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Genus distributions of orientable embeddings for two types of graphs

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Abstract

On the basis of the joint tree model introduced by Liu in 2003, the genus distributions of the orientable embeddings for further types of graphs can be obtained. These are apparently not easily obtained using overlap matrices, the formula of Jackson, etc. In this paper, however, by classifying the associated surfaces, we calculate the genus distributions of the orientable embeddings for two new types of graphs, namely, generalized necklaces and circulant necklaces. These are different from the graphs whose embedding distributions by genus have been obtained to date.

1 Introduction

The derivation of the embedding distribution of a graph is a newly thriving aspect of topological theory. Until now, many authors have computed the genus polynomials of several types of graphs with different methods. Gross et al. [5] did it for bouquets of circles using the formula of Jackson [6]; Gross et al. [4] for necklaces; Furst et al. [2] for closed-end ladders and cobblestone paths using combinatorial methods. Later, Chen et al. [1] did this for necklaces, closed-end ladders and cobblestone paths using overlap matrices. In 2003, Liu set up the joint tree model [8], such that the genus polynomials of more types of graphs can be obtained, such as [7, 12–13].

In this paper, on the basis of the joint tree model, by classifying the associated surfaces of a graph, we obtain the genus distributions of the orientable embeddings for two new types of graphs, which are generalized necklaces and circulant necklaces.

Suppose that the "beads" of a necklace were placed along a path instead of along a cycle. Then the genus distribution formula would follow easily from the bar-amalgamation formula [3]. The difficulty in deriving a formula for necklaces is the extra edge that changes a path of beads into a necklace.

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An important advance of this paper is coping with the extra edge by basing the calculation on the explicit choice of a spanning tree. Although one might derive some of its formulas by direct consideration of a recursion, the focus on spanning tree selection indicates a direction for further generalization, beyond application in necklaces.

Another advance of this paper is generalizing the class of beads, beyond those appearing in [4], for which genus distribution calculations are tractable. This permits one to obtain formulas for infinite families of regular graphs with degree greater than 3 and 4 and with arbitrarily many vertices.

In what follows, we will introduce some definitions and results.

A graph is always considered to be connected. A *linear order* X is a sequence of letters such that if $X = ab \dots z$, then it is indicated that $a \prec b \cdots \prec z$. A *reverse order* $\hat{X}$ of X is the linear order such that $\hat{X} = z \dots ba$. A linear order Y is called a *suborder* of X if and only if each letter on Y is also on X and if $a \prec b$ in X, then $a \prec b$ in Y. Let $Y \subseteq A$ mean that Y is a suborder of some linear order in the set A. A *supplementary order* $\bar{Y}$ of Y corresponding to X is a suborder of X such that $a \in \bar{Y}$ for each $a \notin Y$ and that $a \notin \bar{Y}$ for each $a \in Y$.

A *surface* is a compact 2-dimensional manifold without boundary. It can be represented by a regular polygon with even number of sides on the plane, where each pair of sides can be pasted according to a given direction. Further, an *orientable surface* can be represented by a cyclic order P of letters satisfying the following conditions [10]:

Con1. If $a \in P$, then $a^- \in P$.
Con2. For each letter a on P, both a and a^- occur once on P.

Let $\gamma(S)$ be the genus of surface S and $\mathcal{S}$ be the set of surfaces. On $\mathcal{S}$, an *elementary transformation* [10] is defined by the following three operations:

Op.1 $\forall S \in \mathcal{S},\ S = Aaa^-B,\ A \neq 0$, or $B \neq 0 \Longleftrightarrow S = AB$.
Op.2 $\forall S \in \mathcal{S},\ S = AabBb^-a^-C \Longleftrightarrow S = AaBa^-C$.
Op.3 $\forall S \in \mathcal{S},\ S = AaBCa^-D \Longleftrightarrow S = BaADa^-C$.

If two surfaces S_1 and S_2 can be converted from one to another by finite sequences of elementary transformations, then they are said to be *convertible*. It is easily seen that the convertibility between two surfaces is an *equivalence*, denoted by $S_1 \sim S_2$. Note that S_1 and S_2 have the same orientability and genus.

According to the operations, the following lemma is obtained.

