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Skolem-labeling of generalized three-vane windmills

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Abstract

A graph on $2n$ vertices can be Skolem-labeled if the vertices can be given labels from $\{1, \ldots, n\}$ such that each label i is assigned to exactly two vertices and these vertices are at distance i. Mendelsohn and Shalaby have characterized the Skolem-labeled paths, cycles and windmills (of fixed vane length). In this paper, we obtain necessary conditions for the Skolem-labeling of generalized k-windmills in which the vanes may be of different length. We show that these conditions are sufficient in the case where $k = 3$ and conjecture that any generalized k-windmill, $k > 3$, can be Skolem-labeled if and only if it satisfies these necessary conditions.

1 Introduction

Skolem-type sequences are integer sequences which contain two occurrences of each distinct entry, n, located n positions apart. These sequences have well-known connections with Steiner triple systems and with solutions to Heffter's difference problem.

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In 1991, Mendelsohn and Shalaby [5] generalized this idea to graphs and noted that the Skolem-labeling of a graph could be used to design schemes for testing a communications network for node, link and distance reliability. In essence, a Skolem-labeled graph is a higher dimensional analogue of a Skolem sequence. Each label, n, is an integer which is used to label two vertices located at distance n. They also provided a characterization of the paths and cycles that can be Skolem-labeled. In [2], Baker, Bonato and Kergin approached the problem from the opposite direction and considered a two-dimensional analogue of a Skolem sequence. In doing so, they actually provided necessary and sufficient conditions for the existence of a Skolem-labeling of a $2 \times n$ ladder graph.

In [6], Mendelsohn and Shalaby extended this work to k-windmills; i.e., trees with k disjoint paths of equal length emanating from a central vertex. They showed that k must equal 3 and that the 3-windmills that can be Skolem-labeled are precisely those that meet a particular parity condition. One obvious generalization is to the more realistic situation of generalized k-windmills, where the vanes need not be of the same length. Once this length restriction is removed, there are generalized k-windmills which can be Skolem-labeled for each value k.

In this paper, we explore the parity and nondegeneracy conditions which are necessary for the Skolem-labeling of generalized k-windmills. We then prove that in the case of generalized 3-windmills, these conditions are also sufficient.

2 Skolem-type Sequences

2.1 Definitions and Existence Results

Skolem and other related sequences are tools used in the Skolem-labeling of graphs, so we provide a list of definitions and existence results.

A *Skolem-type sequence* is a sequence $(s_i)_{i \in I}$ of integers from a set J with the Skolem property:

$$\text{for every } j \in J, \text{ there exists a unique } i \in I \text{ such that } s_i = s_{i+j} = j.$$

For a *Skolem sequence of order n*, denoted $\mathcal{S}_n$, $J = \{1, \ldots, n\}$ and $I = \{1, \ldots, 2n\}$. Such a sequence exists if and only if $n \equiv 0, 1 \pmod 4$ [10].

For a *k-extended Skolem sequence of order n* , denoted k-ext $\mathcal{S}_n$, which has an empty space (called a *hook* or *zero*) in position k, $J = \{1, \ldots, n\}$ and $I = \{1, \ldots, 2n+1\} \setminus \{k\}$. Such a sequence exists [1], [7] if and only if either:

$$k \text{ is odd and } n \equiv 0, 1 \pmod 4), \text{ or } k \text{ is even and } n \equiv 2, 3 \pmod 4).$$

A *hooked Skolem sequence*, $h\mathcal{S}_n$, is just a $2n$-extended Skolem sequence.

The sequence is an *m-near Skolem [hooked Skolem] sequence of order n*, denoted m-near $\mathcal{S}_n$ [m-near $h\mathcal{S}_n$] if $J = \{1, \ldots, n\} \setminus \{m\}$. An m-near Skolem sequence of order n exists [8] if and only if either:

$$m \text{ is odd and } n \equiv 0, 1 \pmod 4, \text{ or } m \text{ is even and } n \equiv 2, 3 \pmod 4.$$

For an m-near hooked Skolem sequence, the parity of m above is reversed.

If $J = \{d, \ldots, m+d-1\}$, the sequence is a *[hooked] Langford sequence of length m and defect d*, $\mathcal{L}_d^m$ [$h\mathcal{L}_d^m$]. A Langford sequence of length m and defect d exists [9] if and only if

1) $m \geq 2d - 1$ (*the size constraint*) and

2) $m \equiv 0, 1 \pmod 4$ for d odd or $m \equiv 0, 3 \pmod 4$ for d even.

A hooked Langford sequence of length m and defect d exists [9] if and only if

1) $m(m + 1 - 2d) + 2 \geq 0$ and

2) $m \equiv 2, 3 \pmod 4$ for d odd or $m \equiv 1, 2 \pmod 4$ for d even.

A *k-extended Langford sequence*, k-ext $\mathcal{L}_d^m$, is defined in the obvious way. The following conditions are necessary for the existence of a k-ext $\mathcal{L}_d^m$ [4]:

1) $m \geq 2d - 3$ and $m(2d - 1 - m)/2 + 1 \leq k \leq m(m - 2d + 5)/2 + 1$

2) $(m, k) \equiv (0, 1), (1, d), (2, 0), (3, d + 1) \pmod{(4, 2)}$.

These conditions are sufficient for small defects, $d = 1, 2, 3, 4$, or $d \leq (m + 4)/8$ and for large defects $d = (m + 1)/2, m/2, (m - 1)/2$ [3], [4].

2.2 A useful symmetric Langford sequence

Define $\mathcal{A}_d^{2d-1}$ to be the sequence with:

i in positions i and $2i$, for $i = d, d + 1, \ldots, 2d - 1$, and

$2d + i$ in positions $1 + i$ and $2d + 2i + 1$, for $i = 0, 1, \ldots, d - 2$.

For example, $\mathcal{A}_3^5$ is the sequence 6 7 3 4 5 3 6 4 7 5.

This sequence has some interesting properties.

1) Each of the entries, $d, \ldots, 3d - 2$ occurs once in the first half of the sequence and once in the second. In fact, a Langford sequence, $\mathcal{L}_d^m$, can only have this symmetric property if $m = 2d - 1$. To see this, note that an entry j occurs in positions a_j in the first half of the sequence and $a_j + j$ in the second half, so

$$m(m + 2d - 1)/2 \quad = \quad \sum_{j=d}^{m+d-1} j$$

$$\begin{aligned} &= \sum_{j=d}^{m}(a_j+j) - \sum_{j=d}^{m} a_j \\ &= \sum_{i=m+1}^{2m} i - \sum_{i=1}^{m} i \\ &= m^2 \end{aligned}$$

2) The second occurrence of an entry $p \in \{d, d+1, \ldots, 2d-1\}$ can be moved to the beginning of the sequence to create a $(2p+1)$-ext $\mathcal{L}_d^{2d-1}$. The reverse of this sequence is a $(4d-2p-1)$-ext $\mathcal{L}_d^{2d-1}$.

