

Axiomatization of the center function on trees

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Abstract

We give a new, short proof that four certain axiomatic properties uniquely define the center of a tree.

1 Introduction

For a given connected graph G , a *center vertex* minimizes the maximum distance to the other vertices, while a *median vertex* minimizes the average distance to the other vertices. In particular, the *center* of G , denoted by $C(G)$, consists of all center vertices, i.e., if $e(x) = \max\{d(x, y) \mid y \in V(G)\}$, where $d(u, v)$ denotes the distance between u and v , then $C(G) = \{x \in V(G) \mid e(x) \leq e(z), \text{ for all } z \in V(G)\}$. The *median* of G is defined similarly using sums of distances to the other vertices. These are but two of the many notions of centrality developed for graphs, and in particular for trees (see the references in [3]).

Many centrality notions can be phrased in terms of a location function or consensus function. A consensus function is a model to describe a rational way to obtain consensus among a group of agents or clients. The input of the function consists of certain information about the agents, and the output concerns the issue about which consensus should be reached. The rationality of the process is guaranteed by the fact that the consensus function satisfies certain “rational” rules or “consensus axioms”. A typical question in consensus theory is which set of axioms characterizes a given consensus function; for references see [1, 2, 4]. From this viewpoint the center function on a graph is the location function for which the input are the locations of the agents in the graph and the output are the vertices that minimize the maximum distance to the locations of the agents. An axiomatic characterization of the median function was done in [1, 4]. We treat the center function in this note, thus providing a short proof of the result in [2].

We first recall some well-known facts about the center of a tree T that we will use. For a vertex x and a subset S of vertices of T , the distance $d(x, S)$ from x to S is the distance from x to a vertex in S closest to x .

Fact 1 $C(T)$ consists of a single vertex or two adjacent vertices.

Fact 2 If a longest path P in T has endvertices x and y , then $C(T) \subseteq V(P)$ and $d(x, C(T)) = d(y, C(T))$.

Fact 3 If Q denotes the path between two vertices x and x' in $V(T)$, and S is a set of one vertex or two adjacent vertices of $V(Q)$ such that $d(x, S) = d(x', S)$, then $C(Q) = S$.

For a given tree T of order n , a *profile* on T is an ordered k -tuple of (not necessarily distinct) vertices of T , for some $k \geq 1$. If π is such a profile on T , then we will write $\{\pi\}$ for the set of distinct vertices that appear in π . A *location function* on T is a function L_T defined on the set of all profiles on T such that $L_T(\pi)$ is a non-empty subset of $V(T)$, for each profile π on T . For vertex x in $V(T)$ and profile $\pi = (y_1, y_2, \dots, y_k)$, we let $e(T, \pi, x)$ denote the integer $\max\{d(x, y_i) \mid 1 \leq i \leq k\}$. Then an example of a location function on T is the center function Cen_T on T . It is defined by

$$Cen_T(\pi) = \{x \in V(T) \mid e(T, \pi, x) \leq e(T, \pi, z), \text{ for every } z \in V(T)\},$$

for every profile π on T . A useful, straightforward fact about this function is:

Fact 4 If π is a profile on T such that $\{\pi\} = V(T)$, then $Cen_T(\pi) = C(T)$.

We will abbreviate L_T by L and Cen_T by Cen .

Inspired by earlier work on axiomatic studies of location and consensus functions (see [1, 4]), McMorris, Roberts and Wang [2] presented four natural axioms for location functions on trees that we list below, and they proved that a location function on a tree T satisfies these four axioms if and only if it is the center function on T . For other references on centrality axiomatics, see the list of references in [2].

As usual, for a tree T , if $S \subseteq V(T)$, then $T[S]$ denotes the tree induced by S , i.e., the intersection of all subtrees of T that contain every vertex in S . And, for a profile π on T , $\pi - x$ denotes the profile obtained from π by deleting all occurrences of x in π .

For a tree T , the four axioms in [2] for a location function on T that we wish to consider are:

Middleness (M): For any $x, y \in V(T)$, if Q denotes the path between x and y , then $L((x, y)) = C(Q)$.