Lemma 1.1 [10] *$AaBbCa^-Db^-E \sim ADCBEaba^-b^-$, where $a, b, a^-, b^- \notin ABCDE$.*

Then by applying these operations above, each orientable surface is equivalent to only one of the following canonical forms:

$$S_i = \begin{cases} a_0 a_0^-, & \text{if the surface is sphere;} \\ \Pi_{k=1}^{i} a_k b_k a_k^- b_k^-, & \text{if the genus of a surface is } i. \end{cases}$$

Suppose that there are n sets of linear orders, say $A_1, A_2, \ldots, A_n$. Let $aX_1^n a^- X_2^n S_k$ and $X_1^n X_2^n S_l$ be surfaces, where $X_1^n = Z_1 Z_2 \ldots Z_n$, $X_2^n = \hat{\hat{Z}}_n \ldots \hat{\hat{Z}}_2 \hat{\hat{Z}}_1$ and $Z_l \subseteq A_l$ for $1 \leq l \leq n$. By $A_{(n,\ k)}$, we mean a set constituted by such elements as $aX_1^n a^- X_2^n S_k$, taken over all $Z_l \subseteq A_l$ for $1 \leq l \leq n$. Use $B_{(n,\ l)}$ to denote a set as $\{X_1^n X_2^n S_l\}$. And the letters on S_k or S_l do not appear on X_1^n and X_2^n. Note that $A_{(n,\ k)}$ as a form is meant the different set when A_l varies for $1 \leq l \leq n$. So is $B_{(n,\ l)}$.

Lemma 1.2 [8] *Let S_1 and S_2 be surfaces, $a, b, a^-, b^- \notin S_2$. If $S_1 \sim S_2 aba^- b^-$, then $\gamma(S_1) = \gamma(S_2) + 1$.*

Lemma 1.3 *Let $S \in A_{(n,\ 0)}$ and S^0 be the surface obtained by deleting a and a^- from S. Then*

$$\gamma(S) = \begin{cases} \gamma(S^0), & \text{if } S \in A_{(n-1,k)}; \\ \gamma(S^0) - 1, & \text{if } S \in B_{(n-1,l)}. \end{cases}$$

where k and l are positive integers or zero.

An *embedding* (or *cellular embedding* in early references) of a graph G into a surface S is a homeomorphism τ: $G \to S$, such that each component of $S - \tau(G)$ is homeomorphic to an open disc. Two embeddings τ_1: $G \to S$ and τ_2: $G \to S$ are *the same* if there is a homeomorphism h: $S \to S$ such that $h\tau_2 = \tau_1$. The embedding is called *orientable* if S is orientable. Throughout this article, whenever we use the term embedding, we are referring to an orientable embedding. By the *maximum (minimum) genus* of a graph G, we mean the maximum (minimum) genus of the surface into which G has an embedding.

A *rotation* σ_v at a vertex v is a cyclic permutation of edges incident with v. Let $\sigma = \Pi_{v \in V(G)} \sigma_v$ be a rotation system of G. Let T be a spanning tree of G. A *joint tree*[8] $\tilde{T}_\sigma$ can be got by splitting every cotree edge into two semiedges denoted by a same letter with a choice of indices: $+$(always omitted) or $-$. Based on $\tilde{T}_\sigma$, write down the letters with indices according to a fixed orientation(clockwise or counterclockwise) to obtain a cyclic order of $2\beta(G)$ letters. It represents a surface, called an *associated surface.* If two associated surfaces of G have the same cyclic order with the same indices, then they are said to be the *same.* Otherwise, distinct. So an embedding of a graph into a surface can be represented by a joint tree of it, further by an associated surface of it, where $\beta(G)$ is the number of the cotree edges.

From [9], for a fixed spanning tree T of the graph G, there is a 1–to–1 correspondence between the associated surfaces and the embeddings of G.

It is soon seen that the problem of determining the genus distribution of all embeddings for a graph is transformed into that of finding the number of all distinct associated surfaces in each equivalent class.

An example should serve to clarify the definitions above. For a necklace of 3 beads N_3, the spanning tree is presented with thick lines as shown in Fig. 1.1 and a joint tree of N_3 in Fig.1.2. Denote cotree edge v_1v_2 by a_1, v_3v_4 by a_2, v_5v_6 by a_3. Let joint trees of N_3 have a clockwise rotation at each vertex. Then an associated surface can be shown as $S = aa_1a_2a_2^-a_3^-a^-a_3a_1^-$. According to the rotation at each vertex of a joint tree, all associated surfaces can be found.

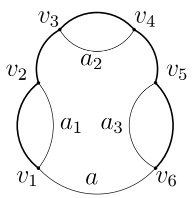

Fig.1.1 N_3

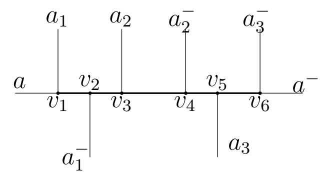

Fig.1.2 A joint tree of N_3

For a graph G, let $g_i(G)$ be the number of distinct embeddings for G into the orientable surface of genus i for $i \geq 0$. The embedding genus distribution of G is:

$$g_0(G), g_1(G), g_2(G), \ldots .$$

Then the *genus polynomial* of G is:

$$f_G(x) = \sum_{i=0}^{\infty} g_i(G)x^i.$$

For convenience, throughout this article, we write $g_i(n)$ instead of $g_i(G)$, where n is variant of G. To understand some definitions mentioned above, also see [11].