3) This sequence can be used to create a new sequence with multiple holes in the middle by adding a fixed $k \in \mathbf{N}$ to each entry and inserting k holes in the middle. This sequence is denoted by $\mathcal{A}_d^{2d-1} + k$.

For example, 67345|36475

46734536-75 is a 9-ext $\mathcal{L}_3^5$, 57-63543764 is a 3-ext $\mathcal{L}_3^5$

$\mathcal{A}_3^5 + 2$ is 89567 $- -$ 58697.

Property 3) will be extremely useful in some of the labeling techniques that follow.

3 Skolem-labeled windmills

A *k-windmill* is a tree consisting of k paths of equal positive length, called *vanes*, which meet at a central vertex called the *pivot*. For clarity, we will often refer to these windmills as *ordinary windmills*.

A *generalized k-windmill* (*gk*-windmill) is a windmill in which the k vanes may be of different positive lengths.

A graph on $2n$ vertices can be (*weakly*) *Skolem-labeled* if each of the vertices can be assigned a label from the set $J = \{1, \ldots, n\}$ such that exactly two vertices at distance j are labeled j, for each $j \in J$. The Skolem-labeling is *strong* if the removal of any edge destroys the Skolem-labeling, see the figures below.

Figure 1: A Skolem-labeling that is not strong.

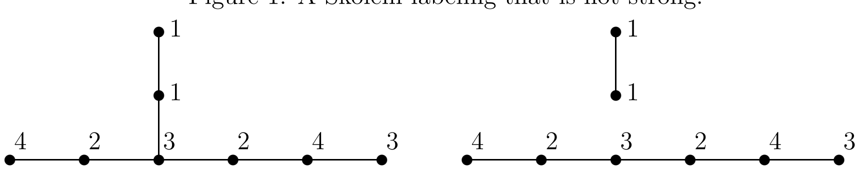
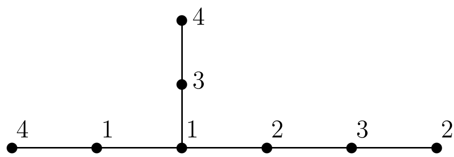


Figure 2: A strong Skolem-labeling for the same graph.



3.1 Elementary properties

A gk-windmill, which must contain at least $k+1$ vertices, can only be Skolem-labeled if $|V|$ is even. In addition, in order to use the label n, there must be a path of length at least n. (This is the part of the Degeneracy Condition of [6] that applies to the gk-windmills.)

For $g3$- and $g4$-windmills, this will always be the case as the path along the longest two vanes is of length at least $\lceil 2(2n-1)/4 \rceil \geq n$.

An ordinary (i.e., not generalized) k-windmill can only be Skolem-labeled if $(2n-1)/k$ is an integer and if the length of the longest path $2(2n-1)/k$ is greater than or equal to n. So only 3-windmills can be Skolem-labeled.

3.2 Skolem parity

In [6], the authors defined the following Skolem parity condition and showed that it was necessary for the existence of a Skolem-labeling of any tree.

The *Skolem parity* of a vertex u of a tree $T = (V, E)$ is

$$\sum_{v \in V} d(u, v) \pmod 2,$$

where $d(u, v)$ is the length of the path from u to v.

Lemma 1 [6] *If T is a tree on $2n$ vertices, then the Skolem parity is independent of the choice of vertex u.*

Lemma 2 (Skolem parity condition) [6] *If T is a Skolem-labeled tree on $2n$ vertices, then either*

1) the Skolem parity of T is even and $n \equiv 0, 3 \pmod 4$ or

2) the Skolem parity of T is odd and $n \equiv 1, 2 \pmod 4$.

In the case of gk-windmills, the Skolem parity condition reduces to the following simple condition.

Theorem 3 *If G is a Skolem-labeled gk-windmill with $2n$ vertices and k vanes, m of which are of odd length, then either:*

1) $n \equiv 0, 1$ (mod 4) and $m \equiv 1$ (mod 4) or

2) $n \equiv 2, 3$ (mod 4) and $m \equiv 3$ (mod 4).

Proof. Suppose $G = (V, E)$ is a Skolem-labeled gk-windmill with vanes of length $x_1, \ldots, x_k$. Using the pivot p to calculate the Skolem parity, we obtain

$$\begin{aligned} \sum_{v \in V} d(p, v) &= \sum_{i=1}^{k} x_i(x_i + 1)/2 \\ &= 1/2[\sum x_i^2 + \sum x_i] \\ &= 1/2[\sum x_i^2 + (2n - 1)] \\ &= 1/2[\sum x_i^2 - 1] + n \end{aligned}$$

Since this is an integer, the number of odd vanes must be odd. Then by Lemma 2,

number of odd vanes $\equiv 1$ (mod 4) $\Longleftrightarrow \sum x_i^2 - 1 \equiv 0$ (mod 4) $\Longleftrightarrow n \equiv 0, 1$ (mod 4)

number of odd vanes $\equiv 3$ (mod 4) $\Longleftrightarrow \sum x_i^2 - 1 \equiv 2$ (mod 4) $\Longleftrightarrow n \equiv 2, 3$ (mod 4).

Therefore, an ordinary k-windmill, G can only be Skolem-labeled if its $k = 3$ equal vanes all have odd length $m = (2n-1)/3$. Hence $n \equiv 2, 3$ (mod 4) and $2n \equiv 1$ (mod 3), so $2n \equiv 4, 22$ (mod 24) and $m \equiv 1, 7$ (mod 8) as in [6].

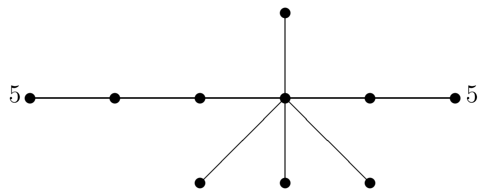
3.3 Nondegeneracy condition

In general, the conditions that we have identified above are not sufficient to guarantee that a gk-windmill can be Skolem-labeled. Although having a path of length at least n guarantees that the label n can be placed, it does not guarantee that $n-1$ can also be placed. The graph given below meets the Skolem parity condition since $n = 5 = m$ and it contains a path of length $n = 5$; however, it cannot be Skolem-labeled, so an additional condition is required.

Theorem 4 (Nondegeneracy condition) *If G is a Skolem-labeled gk-windmill with $2n$ vertices and vanes of length $x_1, \ldots, x_k$, then*

$$n(n+1) \leq \sum_{i=1}^{k} x_i(x_i + 1).$$

Figure 3: A graph in which 4 cannot be placed.



Proof. Let $G = (V, E)$ be a Skolem-labeled gk-windmill with $2n$ vertices, vanes $y_1, \ldots, y_k$ of length $x_1, \ldots, x_k$, respectively, and pivot p. Each vertex $v \neq p$ can be denoted by a pair (i, j) where v is on vane y_i and $j = d(v, p)$. Let p be denoted by $(0, 0)$.