Quasi-Consistency (QC): For any two profiles π and ρ on T , if $L(\pi) = L(\rho)$, then $L(\pi\rho) = L(\pi)$. (Here $\pi\rho$ denotes the profile on T consisting of the terms π of followed by the terms of ρ .)

Redundancy (R): For any profile on T , if x is a vertex of the subtree $T[\{\pi - x\}]$, then $L(\pi - x) = L(\pi)$.

Population Invariance (PI): For any two profiles π and ρ on T , if $\{\pi\} = \{\rho\}$, then $L(\pi) = L(\rho)$.

Note that (M) implies that $L((x, x)) = \{x\}$. Combined with (QC) we get $L((x)) = L((x, x)) = \{x\}$ as well. Let T_π denote $T[\{\pi\}]$. The redundancy axiom (R) implies that any internal vertex x of the subtree T_π can be added to π or deleted from π without changing the output of L . So, basically, $L(\pi)$ is determined by the leaves of subtree T_π .

It is not difficult to show that, for a tree T , the center function Cen satisfies the four axioms. The purpose of this note is to give a new, short proof that, for a tree T , any location function on T that satisfies the four axioms must be the center function on T .

Theorem. *Let T be a tree. If L is a location function on T that satisfies Middleness (M), Quasi-Consistency (QC), Redundancy (R), and Population Invariance (PI), then $L(\pi) = Cen(\pi)$, for every profile π on T .*

Proof. Fix a tree T and a profile π on T . Clearly, $\{\pi\} \subseteq V(T_\pi)$. In fact, every pendant vertex of T_π is in $\{\pi\}$, and, without loss of generality because of (R), we may assume that $\{\pi\} = V(T_\pi)$.

Let a_1 and a_2 be the endpoints of a longest path P in T_π . By Facts 4 and 2, $Cen(\pi) = C(T_\pi) \subseteq V(P)$. For any $x \in V(T_\pi)$, let Q be the path from x to the set $Cen(\pi)$, and let z denote the first vertex of Q that is also on P (proceeding from x to P). Without loss of generality, we may assume that $d(a_1, z) \leq d(a_2, z)$. As P is a longest path, $d(x, z) \leq d(a_1, z)$. So, $d(x, Cen(\pi)) = d(x, z) + d(z, Cen(\pi)) \leq d(a_1, z) + d(z, Cen(\pi)) = d(a_1, Cen(\pi)) = d(a_2, Cen(\pi))$. The last equality follows from Fact 2. Now $Cen(\pi) = C(T_\pi)$ is a set of one or two adjacent vertices in $V(P)$ by Fact 1. As $d(\cdot, Cen(\pi))$ yields every integer value between 0 and $d(a_2, Cen(\pi))$ on the path P from $Cen(\pi)$ to a_2 , there must be a vertex x' on P so that $d(x, Cen(\pi)) = d(x', Cen(\pi))$. Note that $x' = x$ if and only if $Cen(\pi) = \{x\}$, otherwise we have $x' \neq x$. Let P_x denote the path between x and x' . Fact 3

implies that $C(P_x) = Cen(\pi)$. Axiom (M) implies that $L((x, x')) = C(P_x)$. So $L((x, x')) = Cen(\pi)$. Take any two distinct vertices x_1 and x_2 in T . Then we have $L((x_1, x'_1)) = L((x_2, x'_2)) = Cen(\pi)$. So, by (QC), we have $L((x_1, x'_1, x_2, x'_2)) = Cen(\pi)$. If $V(T_\pi) = \{x_1, x_2, \dots, x_n\}$, then repeated use of Axiom (QC) yields $L((x_1, x'_1, x_2, x'_2, \dots, x_n, x'_n)) = Cen(\pi)$. Note that $\{(x_1, x'_1, x_2, x'_2, \dots, x_n, x'_n)\} = \{\pi\}$, so by (PI), we have $L((x_1, x'_1, x_2, x'_2, \dots, x_n, x'_n)) = L(\pi)$. Consequently, $L(\pi) = Cen(\pi)$, as desired. \square

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