2 Generalized necklaces

Given an n-cycle C, for any number k, replace every other edge with a multi-edge of the same multiplicity $j \geq 1$ and then add $(k-j-1)/2$ loops at each vertex of C to obtain a new graph G called a *generalized necklace*, so that the resulting degree is k. When $j = 1$, G denoted by G_n^k has no multi-edge. When $j \geq 2$, n must be even and G is denoted by $\tilde{G}_{n/2}^k$. For $\tilde{G}_{n/2}^k$, depending on j, there may be more than one such graph. Note that G_n^4 and $\tilde{G}_n^3$ are n-vertex necklaces of type $(0, n)$ and $(n, 0)$, respectively, as defined in [4]. The following figures illustrate four generalized necklaces.

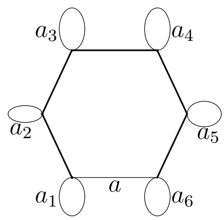

Fig.2.1 G_6^4

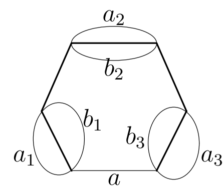

Fig.2.2 $\tilde{G}_3^4$

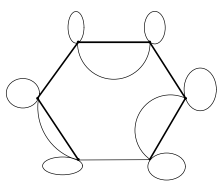

Fig.2.3 $\tilde{G}_3^5$

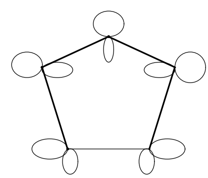

Fig.2.4 G_5^6

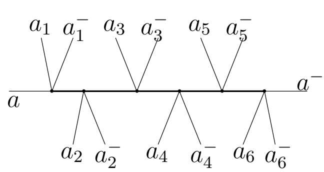

Fig.2.5 A joint tree of G_6^4

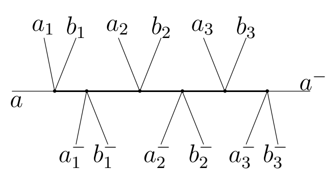

Fig.2.6 A joint tree of $\tilde{G}_3^4$

Theorem 2.1 $f_{G_n^4}(x) = \sum_{i=0}^{1} (i6^n + (1-2i)4^n)x^i$,

$$f_{\tilde{G}_n^4}(x) = \sum_{i=0}^{n} \frac{n!3^{i-1}}{i!(n+1-i)!}(i3^{n-i+1} + 3n - 4i + 3)6^n x^i.$$

Proof. Firstly, choose a spanning tree of G_n^4 by deleting a random edge, denoted by a, from C as indicated with thick lines in Figs. 2.1 and 2.5, and use distinct letters $a_1, a_2, \ldots, a_n$ to denote other cotree edges, which are loops. Let joint trees of G_n^4 have a clockwise rotation at each vertex. Then

$$X_1^n = C_1C_2\ldots C_n, \qquad X_2^n = \hat{\hat{C}}_n\ldots\hat{\hat{C}}_2\hat{\hat{C}}_1,$$

where $C_l \subseteq \{a_la_l^-, a_l^-a_l\}$ for $1 \le l \le n$.

$$X_1^{n-1} = C_1C_2\ldots C_{n-1}, \quad X_2^{n-1} = \hat{\hat{C}}_{n-1}\ldots\hat{\hat{C}}_2\hat{\hat{C}}_1.$$

So the set of associated surfaces of G_n^4 is $A_{(n,0)}$. The set can be classified into the following six sets according to the nth vertex of the joint tree.

$$\{aX_1^{n-1}a_na_n^-a^-X_2^{n-1}\} \qquad \{aX_1^{n-1}a_n^-a_na^-X_2^{n-1}\}$$

$$\{aX_1^{n-1}a_na^-a_n^-X_2^{n-1}\} \qquad \{aX_1^{n-1}a_n^-a^-a_nX_2^{n-1}\}$$
$$\{aX_1^{n-1}a^-a_na_n^-X_2^{n-1}\} \qquad \{aX_1^{n-1}a^-a_n^-a_nX_2^{n-1}\}$$

By deleting a and a^- from these sets, we get classifying sets of $B_{(n,0)}$.

