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A characterization of bipartite graphs with independence number half their order

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Abstract

Let $n(G)$ and $\alpha(G)$ be the order and the independence number of a graph G, repsectively. If G is bipartite graph, then it is well-known and easy to see that $\alpha(G) \geq \frac{n(G)}{2}$. In this paper we present a constructive characterization of bipartite graphs G for which

$$\alpha(G) = \left\lceil \frac{n(G)}{2} \right\rceil .$$

1 Terminology and introduction

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *neighborhood* $N(v) = N_G(v)$ of the vertex v in a graph G consists of the vertices adjacent to v. The vertex v is a *leaf* of G if $d_G(v) = 1$, where $d(v) = d_G(v) = |N_G(v)|$ is the *degree* of $v \in V(G)$. We write K_n for the *complete graph* of order n. If G is a graph and $A \subseteq V(G)$, then we denote by $q(G - A)$ the number of odd components in the subgraph $G - A$.

If M is a maximum matching in a graph G, then $\alpha_0(G) = |M|$ is the *edge independence number*. A matching M of a graph G is *perfect* or *almost perfect* if $2|M| = n(G)$ or $2|M| = n(G) - 1$, respectively.

A set $D \subseteq V(G)$ is a *covering* of G if every edge of G has at least one end in D. The *covering number* $\beta = \beta(G)$ of G is the cardinality of a smallest covering of G.

A set I of vertices in a graph G is independent if every two vertices of S are not adjacent in G. The *independence number* $\alpha = \alpha(G)$ of a graph G is the maximum cardinality among the independent sets of vertices in G.

For detailed information on domination, irredundance, and related topics see the comprehensive monograph [3] by Haynes, Hedetniemi, and Slater.

If G is a bipartite graph with the partite sets V_1 and V_2, then V_1 as well as V_2 are independent sets and coverings. Thus

$$\alpha(G) \geq \frac{n(G)}{2} \geq \beta(G)$$

for each bipartite graph. In this paper we will discuss the question, for which bipartite graphs G the identity $\alpha(G) = \left\lceil \frac{n(G)}{2} \right\rceil$ holds.

Theorem 1.1 (König [4] 1931) *If G is a bipartite graph, then $\beta(G) = \alpha_0(G)$.*

Theorem 1.2 (Gallai [2] 1959) *If G is a graph, then $\alpha(G) + \beta(G) = n(G)$.*

Proofs of Theorems 1.1 and 1.2 can also be found in the book by Volkmann [5]. These two results imply easily the next corollary.

Corollary 1.3 *If G is a bipartite graph, then*

$$\alpha(G) = \left\lceil \frac{n(G)}{2} \right\rceil$$

if and only if G has a perfect matching if $n(G)$ is even and G has an almost perfect matching if $n(G)$ is odd.

In addition, Chunghaisan (cf. [1], p. 80) has shown that a tree T has a perfect matching if and only if $q(G - v) = 1$ for every $v \in V(T)$.

So far as I know, no one has given a contructive characterization of bipartite graphs G for which $\alpha(G) = \lceil \frac{n(G)}{2} \rceil$. In this note we will present such a constructive characterization.

2 The characterizations

For a constructive characterization of trees T for which $\alpha(T) = \lceil \frac{n(T)}{2} \rceil$, we introduce the following operation.

Operation: Let w be an arbitrary vertex of a tree T_w and let v be a vertex of the complete graph K_2. Let T be obtained from $T_w \cup K_2$ by adding the edge vw.

We now define the families $\mathcal{T}_1$ and $\mathcal{T}_2$ as follows:

$T \in \mathcal{T}_1$ if and only if $T = K_2$ or T is obtained from K_2 by a finite sequence of operations above.

$T \in \mathcal{T}_2$ if and only if $T = K_1$ or T is obtained from K_1 by a finite sequence of operations above.

Theorem 2.1 *Let T be a tree of even order n. Then $\alpha(T) = \frac{n}{2}$ if and only if $T \in \mathcal{T}_1$.*

Proof. Assume that $T \in \mathcal{T}_1$. The definition of $\mathcal{T}_1$ easily shows that T has a perfect matching. Therefore Corollary 1.3 implies the desired result.

