference diagonal $d=i-j$, will attack all the sum diagonals $s+2\alpha n$ and all the difference diagonals $-s-1+m-n+2\alpha n$, for α any integer (because of attacks successively carrying over from the original sum diagonal), and will attack all the sum diagonals $-d+n-1+m+2\alpha n$ and all the difference diagonals $d+2\alpha n$, for α any integer (because of attacks successively carrying over from the original difference diagonal). We summarize this in the following remark.

Remark 4. *On the $m \times n$ Möbius board, a queen in the cell (i,j) will attack the sum diagonals*

- $i+j+2\alpha n$
- $-i+j+n-1+m+2\alpha n$

and the difference diagonals

- $i-j+2\alpha n$
- $-i-j-1+m-n+2\alpha n$

for all integers α.

Theorem 5. *For all m and n such that $m \geq 2n^2$ and $m \not\equiv n \pmod 2$, n nonattacking queens can be placed on the $m \times n$ Möbius board.*

Proof. Let $0 \leq \lambda < 2n$ be the remainder of $\frac{m-n-1}{2}$ after division by $2n$. Since $m \geq 2n^2$, we can put queens on the n rows $\lambda, \lambda+2n, \ldots, \lambda+(n-1)2n$, in the cells $\{(\lambda+2nj, j) \mid j=0,1,\ldots,n-1\}$. All the queens are in distinct columns, so there are no column attacks. Moreover, all the queens are on distinct rows, so if there were row attacks they would have to be carry over row attacks. But, say a queen $(\lambda+2nj, j)$ had a carry over row attack to a distinct queen $(\lambda+2nl, l)$: thus $m-1-\lambda-2nj=\lambda+2nl$, implying $n \equiv 0 \pmod{2n}$, a contradiction. Hence there are no (carry over) row attacks between queens.

We have that an arbitrary queen $(\lambda+2nj, j)$ attacks the two sets of sum diagonals $\{\lambda+2nj+j+2\alpha n\}$ and $\{-\lambda+2nj+j+n-1+m+2\alpha n\}$, for α any integer. But

$\lambda \equiv \frac{m-n-1}{2} \pmod{2n}$, and thus we find that these are the same sets of sum diagonals, hence this queen attacks exactly the sum diagonals $\{\frac{m-n-1}{2}+2nj+j+2\alpha n\}$. Similarly, this queen attacks exactly the difference diagonals $\{\frac{m-n-1}{2}+2nj-j+2\alpha n\}$. Say that two distinct queens $(\lambda+2nj, j)$ and $(\lambda+2nl, l)$ had a sum diagonal or carry over sum diagonal attack: then for some α and β, $\frac{m-n-1}{2}+2nj+j+2\alpha n = \frac{m-n-1}{2}+2nl+l+2\beta n$, i.e. $\frac{m-n-1}{2} + j \equiv \frac{m-n-1}{2} + l \pmod{2n}$, so $j \equiv l \pmod{2n}$, a contradiction. Thus there are no attacks on the original sum diagonal or any carry over sum diagonals. Similarly, there are no attacks on the original difference diagonal or any carry over difference diagonals. Hence there are no attacks between any of these n queens, so they are a placement of n nonattacking queens on the $m \times n$ Möbius board. □

The density of the (m, n) covered by Theorem 5 is 0.

Lemma 6. *For all m and n with m such that $m \equiv 1 \pmod{2}$, on the $m \times n$ Möbius board a queen in the cell $(\frac{m-1}{2}, j)$ attacks precisely the same sum and difference diagonals as a queen in the cell $(\frac{m-1}{2} + \alpha n, j)$, for any integer α.*

Proof. The sum diagonals attacked by a queen in the cell $(\frac{m-1}{2}, j)$ are $\{\frac{m-1}{2}+j+2\beta n\}$ and $\{-\frac{m-1}{2} + j + n - 1 + m + 2\beta n\}$, for β any integer. But the sum diagonals attacked by a queen in the cell $(\frac{m-1}{2} + \alpha n, j)$ for α a fixed arbitrary integer are $\{\frac{m-1}{2} + \alpha n + j + 2\beta n\}$ and $\{-\frac{m-1}{2} - \alpha n + j + n - 1 + m + 2\beta n\}$, for all integers β. If α is even these are precisely the same sets of diagonals. If α is odd these are the same but in reverse order. Similarly, a queen in the cell $(\frac{m-1}{2}, j)$ attacks the same difference diagonals as a queen in the cell $(\frac{m-1}{2} + \alpha n, j)$ for any integer α. □