According to Op.1 and Lemmas 1.1–1.3,

$$\begin{aligned}\gamma(aX_1^{n-1}a_na_n^-a^-X_2^{n-1}) &= \gamma(aX_1^{n-1}a^-X_2^{n-1}),\\ \gamma(aX_1^{n-1}a_na^-a_n^-X_2^{n-1}) &= \gamma(X_1^{n-1}X_2^{n-1}aa_na^-a_n^-)\\ &= \gamma(X_1^{n-1}X_2^{n-1})+1.\end{aligned}$$

Of course, $g_i(n)$ is equal to the number of associated surfaces of genus i in $A_{(n,\ 0)}$. And we use $g_i^0(n)$ to denote the number of surfaces of genus i in $B_{(n,\ 0)}$. So the following equations hold.

$$\begin{cases} g_i(n) = 4g_i(n-1) + 2g_{i-1}^0(n-1) & (2.1)\\ g_i^0(n) = 6g_i^0(n-1) & (2.2)\\ g_0(0) = 1 & (2.3)\\ g_0^0(0) = 1 & (2.4)\\ g_i^0(0) = 0, \quad i>0 & (2.5)\end{cases}$$

From (2.2–2.5), $\qquad g_i^0(n) = 6^n,$

then $\qquad g_i(n) = 4g_i(n-1) + 2\cdot 6^{n-1}.$

So $\qquad g_i(n) = i\cdot 6^n + (1-2i)\cdot 4^n,$

thus $\qquad f_{G_n^4}(x) = \sum_{i=0}^{1}(i6^n + (1-2i)4^n)x^i.$

For $\tilde{G}_n^4$, choose all edges of cycle C except one, which is not multi-edge and denoted by a, to obtain a spanning tree. Then label other cotree edges by distinct letters $a_1, b_1, \dots, a_n, b_n$ (see Figs. 2.2 and 2.6). Let joint trees of $\tilde{G}_n^4$ also have a clockwise rotation at each vertex. Let

$$X_1^n = F_1F_2\dots F_n, \qquad X_2^n = \hat{\bar{F}}_n\dots\hat{\bar{F}}_2\hat{\bar{F}}_1,$$

where $F_l \subseteq \{a_lb_la_l^-b_l^-, b_la_la_l^-b_l^-, a_lb_lb_l^-a_l^-, b_la_lb_l^-a_l^-\}$ for $1\le l\le n$. So the set of associated surfaces of $\tilde{G}_n^4$ is $A_{(n,\ 0)}$. The set can be classified into 36 sets according to the rotation of the $(2n-1)$th and $2n$th vertex of the joint tree, which can be represented as follows:

$$\{aX_1^{n-1}Y_1Y_2a^-\bar{Y}_2\bar{Y}_1X_2^{n-1}\} \;\; \{aX_1^{n-1}\hat{\bar{Y}}_1\hat{\bar{Y}}_2a^-\hat{Y}_2\hat{Y}_1X_2^{n-1}\}$$

where $Y_2 \subseteq \{a_n^-b_n^-, b_n^-a_n^-\}$, $Y_1 \subseteq \{a_nb_n, b_na_n\}$ and the number of the letters on Y_1 is 1 or 2. By deleting a and a^- from 36 sets, we get the classifying sets of $B_{(n,\ 0)}$.

By reducing these sets and applying Lemmas 1.1–1.3, we obtain the following equations:

$$\begin{cases} g_i(n) = 6g_i(n-1) + 18g_{i-1}(n-1) + 12g^0_{i-1}(n-1) & (2.6) \\ g^0_i(n) = 18g^0_i(n-1) + 18g^0_{i-1}(n-1) & (2.7) \\ g_0(0) = 1 & (2.8) \\ g^0_0(0) = 1 & (2.9) \\ g^0_i(0) = 0, \quad i > 0 & (2.10) \end{cases}$$

From (2.7–2.10), $g^0_i(n) = \binom{n}{i} 18^n;$

then $g_i(n) = 6g_i(n-1) + 18g_{i-1}(n-1) + 12\dfrac{(n-1)!}{(i-1)!(n-i)!}18^{n-1}.$

So $g_i(n) = \dfrac{n!3^{i-1}}{i!(n+1-i)!}(i3^{n-i+1} + 3n - 4i + 3)6^n.$

Thus $f_{\tilde{G}^4_n}(x) = \displaystyle\sum_{i=0}^{n} \frac{n!3^{i-1}}{i!(n+1-i)!}(i3^{n-i+1} + 3n - 4i + 3)6^n x^i.$ □

Note that the first formula of Theorem 2.1 is consistent with a special case of Theorem 4 in [4]. Using the same method as Theorem 2.1, we also can get the following theorem.

Theorem 2.2. $f_{G^6_n}(x) = \sum_{i=0}^{n} \frac{(n-1)!}{i!(n-i+1)!}(n^2 - 2ni + n + ni2^{n-i+1})40^n x^i,$

$$f_{\tilde{G}^5_n}(x) = \sum_{i=0}^{n+1} \frac{n!32^{i-1}8^{n-i+1}}{i!(n-i+1)!}[i10^{i-1}32^{n-i+1} + 7^{i-1}9^{n-i}(28n - 37i + 28)]x^i.$$

Proof. For G^6_n, $g_i(n)$ satisfies the following equations:

$$\begin{cases} g_i(n) = 40(g_i(n-1) + g_{i-1}(n-1) + g^0_{i-1}(n-1)) \\ g^0_i(n) = 80g^0_i(n-1) + 40g^0_{i-1}(n-1) \\ g_0(0) = 1 \\ g^0_0(0) = 1 \\ g^0_i(0) = 0, \quad i > 0. \end{cases}$$