Since G is Skolem-labeled, each element $m \in \{1, \ldots, n\}$ is associated with 2 vertices (i, j), (i', j') where $d((i, j), (i', j')) = m$. Then

$$m = \left\{ \begin{array}{ll} j + j' & \text{if } i \neq i' \\ \mid j - j' \mid & \text{if } i = i'. \end{array} \right\}$$

Summing over all the labels, we obtain

$$n(n+1)/2 = \sum_{m=1}^{n} m = \sum_{i \neq i'} (j + j') + \sum_{i = i'} \mid j - j' \mid \leq \sum_{m=1}^{n} (j + j').$$

Since this last sum is just the sum of the distances from each of the vertices to the pivot, we could calculate this vane-by-vane, so

$$n(n+1)/2 \leq \sum_{m=1}^{n} (j + j') = \sum_{i=1}^{k} x_i(x_i + 1)/2.$$

Theorem 5 *Any $g3$- or $g4$-windmill satisfies the nondegeneracy condition.*

Proof. Let G be a gk-windmill with $2n$ vertices and vanes of length $x_1, \ldots, x_k$. Since

$$\sum_{i=1}^{k} x_i(x_i + 1)/2 = \sum_{i=1}^{k} \sum_{j=1}^{x_i} j,$$

and

$$\sum_{j=1}^{x_k} j + \sum_{j=1}^{x_t} j \leq \sum_{j=1}^{x_k - 1} j + \sum_{j=1}^{x_t + 1} j, \text{ if } x_k \leq x_t,$$

$\sum_{i=1}^{k} x_i(x_i + 1)/2$ attains a minimum when the vertices are as evenly distributed among the vanes as possible.

If $k = 3$, n must be at least 2 and

$$\sum_{i=1}^{3} x_i(x_i+1) \geq 3(\frac{2n-1}{3})(\frac{2n-1}{3}+1) \geq n(n+1).$$

If $k = 4$, $\frac{2n-1}{4}$ is never an integer. If $n = 2s$, the most even distribution of the vertices would be $s, s, s, s-1$; if $n = 2s+1$, it is $s+1, s, s, s$. Hence, in each of these cases,

$$\sum_{i=1}^{k} x_i(x_i+1) \geq n(n+1).$$

Remark 1 Once $k > 4$, however, the nondegeneracy condition is not automatically satisfied. A $g5$-windmill with vanes of lengths $2, 2, 2, 2, 1$ fails the nondegeneracy condition as does the $g6$-windmill illustrated above.

Remark 2 Note that $n(n+1) = \sum_{i=1}^{k} x_i(x_i+1)$ only when no label appears twice on the same vane. This implies that 1 must be used to label the pivot plus one adjacent vertex and the two 2's must straddle the pivot, so the only $g3$-windmill of this type is the ordinary 3-windmill with vanes of length 1.

In the remainder of the paper, we show that every $g3$-windmill that satisfies the Skolem parity condition can be Skolem-labeled. We also make the following conjecture.

Conjecture 1 *Any gk-windmill that satisfies the Skolem parity and nondegeneracy conditions can be Skolem-labeled.*

4 Labeling techniques for $g3$-windmills

Let $G = W(n : x, y, z)$ be a generalized 3-windmill, on $2n$ vertices, with vanes X, containing x vertices, Y containing y and Z containing z vertices, where $x \geq y \geq z$. Then

$$2n = x + y + z + 1.$$

For ease in identifying the vertices, we place the graph on a grid and use the following coordinate system:

X contains vertices $(1, z+1)$ to $(x, z+1)$,

Y contains vertices $(x+2, z+1)$ to $(x+1+y, z+1)$,

Z contains vertices $(x+1, 1)$ to $(x+1, z)$

p, the pivot, is located at $(x+1, z+1)$.

4.1 Pruning

Let G be a generalized windmill. If we can use the largest labels to label the vertices at the extreme ends of two vanes, we can reduce the problem to finding a Skolem labeling for a smaller tree. In essence, we will have pruned the original tree. In this section, we define a pruning algorithm that works for $g3$-windmills. We note that variations of the pruning algorithm work for other gk-windmills.

Let G be a $g3$-windmill on $2n$ vertices with $x < n$. Note that $x \geq \frac{2n-1}{3}$. Define $d = n - x$ and construct $\mathcal{A}_d^{2d-1}$. This sequence has largest entry $3d - 2$. Since the largest label to be used is n, define $k = n - 3d + 2$, which is greater than zero as $x \geq \frac{2n-1}{3}$. The sequence $\mathcal{A}_d^{2d-1} + k$ has length $2(2d-1) + k = 4d - 2 + n - 3d + 2 = 2n - x = y + z + 1$ and contains entries $d + k = n - 2d + 2, \ldots, n$ which are placed in the $2d - 1$ positions at either end of the sequence. The middle k positions are empty. If we use this sequence to label the path consisting of Y, the pivot and Z, then the last $2d - 1$ positions of Y and Z will be labeled and we are left with a tree on $2n - 2(2d-1) = 3x - y - z + 1$ vertices. Note that the pivot is never labeled in this procedure since $2d-1 = 2n-2x-1 = x+y+z+1-2x-1 = y+z-x = z-(x-y) \leq z$, so we are left with either a $g3$-windmill or a path.

Example 1 Let $G = W(12 : 9, 8, 6)$. Then $d = 3$ and $k = 5$. We use the sequence, $\mathcal{A}_3^5 + 5$, which is

$$11\ 12\ 8\ 9\ 10 - - - - - 8\ 11\ 9\ 12\ 10,$$

to assign labels to the 5 vertices at the ends of the YZ-path. Once we remove these vertices the resulting graph is $W(7 : 9, 3, 1)$.

Theorem 6 *Let G be a $g3$-windmill on $2n$ vertices, with $x < n$, and G' be the tree produced by pruning G. Then G satisfies the Skolem parity condition if and only if G' is either a $g3$-windmill which satisfies the Skolem parity condition or a path which can be Skolem-labeled.*

Proof. Let G be a $g3$-windmill on $2n$ vertices with $x < n$ and G' the tree produced by pruning G. Then G' contains $2n' = 4x - 2n + 2$ vertices arranged on vanes of length $x' = x, y' = y - 2d + 1$ and $z' = z - 2d + 1$. Note that y' and z' have the opposite parity to y and z. This tree will be a $g3$-windmill unless $z = 2d - 1$.

Suppose that G satisfies the Skolem parity condition.

If $n \equiv 2$ or $3 \pmod 4$, then x, y and z are all odd. After pruning, only x' is odd and $n' = 2x' - n + 1 \equiv 1$ or $0 \pmod 4$, respectively. Then if $z' > 0$, G' is a $g3$-windmill which satisfies the Skolem parity condition. If $z' = 0$, then G' can be labeled by a Skolem sequence of order n'.

If $n \equiv 0$ or $1 \pmod 4$ and x is odd, then y and z are even. After pruning, x', y' and z' are all odd and $n' = 2x' - n + 1 \equiv 3$ or $2 \pmod 4$, respectively, so G' is a $g3$-windmill which satisfies the Skolem parity condition.