Theorem 7. *For all m and n such that $m \geq 2n^2$ and $m \equiv n \equiv 1 \pmod{2}$, n nonattacking queens can be placed on the $m \times n$ Möbius board.*

Proof. Let $0 \leq \lambda < 2n$ be the remainder of $\frac{m-1}{2}$ after division by $2n$. Since $m \geq 2n^2$, we can put queens on the n rows $\lambda+\epsilon_0 n, \lambda+2n+\epsilon_1 n, \ldots, \lambda+(n-1)2n+\epsilon_{n-1} n$, in the cells $\{(\lambda+2nj+\epsilon_j n, j) \mid j = 0, 1, \ldots, n-1\}$, for $\epsilon_j \in \{0, 1\}$; we will choose the ϵ_j later. All the queens are in distinct columns so there are no column attacks. Clearly all the queens will be in distinct rows, so if there were row attacks they would have to be carry over row attacks. Suppose a queen $(\lambda+2nj+\epsilon_j n, j)$ had a carry over row attack to a distinct queen $(\lambda+2nl+\epsilon_l n, l)$: thus $m-1-\lambda-2nj-\epsilon_j n = \lambda+2nl+\epsilon_l n$, which implies $n(\epsilon_j + \epsilon_l) \equiv 0 \pmod{2n}$. Now, since each queen can have at most one carry over row attack to another queen, we can choose the ϵ_j and ϵ_l such that $\epsilon_j + \epsilon_l \equiv 1 \pmod{2}$, preventing these row attacks.

Suppose that a queen $(\lambda+2nj+\epsilon_j n, j)$ attacked another queen $(\lambda+2nl+\epsilon_l n, l)$ on a sum diagonal. By Lemma 6 this is equivalent to the queen $(\lambda+2nj, j)$ attacking the queen $(\lambda+2nl, l)$ on a sum diagonal. The case of $\lambda+2nj+j+2\alpha n = \lambda+2nl+l+2\beta n$ implies $j \equiv l \pmod{2n}$, a contradiction. The case of $-\lambda - 2nj + j + n - 1 + m + 2\alpha n = \lambda + 2nl + l + 2\beta n$ implies $j \equiv n + l \pmod{2n}$, and since $j \not\equiv l \pmod{n}$ this is a contradiction. Hence there are no sum diagonal attacks between any queens. Similarly, there are no difference diagonal attacks between any queens. □

The density of the (m,n) covered by Theorem 7 is 0.

If we are a bit more careful we can increase the range of permissible n, in certain cases.

Theorem 8. *For all m,n and k such that*

- *m,n and k are odd;*
- *$m \geq 4n$;*
- *$\gcd(k,2n)=1$; and*
- *$\gcd(\frac{k+1}{2},n)=\gcd(\frac{k-1}{2},n)=1$*

then the placement

$$\{(kj \pmod{2n}, j) \mid j=0,1,\ldots,n-1\}$$

is a solution for the $m\times n$ Möbius board.

Proof. Suppose that a queen $(kj \pmod{2n}, j)$ attacked a distinct queen $(kl \pmod{2n}, l)$ on its original row. Then $kj \pmod{2n} = kl \pmod{2n}$, hence $kj \equiv kl \pmod{2n}$, thus $j \equiv l \pmod{2n}$, contradicting that $j \not\equiv l \pmod{n}$. Because $m \geq 4n$, there can be no carry over row attacks (since all the queens are placed on the top half of the board).

Suppose that a queen $(kj \pmod{2n}, j)$ attacked a distinct queen $(kl \pmod{2n}, l)$ on a sum diagonal $(kj \pmod{2n}) + j + 2\alpha n$. Then $kj + j \equiv kl + l \pmod{2n}$ and $j(k+1) \equiv l(k+1) \pmod{2n}$, thus $j(\frac{k+1}{2}) \equiv l(\frac{k+1}{2}) \pmod{n}$, and since $\gcd(\frac{k+1}{2},n) = 1$, then $j \equiv l \pmod{n}$, a contradiction.