For $\tilde{G}_n^5$, $g_i(n)$ satisfies the following equations:

$$\begin{cases} g_i(n) = 72g_i(n-1) + 224g_{i-1}(n-1) + 184g_{i-1}^0(n-1) + 96g_{i-2}^0(n-1) \\ g_i^0(n) = 256g_i^0(n-1) + 320g_{i-1}^0(n-1) \\ g_0(0) = 1 \\ g_0^0(0) = 1 \\ g_i^0(0) = 0, \quad i > 0. \end{cases} \qquad \square$$

Generally, for G_n^{2k+2}, every vertex has k loops. Using the same method as above, we get the set of associated surfaces $A_{(n,\ 0)}$, where $X_1^n = G_1G_2\ldots G_n$, $X_2^n = \hat{\tilde{G}}_n \ldots \hat{\tilde{G}}_2\hat{\tilde{G}}_1$, $G_l \subseteq A_l$ for $1 \le l \le n$ and A_l is a set of cyclic permutations on $\{a_{n_1}a_{n_1}^- \ldots a_{n_k}a_{n_k}^-\}$.

So by reducing these sets and applying Lemmas 1.1–1.3, $A_{(n,0)}$ can be classified into such sets as $A_{(n-1,k)}$ and $B_{(n-1,l)}$ of different genus, $B_{(n,0)}$ into such sets as $B_{(n-1,l)}$ of different genus. The same discussion can be done on $\tilde{G}_n^k$. So it is obvious that we obtain the following result.

Theorem 2.3 *Let $g_i(n)$ be the number of embeddings for $G_n^k(\tilde{G}_n^k)$ into an orientable surface of genus i. Then $g_i(n)$ is the linear combination with integral coefficients of $g_j(n-1)$ and $g_k^0(n-1)$, and $g_i^0(n)$ is that of $g_k^0(n-1)$, for $i, j, k \ge 0$ and $0 \le j, k \le i$.*

3 Circulant necklaces

Suppose that uv is an edge. Add vertices $u_1^1, u_2^1, \ldots, u_m^1, v_1^1, v_2^1, \ldots, v_m^1, u_1^2, u_2^2, \ldots, u_m^2$, $v_1^2, v_2^2, \ldots, v_m^2, \ldots, u_1^n, u_2^n, \ldots, u_m^n, v_1^n, v_2^n, \ldots, v_m^n$ between u and v in such a sequence and connect $u_l^j v_l^j (1 \le l \le m, 1 \le j \le n)$ to obtain a graph, denoted by L_n^m. By amalgamating u and v, we obtain a new graph called a *circulant necklace*, denoted by S_n^m (see Figs. 3.1 and 3.2).

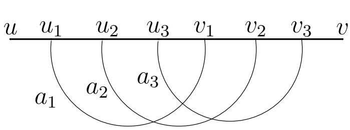

Fig.3.1 L_1^3

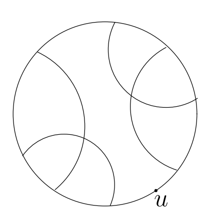

Fig.3.2 S_2^2

For L_1^m, let the path $uu_1^1u_2^1\ldots u_m^1v_1^1v_2^1\ldots v_m^1v$ be a spanning tree. Label the cotree edge $u_l^1v_l^1$ by a_l^1 for $1 \le l \le m$, where $a_1^1, a_2^1, \ldots, a_m^1$ are distinct letters. Let joint trees of L_1^m have a clockwise rotation at each vertex. Let

$$Y_1^m = E_m E_{m-1} \dots E_1, \qquad Y_2^m = \hat{\hat{E}}_1 \hat{\hat{E}}_2 \dots \hat{\hat{E}}_m,$$

$$Y_3^m = D_1 D_2 \dots D_m, \qquad Y_4^m = \hat{\hat{D}}_m \hat{\hat{D}}_{m-1} \dots \hat{\hat{D}}_1,$$

where $E_l \subseteq \{a_l\}, D_l \subseteq \{a_l^-\}$ for $1 \leq l \leq m$.

Then the set of associated surfaces of L_1^m is $\{Y_1^m Y_2^m Y_3^m Y_4^m\}$, and the set can be classified into two $\{a_m Y_2^{m-1} a_m^- Y_3^{m-1} Y_4^{m-1} Y_1^{m-1}\}$ and two $\{a_m Y_3^{m-1} a_m^- Y_4^{m-1} Y_1^{m-1} Y_2^{m-1}\}$. Let $h_i(m)$ be the number of the surfaces of genus i in $\{Y_1^m Y_2^m Y_3^m Y_4^m\}$.