Conversely, assume that $\alpha(T) = \frac{n(T)}{2}$. According to Corrolary 1.3, the tree T has a perfect matching M. We proceed by induction on the order $n = n(T)$. If $n = 2$, then $T = K_2 \in \mathcal{T}_1$. Now assume that $n \geq 4$ is an even integer and that $\alpha(T) = \frac{n}{2}$. Let $P = x_1x_2 \ldots x_t$ be a longest path in T. Since T has a perfect matching M and as P is a longest path in T, we conclude that $d_T(x_2) = 2$ and thus $x_1x_2 \in M$. This implies that $T' = T - \{x_1, x_2\}$ is also a tree of even order with the perfect matching $M - \{x_1x_2\}$ and so $\alpha(T') = \frac{n(T')}{2}$. In view of the induction hypothesis, T' belongs to the family $\mathcal{T}_1$. By the definition of $\mathcal{T}_1$, it follows that $T \in \mathcal{T}_1$, and the proof is complete. □

Theorem 2.2 *Let T be a tree of odd order n. Then $\alpha(T) = \frac{n+1}{2}$ if and only if $T \in \mathcal{T}_2$.*

Proof. Assume that $T \in \mathcal{T}_2$. The definition of $\mathcal{T}_2$ shows that T has an almost perfect matching. Therefore Corollary 1.3 implies the desired result.

Conversely, assume that $\alpha(T) = \frac{n(T)+1}{2}$. According to Corrolary 1.3, the tree has an almost perfect matching M. We proceed by induction on the order $n = n(T)$. If $n = 1$ or $n = 3$, then $T \in \mathcal{T}_1$. Now assume that $n \geq 5$ is an odd integer and that $\alpha(T) = \frac{n+1}{2}$. Let $P = x_1x_2 \ldots x_t$ be a longest path in T. Since T has an almost perfect matching M and as P is a longest path in T, we conclude that $d_T(x_2) \leq 3$ and $d_T(x_{t-1}) \leq 3$. In addition, we observe that $d_T(x_2) = 2$ or $d_T(x_{t-1}) = 2$. Otherwise, let $y_2 \neq x_1$ and $y_3 \neq y_t$ be a leaf adjacent with x_2 and x_{t-1}, respectively. This is a contradiction to the hypothesis that T contains an almost perfect matching. Now we assume, without loss of generality, that $d_T(x_2) = 2$. If $d_T(x_{t-1}) = 3$, then it follows that $x_1x_2 \in M$. This implies that $T' = T - \{x_1, x_2\}$ is also a tree of odd order with the almost perfect matching $M - \{x_1x_2\}$ and so $\alpha(T') = \frac{n(T')+1}{2}$. In view of the induction hypothesis, T' belongs to the family $\mathcal{T}_2$. By the definition of $\mathcal{T}_2$, it follows that $T \in \mathcal{T}_2$. In the remaining case that $d_T(x_2) = 2$ and $d_T(x_{t-1}) = 2$, we can assume, without loss of generality, that $x_1x_2 \in M$, and we obtain the desired result analogously. □

Observation 2.3 If G is a connected graph with a maximum matching M, then G contains a spanning tree with the maximum matching M.

Proof. If G is itself a tree, then this observation is trivial. If G is not a tree, simply remove edges lying on cycles in $G - M$, one at time, until only bridges remain. □

Theorem 2.4 *Let G be a bipartite graph of even order n. Then $\alpha(G) = \frac{n}{2}$ if and only if G has a spanning tree $T \in \mathcal{T}_1$.*

Proof. If $\alpha(G) = \frac{n}{2}$, then it follows from Corollary 1.3 that G has a perfect matching M. Combining this with Observation 2.3, we find that there exists a spanning tree T of G with the perfect matching M. According to Corollary 1.3, we conclude that

$\alpha(T) = \frac{n(T)}{2} = \frac{n}{2}$, and hence Theorem 2.1 implies that $T \in \mathcal{T}_1$.

Conversely, assume that G has a spanning tree $T \in \mathcal{T}_1$. In view of Theorem 2.1, we have $\alpha(T) = \frac{n(T)}{2}$. Since $\alpha(G) \geq \frac{n}{2} = \frac{n(T)}{2}$, we can immediately deduce that $\alpha(G) = \frac{n}{2}$. $\square$

Applying Theorem 2.2 instead of Theorem 2.1, we can prove the next result analogously to the proof of Theorem 2.4.

Theorem 2.5 *Let G be a bipartite graph of odd order n. Then $\alpha(G) = \frac{n+1}{2}$ if and only if G has a spanning tree $T \in \mathcal{T}_2$.*

Remark 2.6 The complete graph and other examples show that that neither Theorem 2.4 nor Theorem 2.5 is valid for non-bipartite graphs in general.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, The Macmillan Press Ltd., London and Basingstoke (1976).

[2] T. Gallai, Über extreme Punkt- und Kantenmengen, *Ann. Univ. Sci. Budapest, Eötvös Sect. Math.* **2** (1959), 133–138.

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[4] D. König, Graphen und Matrizen, *Mat. Fiz. Lapok* **38** (1931), 116–119.

[5] L. Volkmann, *Foundations of Graph Theory*, Springer-Verlag, Wien New York, 1996 (in German).

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