Suppose that a queen $(kj \pmod{2n}, j)$ attacked a distinct queen $(kl \pmod{2n}, l)$ on a sum diagonal $-(kj \pmod{2n})+j+n-1+m+2\alpha n$. Then $-kj+j+n-1+m = kl+l+2\beta n$ for some integer β, and so $(k+1)l+(k-1)j-m+1 = (2\beta-1)n$. But the left-hand of this equation is even while the right-hand side of this equation is odd, a contradiction.

Similarly there are no attacks along any difference diagonals, so this is a solution for the $m\times n$ Möbius board. □

In particular if $3 \nmid n$ then $k=3$ works. When n is divisible by 3, since at least one of $k-1$, k or $k+1$ will also be divisible by 3, we will not be able to use this theorem.

Let $S = \{(m,n) \mid m,n \equiv 1 \pmod 2, m \geq 4n, 3 \nmid n\}$. First we note that

$$\begin{aligned}\sum_{\substack{m\leq x,\\ m\equiv 1 \pmod 2}} m &= \sum_{j=0}^{\lceil \frac{x}{2}\rceil -1} (2j+1)\\ &= \left\lceil \frac{x}{2}\right\rceil - 1 + \left\lceil \frac{x}{2}\right\rceil \Big(\left\lceil \frac{x}{2}\right\rceil - 1\Big)\\ &= \frac{x^2}{4} + O(x).\end{aligned}$$

Then,

$$\begin{aligned}|S_x| &= \sum_{\substack{m\le x,\\ m\equiv 1 \pmod 2}} \sum_{\substack{n\le \frac{m}{4},\\ n\equiv 1,5 \pmod 6}} 1\\ &= \sum_{\substack{m\le x,\\ m\equiv 1 \pmod 2}} \left(\frac{m}{12}+O(1)\right)\\ &= \frac{x^2}{48}+O(x).\end{aligned}$$

Therefore the density of the (m,n) covered by Theorem 8 is $\frac{1}{48}$.

In summary, Theorems 2, 3 and 8 yield solutions for $m\times n$ Möbius boards with density $\frac{1}{4}+\frac{1}{4}+\frac{1}{48}=\frac{25}{48}$. In the following theorem we find a set of $m\times n$ Möbius boards for which there do not exist solutions.

Theorem 9. *When $m\ge 2n$ and $m\equiv n\equiv 0 \pmod 2$ then it is impossible to place n nonattacking queens on the $m\times n$ Möbius board.*

Proof. We suppose that we can place n nonattacking queens on the $m\times n$ Möbius board, and we will derive a contradiction. We say that two sum diagonals c and c' are distinct when $c\not\equiv c' \pmod{2n}$. Since $m\equiv n\equiv 0 \pmod 2$, Remark 4 shows us that there are exactly $2n$ distinct sum diagonals. If there are n queens placed on the $m\times n$ Möbius board then the pairs of sum diagonals attacked by them must form a complete residue system modulo $2n$. Thus the sum of the sum diagonals over all queens is equal to

$$\sum_{i=0}^{2n-1} i=\frac{(2n-1)2n}{2}\equiv n \pmod{2n}.$$

But if the queens are placed in cells (i_s,s) for $0\le s<n$ then the sum of the sum diagonals over all queens is

$$\begin{aligned}\sum_{s=0}^{n-1}((i_s+s)+(-i_s+s+n-1+m)) &\equiv 2\left(\sum_{s=0}^{n-1} s\right)+n^2-n+mn \pmod{2n}\\ &\equiv n^2-n+n^2-n+mn \pmod{2n}\\ &\equiv 0 \pmod{2n},\end{aligned}$$

because m is even. Thus a contradiction is reached. ◻

Let $S = \{(m,n) | m, n \equiv 0 \pmod 2, m \geq 2n\}$. Then

$$
\begin{aligned}
|S_x| &= \sum_{j=1}^{\lfloor \frac{x}{2} \rfloor} \left\lfloor \frac{1}{2} j \right\rfloor \\
&= \frac{1}{2} \sum_{j=1}^{\lfloor \frac{x}{2} \rfloor} j + O(x) \\
&= \frac{1}{4} \left\lfloor \frac{x}{2} \right\rfloor \left(\left\lfloor \frac{x}{2} \right\rfloor + 1 \right) + O(x) \\
&= \frac{x^2}{16} + O(x).
\end{aligned}
$$

Therefore the above theorem shows that a set of boards with density 1/16 do not have solutions.