Then $h_i(m) =$
$8h_{i_8}(m-1) + 8h_{(i-1)_1}(m-1) + 8h_{(i-1)_2}(m-1) + 8h_{(i-1)_4}(m-1) + 32h_{(i-1)}(m-1)$, where

$$\begin{aligned}
h_{i_8}(m) &= 4h_{(i-1)_8}(m-1) + 4h_{(i-1)}(m-1) + 4h_{(i-2)_1}(m-1) + 4h_{(i-2)_1}(m-1),\\
h_{i_1}(m) &= 4h_{(i-1)_8}(m-1) + 4h_{(i-1)}(m-1) + 4h_{(i-1)_1}(m-1) + 4h_{(i-1)_2}(m-1),\\
h_{i_2}(m) &= 2h_{i_2}(m-1) + 6h_{i_4}(m-1) + 8h_{(i-1)_2}(m-1),\\
h_{i_4}(m) &= 8h_{(i-1)_2}(m-1) + 8h_{(i-1)_4}(m-1). \qquad [12].
\end{aligned}$$

Lemma 3.1 *The maximum and minimum genus of L_1^m is equal to $[\frac{m}{2}]$ and 0, respectively.*

Proof. When $E_l = a_l,\ D_l = a_l^-$ for $1 \leq l \leq m$, $\gamma(Y_1^m Y_2^m Y_3^m Y_4^m) = \gamma(a_m a_{m-1} \dots a_1 a_1^- a_2^- \dots a_m^-) = 0$. When $\hat{\hat{E}}_1 = a_l,\ D_l = a_l^-$ for $1 \leq l \leq m$, $\gamma(Y_1^m Y_2^m Y_3^m Y_4^m) = \gamma(a_1 a_2 \dots a_m a_1^- a_2^- \dots a_m^-) = [\frac{m}{2}]$. Then this lemma is true. □

For S_n^m, choose the path $uu_1^1 u_2^1 \dots u_m^1 v_1^1 v_2^1 \dots v_m^1 u_1^2 u_2^2 \dots u_m^2 v_1^2 v_2^2 \dots v_m^2 \dots u_1^n u_2^n \dots u_m^n v_1^n v_2^n \dots v_m^n$ as a spanning tree. Denote cotree edge $v_m^n u$ by a, $u_l^j v_l^j$ by a_l^j $(1 \leq l \leq m, 1 \leq j \leq n)$, where the letters are distinct. Let joint trees of S_n^m have a clockwise rotation at each vertex. Let

$$X_1^n = B_1 B_2 \dots B_n, \qquad X_2^n = \hat{\hat{B}}_n \dots \hat{\hat{B}}_2 \hat{\hat{B}}_1,$$

where $B_l \subseteq \{a_1^l a_2^l \dots a_m^l a_1^{l-} a_2^{l-} \dots a_m^{l-}\}$ for $1 \leq l \leq n$.

So the set of associated surfaces of S_n^m is $A_{(n,\,0)}$. The set can also be classified into such sets as $A_{(n-1,\,k)}$ and $B_{(n-1,\,l)}$ of different genus. By deleting a and a^- from these sets, we get classifying sets of $B_{(n,\,0)}$.

Lemma 3.2 *The maximum and minimum genus of S_1^m $(m \geq 3)$ is equal to $[\frac{m+1}{2}]$ and 1, respectively.*

Proof. This lemma follows from Lemma 3.1. □

Lemma 3.3 $g_i^0(n) = h_0(m) g_i^0(n-1) + h_1(m) g_{i-1}^0(n-1) + \dots + h_{[\frac{m}{2}]}(m) g_{i-[\frac{m}{2}]}^0(n-1)$,

$$g_0(0) = 1, \qquad g_0(i) = 0, \qquad g_i(n) = g_{i-1}^0(n), \qquad \textit{for } n > i \geq 1, \textit{ when } m \geq 3.$$

Proof. According to Lemma 1.3, this lemma holds. □

Firstly, let $\underline{k} = (k_1, k_2, \ldots, k_{[\frac{m}{2}]-1})$, $\underline{h} = (h_2(m), h_3(m), \ldots, h_{[\frac{m}{2}](m)})$,

$$\underline{k}! = \sum_{i=1}^{[\frac{m}{2}-1]} k_i!, \quad \underline{h}^{\underline{k}} = \sum_{i=1}^{[\frac{m}{2}-1]} h_{i+1}^{k_i(m)}, \quad \underline{a} = (a_1, a_2, \ldots, a_{[\frac{m}{2}]-1}),$$

where
$$a_1 = [\tfrac{1}{2}(i - 1 - 3k_2 - 4k_3 - \cdots - [\tfrac{m}{2}]k_{[\frac{m}{2}]-1})],$$
$$a_2 = [\tfrac{i-1}{3}], \quad \ldots\ldots, \quad a_{[\frac{m}{2}]-1} = [\tfrac{i-1}{[\frac{m}{2}]}].$$