If $n \equiv 0$ or $1 \pmod 4$ and x is even, then one of y and z is even and the other is odd. After pruning, x' will still be even, as will exactly one of y' and z', and $n' = 2x' - n + 1 \equiv 1$ or $0 \pmod 4$, respectively. If $z' > 0$, then G' is a $g3$-windmill which satisfies the Skolem parity condition. If $z' = 0$, then G' can be labeled by a Skolem sequence of order n'.

Now suppose that G' is a $g3$-windmill which satisfies the Skolem parity condition.

If $n' \equiv 2$ or $3 \pmod 4$, then x', y' and z' are all odd, so x is odd and y and z are even. Then $n = 2x - n' + 1 \equiv 1$ or $0 \pmod 4$, respectively. Hence, G satisfies the Skolem parity condition. If $n' \equiv 0$ or $1 \pmod 4$ and x' is odd, then y' and z' are both even and x, y, z are all odd, so $n = 2x - n' + 1 \equiv 3$ or $2 \pmod 4$, respectively. Hence, G satisfies the Skolem parity condition.

If $n' \equiv 0$ or $1 \pmod 4$ and x' is even, then exactly one of y' and z' is even and the other is odd, so x is even and exactly one of y and z is even and the other odd. Then $n = 2x - n' + 1 \equiv 1$ or $0 \pmod 4$, respectively, and G satisfies the Skolem parity condition.

Finally, suppose that G' is a path (so $z = 2d - 1$) which can be Skolem-labeled. So $n' \equiv 0$ or $1 \pmod 4$. Since $2n' = x' + y' + 1$, exactly one of x' and y' must be odd. If x' is odd, then $y = y' + z$ and z are also both odd and $n = 2x - n' + 1 \equiv 3$ or $2 \pmod 4$, respectively. If y' is odd, then $y = y' + z$ is even, x is even, z is odd and $n = 2x - n' + 1 \equiv 1$ or $0 \pmod 4$. Hence G satisfies the Skolem parity condition.

Remark 3 Since a $g3$-windmill can only be pruned if $n > x$, a $g3$-windmill cannot be pruned more than once. After the pruning, $n' = 2x - n + 1 = x - (n - x - 1) \le x = x'$.

4.2 Direct labeling techniques

Let $G = W(n : x, y, z)$ be a $g3$-windmill which satisfies the Skolem parity condition. Then G has exactly one vane of odd length if $n \equiv 0, 1 (mod\ 4)$ and three vanes of odd length if $n \equiv 2, 3 (mod\ 4)$. We provide a number of labeling techniques.

4.2.1 $n \equiv 0, 1 \pmod 4$, z even

In this group, a [near] Skolem sequence is used to label Z, while a [hooked] Langford sequence, [plus the omitted labels from the near Skolem sequence], are used on the XY-path.

a) $n \equiv 0, 1 \pmod 4$, $z \equiv 0, 2 \pmod 8$.

Place a $\mathcal{L}_{(z+2)/2}^{(x+y+1)/2}$ on the XY-path and a $\mathcal{S}_{z/2}$ on Z.

Since $z/2 \equiv 0, 1 \pmod 4$, this Skolem sequence clearly exists, so we need only verify that the Langford sequence exists. Since $n \equiv 0, 1 \pmod 4$, G has exactly one vane, X

or Y, of odd length. In either case, $x \geq y+1 \geq z+1$, which implies $x+y+1 \geq 2z+2$, so the size constraint is satisfied. If $z \equiv 0 \pmod 8$,

$$(z+2)/2 \text{ is odd and } (x+y+1)/2 = (2n-z)/2 \equiv 0 \text{ or } 1 \pmod 4;$$

if $z \equiv 2 \pmod 8$, then

$$(z+2)/2 \text{ is even and } (x+y+1)/2 \equiv 3 \text{ or } 0 \pmod 4.$$

b) $n \equiv 0, 1 \pmod 4$, $z \equiv 2, 4 \pmod 8$, $(2n-8)/3 \geq z$.

Place a $h\mathcal{L}_{(z+4)/2}^{(x+y-1)/2}$ on the XY-path, leaving the vertices $(x-1+y, z+1)$ and $(x+1+y, z+1)$ at the end of Y unlabeled. Label these two vertices 2. Place a 2-near $\mathcal{S}_{(z+2)/2}$ on Z.

Since $(2n-8)/3 \geq z$,

$$(x+y-1)/2 = (2n-z-2)/2 \geq (3z+8-z-2)/2 = z+3 = 2[(z+4)/2]-1.$$

If $z \equiv 2 \pmod 8$, then

$$(z+4)/2 \text{ is odd and } (x+y-1)/2 = (2n-z-2)/2 \equiv 2 \text{ or } 3 \pmod 4;$$

if $z \equiv 4 \pmod 8$, then

$$(z+4)/2 \text{ is even and } (x+y-1)/2 \equiv 1 \text{ or } 2 \pmod 4.$$

c) $n \equiv 0, 1 \pmod 4$, $z \equiv 0, 6 \pmod 8$, $(2n-8)/3 \geq z$.

Place a $\mathcal{L}_{(z+4)/2}^{(x+y-1)/2}$ on the XY-path, leaving the last two vertices of Y unlabeled. Label these vertices 1. Place a 1-near $\mathcal{S}_{(z+2)/2}$ on Z.

As in construction **b**, $(x+y-1)/2 \geq z+3$. If $z \equiv 0 \pmod 8$,

$$(z+4)/2 \text{ is even and } (x+y-1)/2 = (2n-z-2)/2 \equiv 3 \text{ or } 0 \pmod 4;$$

if $z \equiv 6 \pmod 8$,

$$(z+4)/2 \text{ is odd and } (x+y-1)/2 \equiv 0 \text{ or } 1 \pmod 4.$$

4.2.2 $n \equiv 0, 1 \pmod 4$, y even

This is similar to 4.2.1 above except that the [near] Skolem sequence is placed on Y. Since y is even, either x or z must be odd. Existence of the given sequences is verified as in 4.2.1.

a) $n \equiv 0, 1 \pmod 4$, $y \equiv 0, 2 \pmod 8$, $(2n-2)/3 \geq y$.

Place a $\mathcal{L}_{(y+2)/2}^{(x+z+1)/2}$ on the XZ-path and a $\mathcal{S}_{y/2}$ on Y.

b) $n \equiv 0, 1 \pmod 4$, $y \equiv 2, 4 \pmod 8$, $(2n-8)/3 \geq y$.

Place a $h\mathcal{L}_{(y+4)/2}^{(x+z-1)/2}$ on the XZ-path leaving the vertices $(x+1, 3)$ and $(x+1, 1)$ unlabeled. Label them 2. Put a 2-near $\mathcal{S}_{(y+2)/2}$ on Y.

c) $n \equiv 0, 1 \pmod 4$, $y \equiv 0, 6 \pmod 8$, $(2n-8)/3 \geq y$.