This means that although we only construct solutions of 25/48 of all the Möbius boards, we in fact construct solutions of 25/45 = 5/9 of the "potentially solvable" Möbius boards.

In fact a similar proof can also be applied to a $2n-1 \times n$ board when n is even.

Theorem 10. *It is impossible to place $n = \lceil m/2 \rceil$ nonattacking queens on the $m \times n$ Möbius board for n even and $m = 2n-1$.*

Proof. Like the previous theorem, we will say that two sum diagonals c and c' are distinct when $c \not\equiv c' \pmod{2n}$. Among n nonattacking queens on the $n \times (2n-1)$ Möbius board, there must either be a queen on row $(n-2)/2$ or $3n/2 - 1$. Let this queen be in the cell (r', c'). It follows from Remark 4 that this queen will attack exactly one distinct sum diagonal, and that the other $n-1$ queens will each attack 2 distinct sum diagonals. Say the n queens are in the cells (r_i, c_i), $0 \leq i < n$.

Since there are $2n$ sum diagonals and the n queens attack exactly $2n-1$ sum diagonals, there must be a single sum diagonal, say a, which is not attacked. We can now equate the sum of all sum diagonals to the sum of the attacked sum diagonals plus the nonattacked sum diagonal:

$$
\begin{aligned}
\sum_{i=0}^{2n-1} i &\equiv \sum_{i=0}^{n-1} (r_i + c_i - r_i + c_i + n - 1 + m) - (r' + c') + a \pmod{2n} \\
&\equiv \sum_{i=0}^{n-1} (2c_i + 3n - 2) - (r' + c') + a \pmod{2n} \\
&\equiv 2 \cdot \frac{(n-1)n}{2} + (3n-2)n - (r' + c') + a \pmod{2n} \\
&\equiv -n - (r' + c') + a \pmod{2n}.
\end{aligned}
$$

Hence

$$
\frac{(2n-1)2n}{2} \equiv n - (r' + c') + a \pmod{2n},
$$

and thus

$$r' + c' \equiv a \pmod{2n}.$$

This implies that the one nonattacked sum diagonal a is a sum diagonal that we know to be attacked by the queen in cell (r', c'), a contradiction. □

The density of the (m, n) covered by the above theorem is 0.

We can represent a solution for the $n \times n$ standard board as a self-mapping g of $\{0, \ldots, n-1\}$, and to be a solution it is necessary and sufficient that $g(x), g(x) + x$ and $g(x) - x$ all be injective. The following theorem constructs solutions for the Möbius board from solutions for the standard board.

Theorem 11. *For all m and n with m even and $n \geq m - 1$, and any solution g for the $m/2 \times m/2$ standard board, then*

$$\{(2g(j), 2j) \mid j = 0, 1, \ldots, \frac{m}{2} - 1\}$$

is a solution for the $m \times n$ Möbius board.

Proof. Clearly there are no row attacks or carry over row attacks. Clearly there are no column attacks.

Let $(2g(j), 2j)$ and $(2g(l), 2l)$ be distinct queens. For $2g(j)+2j+2\alpha n = 2g(l)+2l$, either $\alpha = 0$ or $\alpha \neq 0$. If $\alpha = 0$, this contradicts that g is a solution for the standard board. If $\alpha \neq 0$, clearly $\alpha > 0$ (as the right-hand side is nonnegative). But $\max_{0 \leq l \leq \frac{m}{2}-1}\{2g(l) + 2l\} = m - 2 < n$, i.e. the maximum of the right-hand side is strictly less than the minimum of the left-hand side, a contradiction. If $2g(j)+2j+2\alpha n = -2g(l)+2l+n-1+m$, the left-hand side is even and the right-hand side is odd, a contradiction. If $-2g(j)+2j+n-1+m+2\alpha n = -2g(l)+2l+n-1+m$, then $g(j) - j - \alpha n = g(l) - l$; if $\alpha = 0$ this contradicts that g is a solution for the standard board, and if $\alpha \neq 0$, this is also a contradiction (if $\alpha < 0$, the left-hand side is strictly greater than the right-hand side, and if $\alpha > 0$ the left-hand side is strictly less than the right-hand side). Thus there are no sum diagonal attacks between queens.