Lemma 3.4 $g_i^0(n) = \sum\limits_{0 \le \underline{k} \le \underline{a}} \dfrac{n! h_0^{b_1} h_1^{b_2} \underline{h}^{\underline{k}}}{\underline{k}! b_1! b_2!},$

where
$$b_1 = n - i + 1 + k_1 + 2k_2 + \cdots + ([\tfrac{m}{2}] - 1)k_{[\frac{m}{2}]-1},$$
$$b_2 = i - 1 - 2k_1 - 3k_2 - \cdots - [\tfrac{m}{2}]k_{[\frac{m}{2}]-1}.$$

Theorem 3.5 $f_{S_n^2}(x) = \sum\limits_{i=0}^{\infty} \dfrac{n! 2^{n-i+1} 8^{i-1} a(n,i)}{i!(n-i+1)!} x^i,$
where $a(n,i) = i4^{n-i+1} + 4n - 5i + 4$.

Proof. Firstly, by choosing a spanning tree of S_n^2 with the same method as above, we get the set of associated surfaces $A_{(n,\ 0)}$ and its classifying set $A_{(n-1,\ k)}$ and $B_{(n-1,\ l)}$. From Lemmas 1.1–1.3 and 3.1–3.2, the following equations can be obtained:

$$\begin{cases} g_i(n) = 2g_i(n-1) + 8g_{i-1}(n-1) + 6g_{i-1}^0(n-1) & (3.1) \\ g_i^0(n) = 8g_i^0(n-1) + 8g_{i-1}^0(n-1) & (3.2) \\ g_0(0) = 1 & (3.3) \\ g_0^0(0) = 1 & (3.4) \\ g_i^0(0) = 0, \quad i > 0 & (3.5) \end{cases}$$

From (3.2–3.5),

$$g_i^0(n) = \frac{n!}{i!(n-i)!} 8^n.$$

Then $g_i(n) = \dfrac{n! 2^{n-i+1} 8^{i-1}}{i!(n-i+1)!} a(n,i)$, where $a(n,i) = i4^{n-i+1} + 4n - 5i + 4$.

Thus

$$f_{S_n^2}(x) = \sum_{i=0}^{\infty} \frac{n! 2^{n-i+1} 8^{i-1} a(n,i)}{i!(n-i+1)!} x^i. \quad \square$$

Through the discussion above, we can get the following theorem.

Theorem 3.6 $f_{S_n^m}(x) = \sum\limits_{i=0}^{\infty} g_i(n) x^i, (m \ge 3)$

where $g_0(n) = \begin{cases} 1, & n = 0; \\ 0, & n > 0. \end{cases},$

$$g_i(n) = \begin{cases} f_i(n) + g^0_{i-1}(n), & n \le i; \\ g^0_{i-1}(n), & n > i. \end{cases} \quad \text{for } i \ge 1,$$

$$f_i(n) = \sum_{k=0}^{i-n} \alpha^{(n-3)}_{i+1-n-k}(a_{k+1} - a_k),$$

$$\alpha^{(i)}_k = \sum_{j=1}^{k} a_j \alpha^{(i-1)}_{k+1-j}, \quad \alpha^{-2}_j = 1, \ \alpha^{-1}_j = a_j, \ a_0 = 0.$$

In the following, choose a spanning tree and obtain the set of associated surfaces and its classifying set of S^i_n $(3 \le i \le 5)$ in the same way as Theorem 3.5. For brevity, in the course of proofs of Corollaries 3.7–3.9, we only give some equations that $g_i(n)$ satisfies.

Corollary 3.7 $f_{S^3_n}(x) = \sum_{i=0}^{\infty} g_i(n)x^i$, *where*

$$g_i(n) = \begin{cases} 0, & 0 \le n \le i-2; \\ 56^{i-1} - 32^{i-1}, & n = i-1; \\ 32^i + 8\dot{5}6^{i-1}i, & n = i; \\ \dfrac{n! 56^{i-1} 8^{n-i+1}}{(i-1)!(n-i+1)!}, & n > i. \end{cases}$$

Proof. The embedding genus distribution $g_i(n)$ of S^3_n satisfies the following equations:

$$\begin{cases} g_i(n) = 32 g_i(n-1) + 8 g^0_{i-1}(n-1) + 24 g^0_{i-2}(n-1) \\ g^0_i(n) = 8 g^0_i(n-1) + 56 g^0_{i-1}(n-1) \\ g_0(0) = 1 \\ g^0_0(0) = 1 \\ g^0_i(0) = 0, \quad i > 0 \end{cases}$$