Place a $\mathcal{L}_{(y+4)/2}^{(x+z-1)/2}$ on the XZ-path leaving vertices $(x+1, 2)$ and $(x+1, 1)$ unlabeled. Label these vertices 1. Put a 1-near $\mathcal{S}_{(y+2)/2}$ on Y.

4.2.3 Long X-vanes

A [hooked] Langford sequence is used to label the long X-vane plus one or two additional vertices and the remaining vertices are covered by an extended Skolem sequence.

a) $n \equiv 2, 3 \pmod 4$, $y+z \equiv 4, 6 \pmod 8$, $x \geq (4n-1)/3$.

Place a $\mathcal{L}_{(y+z+2)/2}^{(n-((y+z+2)/2)+1)}$ on X and the pivot and a $(z+1)$-ext $\mathcal{S}_{(z+y)/2}$ along the ZY-path.

Since $n-(y+z+2)/2+1 = (x+1)/2$, we have $(x+1)/2 \geq y+z+1$ whenever $x \geq (4n-1)/3$. If $y+z \equiv 4 \pmod 8$, then

$$(y+z)/2 \equiv 2 \pmod 4,\ (y+z+2)/2 \text{ is odd and}$$

$$n-(y+z+2)/2+1 \equiv 0 \text{ or } 1 \pmod 4.$$

If $y+z \equiv 6 \pmod 8$, then

$$(y+z)/2 \equiv 3 \pmod 4,\ (y+z+2)/2 \text{ is even and}$$

$$n-(y+z+2)/2+1 \equiv 3 \text{ or } 0 \pmod 4.$$

b) $n \equiv 2, 3 \pmod 4$, $y+z \equiv 0, 2 \pmod 8$, $x \geq (4n-1)/3$.

Place a $h\mathcal{L}_{(y+z+2)/2}^{(n-((y+z+2)/2)+1)}$ on X plus the pivot and vertex $(x+2, z+1)$ of Y and a $(z+2)$-ext $\mathcal{S}_{(z+y)/2}$ along the ZY-path.

If $y+z \equiv 0 \pmod 8$, then

$$(y+z)/2 \equiv 0 \pmod 4,\ (y+z+2)/2 \text{ is odd and}$$

$$n-(y+z+2)/2+1 \equiv 2 \text{ or } 3 \pmod 4.$$

If $y+z \equiv 2 \pmod 8$, then

$$(y+z)/2 \equiv 1 \pmod 4,\ (y+z+2)/2 \text{ is even and}$$

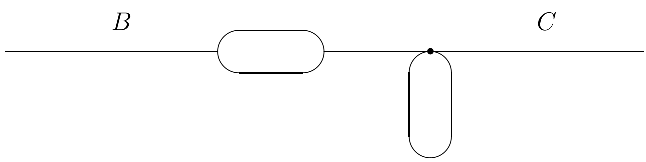
$$n-(y+z+2)/2+1 \equiv 1 \text{ or } 2 \pmod 4.$$

4.2.4 Short Z-vanes

In this construction, we label windmills with relatively short Z-vanes by using: $\mathcal{A}_d^{2d-1} + (n-3d+2)$, for a suitable choice of d given below, to label [most of] Z plus a block of vertices near the middle of X with labels $n-2d+2$ to n, inclusive.

a) $n \equiv 0, 3 \pmod 4$ and $z \equiv 3 \pmod 4$ *or* $n \equiv 1, 2 \pmod 4$ and $z \equiv 1 \pmod 4$.

Let $d = \frac{z+1}{2}$. $\mathcal{A}_d^{2d-1} + (n-3d+2)$ can be used to label Z and some vertices on X, see ovals in the diagram. There are two remaining paths denoted by B and C, see the figure below.



The path labeled B contains

$$x - (2d-1) - (n-3d+1) = x + d - n = \frac{x-y}{2} \text{ vertices.}$$

If $n \equiv 0 \pmod 4$ and $z \equiv 3 \pmod 4$, then x must be even, so $x + d - n$ is even and $x - y \equiv 0 \pmod 4$. This holds in each case.

If $(x-y)/4 \equiv 0$ or $1 \pmod 4$, then $\mathcal{S}_{\frac{x-y}{4}}$ exists and can be used to label the $(x-y)/2$ vertices of B. The path C contains

$$x + y + 1 - z - \left(\frac{x-y}{2}\right) = 2n - 2z - \left(\frac{x-y}{2}\right) \text{vertices}$$

which can be labeled using a $\mathcal{L}_{\frac{x-y}{4}+1}^{n-z-\left(\frac{x-y}{4}\right)}$. This sequence exists for all cases of n and z under consideration provided that

$$\begin{aligned} 2\left(\frac{x-y}{4}+1\right) - 1 &\le n - z - \left(\frac{x-y}{4}\right) \\ \Longleftrightarrow 2x - 2y + 8 - 4 &\le 4n - 4z - x + y \\ \Longleftrightarrow 3x - 3(2n - x - z - 1) + 4z + 4 &\le 4n \\ \Longleftrightarrow 6x + 7z + 7 &\le 10n. \end{aligned}$$

A similar discussion can be used if $(x-y)/4 \equiv 2$ or $3 \pmod 4$. The results are summarized in the following table.

$(x-y)/4$ (mod 4)	B	C	size constraint
0, 1	$\mathcal{S}_{\frac{x-y}{4}}$	$\mathcal{L}_{\frac{x-y}{4}+1}^{n-z-\left(\frac{x-y}{4}\right)}$	$10n \geq 6x+7z+7$
3	1-near $\mathcal{S}_{\frac{x-y}{4}+1}$	1 1 then $\mathcal{L}_{\frac{x-y}{4}+2}^{n-z-\left(\frac{x-y}{4}\right)-1}$	$10n \geq 6x+7z+19$
2	2-near $\mathcal{S}_{\frac{x-y}{4}+1}$	2−2 hooked into $h\mathcal{L}_{\frac{x-y}{2}+2}^{n-z-\left(\frac{x-y}{4}\right)-1}$	$10n \geq 6x+7z+19$

b) $n \equiv 0, 3 (\mathrm{mod}\ 4)$ and $z \equiv 1(\mathrm{mod}\ 4)$ *or* $n \equiv 1, 2(\mathrm{mod}\ 4)$ and $z \equiv 3(\mathrm{mod}\ 4)$.

Let $d = \frac{z-1}{2}$. Then $\mathcal{A}_d^{2d-1} + (n-3d+2)$ can be used to label $2d-1 = z-2$ vertices of Z plus $z-2$ vertices near the middle of X. The remaining vertices of Z are labeled 1 (location given below for each case). There are two possibilities:

i) 1 in $(x+1, z)$ and $(x+1, z-1)$:

Then B contains

$$x-(2d-1)-(n-3d-1) = x+d-n+2 = \frac{x-y+2}{2} \text{ vertices.}$$

In each case, $x+d-n+2$ is even, so $x-y+2 \equiv 0(mod\ 4)$.

ii) 1 in $(x+1, 1)$ and $(x+1, 2)$:

Then B contains

$$x-(2d-1)-(n-3d+1) = x+d-n = \frac{1}{2}(x-y-2) \text{ vertices}$$

and $x-y-2 \equiv x-y+2 \equiv 0(mod\ 4)$.