Similarly there are no difference diagonal attacks between queens. Thus this is a solution for the $m \times n$ Möbius board. □

This construction gives new Möbius board solutions when $n = m - 1$. The rest of the range covered by this construction gives solutions for boards which are at least as wide as they are tall, which were solved in Theorems 2 and 3. However, inequivalent standard solutions will lead to inequivalent Möbius solutions, so this method is of some interest. There are many known methods for constructing n-queens solutions [1].

3 Conclusions and open problems

In this paper we have considered the extension of the n-queens problem to the $m \times n$ Möbius board. We have shown that solutions exist for all $m \times n$ Möbius boards such that $m \leq n$ and $n > 3$, for all $m \times n$ Möbius boards such that $m \geq 2n^2$, m and n not both even, and for all odd m, n with $m \geq 4n$. As well we have shown that when $m \geq 2n$ and $m \equiv n \equiv 0 \pmod 2$, or $m = 2n - 1$ and $n \equiv 0 \pmod 2$, then there does not exist a solution for the $m \times n$ Möbius board. The values with a "Y" that do not directly follow from our results are

- when $m = 2$, $n = 2, 3$: trivially a single queen is possible;
- when $n = 2$, $m = 5, 7$: queens at $(0, 0)$ and $(2, 1)$ solve these cases.

Table 2 summarizes the results of this paper for small boards with $m, n \leq 16$. We have also computed in Table 1 the number $M(m)$ of solutions for the $m \times m$

$m \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
2	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
3	Y	N	N	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
4	Y	N		Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
5	Y	Y			Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
6	Y	N				Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
7	Y	Y		N			Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
8	Y	N		N			Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
9	Y	Y							Y	Y	Y	Y	Y	Y	Y	Y
10	Y	N		N					Y	Y	Y	Y	Y	Y	Y	Y
11	Y	Y				N					Y	Y	Y	Y	Y	Y
12	Y	N		N		N					Y	Y	Y	Y	Y	Y
13	Y	Y											Y	Y	Y	Y
14	Y	N		N		N							Y	Y	Y	Y
15	Y	Y						N							Y	Y
16	Y	N		N		N		N							Y	Y

Table 2: The current status of $m \times n$ Möbius boards with $m, n \leq 16$. Blank indicates the status does not follow from our results.

Möbius board for m from 1 to 16. From this we conjecture the following.

Conjecture 12. *$M(m)$ is nondecreasing. $M(m)/m!$ is nonincreasing.*

However, the behavior of the ratios $M(m)/M(m-1)$ is not obvious, and indeed is certainly not monotonic:

- $M(5)/M(4) = \frac{40}{16} = 2.500000000$

- $M(6)/M(5) = \frac{192}{40} = 4.800000000$
- $M(7)/M(6) = \frac{560}{192} \approx 2.91666667$
- $M(8)/M(7) = \frac{3328}{560} \approx 5.942857143$
- $M(9)/M(8) = \frac{11772}{3328} \approx 3.537259615$
- $M(10)/M(9) = \frac{63840}{11772} \approx 5.423037717$
- $M(11)/M(10) = \frac{259336}{63840} \approx 4.062280702$
- $M(12)/M(11) = \frac{1550976}{259336} \approx 5.980565753$
- $M(13)/M(12) = \frac{7169656}{1550976} \approx 4.622673723$
- $M(14)/M(13) = \frac{42410256}{7169656} \approx 5.915242795$
- $M(15)/M(14) = \frac{234044160}{42410256} \approx 5.518574564$
- $M(16)/M(15) = \frac{1366190592}{234044160} \approx 5.837319726.$

It would be interesting for the behavior of the ratios $M(m)/M(m-1)$ to be explained, in particular, whether the limit of $M(m)/M(m-1)$ exists as $m \to \infty$.

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