□

Corollary 3.8 $f_{S^4_n}(x) = \sum_{i=0}^{\infty} g_i(n)x^i$, *where*

$$g_i(n) = \begin{cases} 0, & n < i - [\frac{i}{2}]; \\ \dfrac{(240n - 145i + 95)n!}{(i-n)!(2n-i+1)!} 50^{2n-i} 95^{i-n-1} + g^0_{i-1}(n), & i - [\frac{i}{2}] \le n < i; \\ 50^i + g^0_{i-1}(i), & n = i; \\ g^0_{i-1}(n), & n > i. \end{cases}$$

$$g_i^0(n) = \sum_{k=0}^{[\frac{i}{2}]} \frac{n!95^k 153^{i-2k} 8^{n-i+k}}{k!(n-i+k)!(i-2k)!}.$$

Proof. The embedding genus distribution $g_i(n)$ of S_n^4 satisfies the following equations:

$$\begin{cases} g_i(n) = 50g_{i-1}(n-1) + 95g_{i-2}(n-1) + 8g_{i-1}^0(n-1) + 103g_{i-2}^0(n-1) \\ g_i^0(n) = 8g_i^0(n-1) + 153g_{i-1}^0(n-1) + 95g_{i-2}^0(n-1) \\ g_0(0) = 1 \\ g_0^0(0) = 1 \\ g_i^0(0) = 0, \quad i > 0. \end{cases}$$

□

Corollary 3.9 $f_{S_n^5}(x) = \sum_{i=0}^{\infty} g_i(n) x^i$, *where*

$$g_i(n) = \begin{cases} 0, & n < i - [\frac{i+1}{2}]; \\ \frac{(876n-470i+406)n!}{(i-n)!(2n-i+1)!} 50^{2n-i} 95^{i-n-1} + g_{i-1}^0(n), & i - [\frac{i+1}{2}] \le n < i; \\ 64^i + g_{i-1}^0(i), & n = i; \\ g_{i-1}^0(n), & n > i. \end{cases}$$

$$g_i^0(n) = \sum_{k=0}^{[\frac{i}{2}]} \frac{n!728^k 288^{i-2k} 8^{n-i+k}}{k!(n-i+k)!(i-2k)!}.$$

Proof. The embedding genus distribution $g_i(n)$ of S_n^5 satisfies the following equations:

$$\begin{cases} g_i(n) = 64g_{i-1}(n-1) + 406g_{i-2}(n-1) + 8g_{i-1}^0(n-1) \\ \qquad\qquad + 224g_{i-2}^0(n-1) + 322g_{i-3}^0(n-1) \\ g_i^0(n) = 8g_i^0(n-1) + 288g_{i-1}^0(n-1) + 728g_{i-2}^0(n-1) \\ g_0(0) = 1 \\ g_0^0(0) = 1 \\ g_i^0(0) = 0, \quad i > 0. \end{cases}$$

□

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References

[1] J. Chen, J.L. Gross and R.G. Rieper, Overlap matrices and total imbedding distributions,*Discrete Math.* **128** (1994), 73–94.

[2] M.L. Furst, J.L. Gross and R. Statman, Genus distribution for two classes of graphs, *J. Combin. Theory Ser. B* **46** (1989), 22–36.

[3] J.L. Gross and M.L. Furst, A hierarchy for imbedding distribution invariants of a graph, *J. Graph Theory* **11** (1987), 205–220.

[4] J.L. Gross, E.W. Klein and R.G. Rieper, On the average genus of a graph, *Graphs Combin.* **9** (1993), 153–162.

[5] J.L. Gross, D.P. Robbins and T.W. Tucker, Genus distribution for bouquets of circles, *J. Combin. Theory Ser. B* **47** (1989), 292–306.

[6] D.M. Jackson, Counting cycles in permutations by group characters, with an application to a topological problem, *Trans. Amer. Math. Soc.* **299** (1987), 785–801.

[7] L.F. Li and Y.P. Liu, Genus distribution of embeddings for a typed of 3-regular graphs (in Chinese), *J. Beijing Jiaotong University* **28** (2004), 39–42.

[8] Y.P. Liu, *Advances in Combinatorial Maps* (in Chinese), Beijing: Northern Jiaotong University Press, 2003.

[9] Y.P. Liu, *Algebraic Principles of Maps* (in Chinese), Beijing: Higher Education Press, 2006.

[10] Y.P. Liu, *Embeddability in Graphs*, Dordrecht/Boston/London: Kluwer, Academic Publisher, 1995.

[11] W.S. Massey, *Algebraic Topology: An Introduction*, Springer-Verlag, New York, 1967.

[12] L.X. Wan and Y.P. Liu, Genus distribution of the orientable embeddings for a type of graphs (in Chinese), *J. Beijing Jiaotong University* **29** (2005), 65–68.

[13] X.M. Zhao and Y.P. Liu, Genus polynomial for tree-like graph (in Chinese), *J. Beijing Jiaotong University* **28** (2004), 7–11.

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