The labelings are summarized in the table below:

$(x-y+2)/4$ (mod 4)	B	C	size constraint
0,3 use i)	1-near $\mathcal{S}_{\frac{x-y+2}{4}+1}$	$\mathcal{L}_{\frac{x-y+2}{4}+2}^{n-z-\left(\frac{x-y+2}{4}\right)+1}$	$10n \geq 6x+7z+17$
1 use ii)	1-near $\mathcal{S}_{\frac{x-y+2}{4}}$	$\mathcal{L}_{\frac{x-y+2}{4}+1}^{n-z-\left(\frac{x-y+2}{4}\right)+2}$	$10n \geq 6x+7z+13$
2 use ii)	$\mathcal{L}_3^{\frac{x-y+2}{4}-1*}$	2−2 hooked into $h\mathcal{L}_{\frac{x-y+2}{4}+2}^{n-z-\left(\frac{x-y+2}{4}\right)+1}$	$10n \geq 6x+7z+17$ $^*22 \leq x-y$

In order to use the last construction, $5 \leq \frac{x-y+2}{4} - 1$, so $22 \leq x-y$ (which forces n to be quite large). However, the only smaller case occurs when $\frac{x-y+2}{4} = 2$, so $x-y=6$. We adapt the construction in a) to cover $W(n : x, x-6, z)$.

Let $d = \frac{z+1}{2}$ and use $\mathcal{A}_d^{2d-1} + (n-3d+2)$ to label the vertices of Z plus some vertices of X. We have used labels $n-2d+2$ to n inclusive. Then B contains

$$x-(2d-1)-(n-3d+1)=x+d-n=\frac{1}{2}(x-y)=3 \text{ vertices.}$$

If $d \neq \frac{n}{4}$, label $(1,z+1)$, $(n-2d+2,z+1)$ with the next largest label, $n-2d+1$, put 1's in $(2,z+1)$ and $(3,z+1)$ and use a $(n-2z-2)$-ext $\mathcal{L}_2^{n-z-2}$ to label the remaining vertices. Otherwise, put $n-2d+1$ in $(3,z+1)$ and $(n-2d+4,z+1)$, 1's in $(1,z+1)$ and $(2,z+1)$ and use a $(n-2z)$-ext $\mathcal{L}_2^{n-z-2}$ for the remaining vertices. The only constraint here is that $3 \leq n-z-2$ or $z \leq n-5$. If $n \geq 8$, then $n-5 \geq \lfloor\frac{n-1}{2}\rfloor \geq z$. Since $y \geq 1$, $7 \leq x < \frac{4n-1}{3}$ and $6 \leq n$. If $n=7$, then $z \leq \lfloor\frac{n-1}{2}\rfloor$ and $z \equiv 1(mod\ 4)$ imply that $z=1 \leq 2=7-5$, which satisfies the constraint.

4.2.5 Long Z-vanes, $n \equiv 2,3(mod\ 4)$

Here we are interested in relatively large values of z, where $x \geq n$. If $n \equiv 2,3(mod\ 4)$, then x, y and z are all odd.

In this group, X, the pivot and part of Z are labeled by a [hooked] Langford sequence of defect d. The label $d-1$ is used to deal with the problem that y and z are odd. The remaining vertices are labeled using smaller sequences.

We illustrate this first with an example. Consider $W(19:21,9,7)$. Use any $\mathcal{L}_7^{13}$ (for example, $\mathcal{A}_7^{13}$) to label X, the pivot and the 4 vertices of Z closest to the pivot.

Use 6 to label vertices (22,3) and (23,8), leaving an even number of unlabeled vertices on both Y and Z.

```
14  15  16  ...  16  10  17  11  6  -  -  -  -  -  -  -  -
                            18
                            12
                            19
                            13
                            6
                            -
                            -
```

Use $\mathcal{S}_1$ and $\mathcal{L}_2^4$ to label the remaining vertices of Z and Y, respectively.

More generally, suppose that $\mathcal{L}_d^{n+1-d}$, for some d, is used to label X, the pivot and the vertices $(x+1,z),\ldots,(x+1,z-d+4)$ of Z; the vertices $(x+1,z-d+3)$ and $(x+2,z+1)$ are labeled $d-1$. Then

$$2n+2-2d=x+1+d-3$$

so

$$3d=x+y+z+3-x+2$$

$$d=\frac{y+z+5}{3}$$

Hence, $y+z\equiv 1(mod\ 3)$; however, both y and z are odd, so $y+z\equiv 4(mod\ 6)$ and d must be odd. This forces $n-1+d\equiv 0,1(mod\ 4)$. Since $n\equiv 2,3(mod\ 4)$, $d\equiv 3(mod\ 4)$ and so $y+z\equiv 4(mod\ 12)$.

For the $\mathcal{L}_d^{n+1-d}$ to exist, $n+1-d\geq 2d-1$ which implies that $x\geq n+2$. To use the sequence in this way, we must also ensure Z is long enough to accommodate the required vertices, so $z\geq d-2=\frac{y+z-1}{3}$ which implies $z\geq\frac{y-1}{2}$. There are $x+z+1-2(n+1-d)-1=2d-y-3\equiv 3-y(mod\ 8)$ unlabeled vertices on Z and $y-1$ on Y.

If $y\equiv 1,3(mod\ 8)$, then $d-\frac{y-3}{2}\equiv 1,0(mod\ 4)$, so $\mathcal{S}_{d-\frac{y+3}{2}}$ can be used to finish labeling Z and $\mathcal{L}_{d-\frac{y+3}{2}+1}^{\frac{y-1}{2}}$ can be used for Y whenever

$$2d-y-3+1\leq\frac{y-1}{2}\text{ or equivalently }11\leq y+4(y-z).$$

Similarly, if $y\equiv 5$ or $7(mod\ 8)$, then $d-\frac{y+3}{2}+1\equiv 0$ or $2(mod\ 4)$, respectively so use a 1-near $\mathcal{S}_{d-\frac{y+3}{2}+1}$ or a 2-near $\mathcal{S}_{d-\frac{y+3}{2}+1}$, respectively. The unused entry (1 or 2) is used to label 2 vertices at one end of Y along with $\mathcal{L}_{d-\frac{y+3}{2}+2}^{\frac{y-3}{2}}$ or $h\mathcal{L}_{d-\frac{y+3}{2}+2}^{\frac{y-3}{2}}$, respectively. Here the constraint is

$$2d-y-3+3\leq\frac{y-3}{2}\text{ or }y+4(y-z)\geq 29.$$

We summarize this labeling.

a) $y+z\equiv 4(\text{mod }12), x\geq n+2$ and $y\geq z\geq\frac{y-1}{2}$

Take $d=\frac{y+z+5}{3}$; use

$\mathcal{L}_d^{n+1-d}$ for X, the pivot and $(x+1,z),\dots,(x+1,z-d+4)$;

$d-1$ for $(x+1,z-d+3)$ and $(x+2,z+1)$; plus

y (mod 8)	end of Z	Y	$y+4(y-z)\geq$
1,3	$\mathcal{S}_{d-\frac{y+3}{2}}$	$\mathcal{L}_{d-\frac{y+3}{2}+1}^{\frac{y-1}{2}}$	11
5	1-near $\mathcal{S}_{d-\frac{y+3}{2}+1}$	$11\mathcal{L}_{d-\frac{y+3}{2}+2}^{\frac{y-3}{2}}$	29
7	2-near $\mathcal{S}_{d-\frac{y+3}{2}+1}$	$2-2$ hooked into $h\mathcal{L}_{d-\frac{y+3}{2}+2}^{\frac{y-3}{2}}$	29

A similar discussion can be used for $y+z\equiv 2,0$ (mod 12).

b) $y+z\equiv 2(\text{mod }12), x\geq n+1$ and $y\geq z>\frac{y}{2}$

Take $d=\frac{y+z+4}{3}$; use

$h\mathcal{L}_d^{n+1-d}$ for X, the pivot and $(x+1,z),\ldots,(x+1,2d-y-2)$;

$d-1$ for $(x+2,z+1)$ and $(x+1,2d-y-1)$, which is the hook of $h\mathcal{L}_d^{n+1-d}$; plus

y (mod 8)	end of Z	Y	$y+4(y-z)\geq$
1,7	$\mathcal{S}_{d-\frac{y+3}{2}}$	$\mathcal{L}^{\frac{y-1}{2}}_{d-\frac{y+3}{2}+1}$	7
3	1-near $\mathcal{S}_{d-\frac{y+3}{2}+1}$	$11\mathcal{L}^{\frac{y-3}{2}}_{d-\frac{y+3}{2}+2}$	25
5	2-near $\mathcal{S}_{d-\frac{y+3}{2}+1}$	$2-2$ hooked into $h\mathcal{L}^{\frac{y-3}{2}}_{d-\frac{y+3}{2}+2}$	25

c) $y+z\equiv 0(\text{mod } 12)$, $x\geq n+6$ and $y\geq z>\frac{y}{2}$

Take $d=\frac{y+z+9}{3}$; use

$\mathcal{L}_d^{n+1-d}$ for X, the pivot and $(x+1,2d-y-1),\ldots,(x+1,z)$ plus

y (mod 8)	$d-1$	end of Z	Y	$y+4(y-z)\geq$
5,7	$(x+1,z-d+4)$, $(x+3,z+1)$	4-ext $\mathcal{S}_{d-\frac{y+3}{2}}$	$h\mathcal{L}^{\frac{y-1}{2}}_{d-\frac{y+3}{2}+1}$ (hook is filled by $d-1$)	27
1,3	$(x+1,z-d+3)$, $(x+2,z+1)$	5-ext $\mathcal{S}_{d-\frac{y+3}{2}}$	$\mathcal{L}^{\frac{y-1}{2}}_{d-\frac{y+3}{2}+1}$	27

The appropriate 4- or 5-extended sequence must exist, so $d-\frac{y+3}{2}\geq 2$ which implies $z\geq\frac{y+3}{2}$. This also guarantees that Z is long enough to accommodate the sequences. However, $z\neq\frac{y+1}{2}$; otherwise $3y+1=2y+y+1=2y+2z=2(y+z)\equiv 0(\text{mod } 24)$; a contradiction. So the construction holds for all $z>\frac{y}{2}$.

Note that if $y+z\equiv 0$ (mod 12), then $x\neq n,n+3,n+4$; otherwise, $n\equiv 3,2,3$ (mod 4), respectively and $y+z=2n-1-x\equiv 2$ (mod 4), a contradiction. The cases $x=n+1,n+2$ are covered in 4.2.7, so the only outstanding case is $x=n+5$.

Now suppose that $x=n+5$. Then $n\equiv 2$ (mod 4) and $y+z=n-6$, so this case applies if $n\equiv 6$ (mod 12). Since $z\leq\frac{y+z}{2}=\frac{n-6}{2}$ and $\frac{n-6}{2}$ is even, $z\leq\frac{n-8}{2}$. Therefore,

$$\begin{aligned}10n-6x-7z &\geq \frac{1}{2}(20n-12n-60-7n+56)\\ &= \frac{1}{2}(n-4).\end{aligned}$$

Since $\frac{1}{2}(n-4)\geq 19$ whenever $n\geq 42$, 4.2.4 can be used for all $n\geq 42$. For each remaining case, $W(n:n+5,n-6-z,z)$, $16<n<42$ (since $x<\frac{4n-1}{3}$), $n\equiv 6(mod\ 12)$, z is odd and $z\leq\frac{n-8}{2}$. This means that $n=30$ and $z\leq 11$ or $n=18$ and $z\leq 5$. In the first case, $10n-6x-7z\geq 24$ if $z\leq 9$ and $10n-6x-7z\geq 21$

is $z \leq 3$, so 4.2.4 can be applied. This leaves $W(30 : 35, 13, 11)$ and $W(18 : 23, 7, 5)$, see Appendix 1.

4.2.6 More Long Z-vanes, $n \equiv 2, 3 \pmod 4$

Once again, consider $n \equiv 2, 3 \pmod 4$, so x,y and z are odd. This labeling is similar to 4.2.5, but one label is moved from a vertex of Z to a vertex of Y to accommodate the label $d - 1$.

Consider, first, $W(19 : 19, 9, 9)$. Use any $\mathcal{L}_7^{13}$ (for example, $\mathcal{A}_7^{13}$) to label X, the pivot and the 6 vertices of Z closest to the pivot.

$$\begin{array}{ccccccccccccccccc}
14 & 15 & 16 & \ldots & 15 & 9 & 16 & 10 & - & - & - & - & - & - & - & - & - \\
 & & & & & & & 17 & & & & & & & & & \\
 & & & & & & & 11 & & & & & & & & & \\
 & & & & & & & 18 & & & & & & & & & \\
 & & & & & & & 12 & & & & & & & & & \\
 & & & & & & & 19 & & & & & & & & & \\
 & & & & & & & 13 & & & & & & & & & \\
 & & & & & & & - & & & & & & & & & \\
 & & & & & & & - & & & & & & & & & \\
 & & & & & & & - & & & & & & & & &
\end{array}$$

Since 7 is the smallest label in $\mathcal{L}_7^{13}$, no label can occur twice on the 6 vertices of Z that we have labeled, so any of these labels could be moved to the corresponding vertex on Y. Move the label 17 from vertex (20,9) to vertex (21,10), label vertices (20,9) and (20,3) with 6 and use $\mathcal{S}_1$, and $\mathcal{L}_2^4$ to label the remaining vertices of Z and Y, respectively.

$$\begin{array}{ccccccccccccccccc}
14 & 15 & 16 & \ldots & 15 & 9 & 16 & 10 & \mathbf{17} & - & - & - & - & - & - & - & - \\
 & & & & & & & \mathbf{6} & & & & & & & & & \\
 & & & & & & & 11 & & & & & & & & & \\
 & & & & & & & 18 & & & & & & & & & \\
 & & & & & & & 12 & & & & & & & & & \\
 & & & & & & & 19 & & & & & & & & & \\
 & & & & & & & 13 & & & & & & & & & \\
 & & & & & & & \mathbf{6} & & & & & & & & & \\
 & & & & & & & 1 & & & & & & & & & \\
 & & & & & & & 1 & & & & & & & & &
\end{array}$$

More generally as in 4.2.5, the value d is key to this labeling. First, use the $2(n + 1 - d)$ entries of $\mathcal{L}_d^{n+1-d}$ to label the $x + 1 + d - 1$ vertices of X, the pivot and $(x + 1, z), \ldots, (x + 1, z - d + 2)$ of Z. Note that only $d - 1$ positions of Z are used, so no entry of $\mathcal{L}_d^{n+1-d}$ can occur twice on Z. Shift the label on vertex $(x+1,z)$ to $(x+2,z+1)$ and label vertices $(x+1,z-d+1)$ and $(x+1,z)$ with $d-1$. Since

$$2n+2-2d = x+1+d-1 \ (*)$$

we have

$$3d = x+y+z+3-x \quad \text{and} \quad d = \frac{y+z+3}{3}.$$

Hence, $y+z \equiv 0 \pmod 3$; however both y and z are odd, so $y+z \equiv 0(mod\ 6)$ and d must be odd. This forces $n+1-d \equiv 0,1(mod\ 4)$. Since $n \equiv 2,3(mod\ 4)$, $d \equiv 3(mod\ 4)$ and so $y+z \equiv 6(mod\ 12)$. The constraints here are:

$$n+1-d \geq 2d-1, \text{ so } x \geq n \text{ by } (*) \text{ and } z > d-1 = \frac{y+z}{3}, \text{ so } z > \frac{y}{2}.$$

There are $x+z+1-2(n+1-d)-1 = 2d-y-3$ unlabeled vertices on z and $y-1$ on Y which we label with appropriate sequences.

We summarize these labelings.

a) $\boldsymbol{y+z \equiv 6 \pmod{12}}$, $\boldsymbol{x \geq n}$ and $\boldsymbol{y \geq z > \frac{y}{2}}$

Take $d = \frac{y+z+3}{3}$; use

$\mathcal{L}_d^{n+1-d}$ for X, the pivot and $(x+1,z),\ldots,(x+1,z-d+2)$;

the label from $(x+1,z)$ for $(x+2,z+1)$;

$d-1$ for $(x+1,z)$ and $(x+1,z-d+1)$; plus

$y \pmod 8$	end of Z	Y	$y+4(y-z) \geq$
1,3	$\mathcal{S}_{d-\left(\frac{y+3}{2}\right)}$	$\mathcal{L}^{\frac{y-1}{2}}_{d-\left(\frac{y+3}{2}\right)+1}$	3
5	1-near $\mathcal{S}_{d-\left(\frac{y+3}{2}\right)+1}$	$11\mathcal{L}^{\frac{y-3}{2}}_{d-\left(\frac{y+3}{2}\right)+2}$	21
7	2-near $\mathcal{S}_{d-\left(\frac{y+3}{2}\right)+1}$	$2-2$ hooked into $h\mathcal{L}^{\frac{y-3}{2}}_{d-\left(\frac{y+3}{2}\right)+2}$	21

To use this construction, $z-d+1 \geq 1$, so $z \geq \frac{y+3}{2}$ and $d-\left(\frac{y+3}{2}\right) \geq 0$. If $y \equiv 7 \pmod 8$, $d-\left(\frac{y+3}{2}\right)+1$ would have to be greater than or equal to 2, so $z \geq \frac{y+9}{2}$; however, if $y \equiv 7 \pmod 8$, $z \neq \frac{y+3}{2}, \frac{y+5}{2}, \frac{y+7}{2}$ since $y+z \equiv 6 \pmod{12}$.

Finally, suppose $z = \frac{y+1}{2}$. Then $y+z = \frac{3y+1}{2} \not\equiv 6(mod\ 12)$ for $y \equiv 1,3,5$ or $7 \pmod 8$. So this case does not apply.

A similar discussion for $y+z \equiv 8,10 \pmod{12}$ gives the following labelings.

b) $\boldsymbol{y+z \equiv 8 \pmod{12}}$, $\boldsymbol{x \geq n+1}$ and $\boldsymbol{y \geq z > \frac{y}{2}}$

Take $d = \frac{y+z+4}{3}$; use

$\mathcal{L}_d^{n+1-d}$ for X, the pivot and $(x+1, z), \ldots, (x+1, z-d+3)$;

the label from $(x+1, z)$ for $(x+2, z+1)$;

$d-1$ for $(x+1, z)$ and $(x+1, z-d+1)$; plus

$y \pmod 8$	end of Z	Y	$y + 4(y-z) \geq$
1,7	$h\mathcal{S}_{d-\frac{y+3}{2}}$	$\mathcal{L}_{d-\frac{y+3}{2}+1}^{\frac{y-1}{2}}$	7
3	1-near $\mathcal{S}_{d-\frac{y+3}{2}+1}$	$11\mathcal{L}_{d-\frac{y+3}{2}+2}^{\frac{y-3}{2}}$	25
5	2-near $h\mathcal{S}_{d-\frac{y+3}{2}+1}$	$2-2$ hooked into $h\mathcal{L}_{d-\frac{y+3}{2}+2}^{\frac{y-3}{2}}$	25 if $z > \frac{y+19}{2}$
5	see below		$\frac{y}{2} < z < \frac{y+19}{2}$

To use this construction for $y \equiv 1$ or $7 \pmod 8$, $z-d+1 \geq 4$, so $z \geq \frac{y+13}{2}$ and $d - \frac{y+3}{2} \geq 2$. However, since $y+z \equiv 8 \pmod{12}$, and $y \equiv 1$ or $7 \pmod 8$ there are no odd values of z, $\frac{y}{2} < z < \frac{y+13}{2}$.

To use the construction for $y \equiv 3 \pmod 8$, $z-d+1 \geq 2$, so $z \geq \frac{y+7}{2}$, however, there are no other values of z, $\frac{y}{2} < z < \frac{y+7}{2}$.

Finally to use this for $y \equiv 5 \pmod 8$, $z+d-1 \geq 6$, so $z \geq \frac{y+19}{2}$ and $d - \frac{y+3}{2} + 1 \geq 4$. There is one additional possible value for z, $z = \frac{y+1}{2}$. In this case, set $d = \frac{y+z+4}{3} = z+1$. $\mathcal{L}_d^{n+1-d}$ can be used to label X, the pivot and all of Z except the vertex $(x+1, 1)$. Since $d > z$, the label in $(x+1, z)$ can be moved to $(x+2, z+1)$. Use $z-1$ to label $(x+1, 1)$ and $(x+1, z)$ and z for $(x+3, z+1)$ and $(x+z+3, z+1)$. This is always possible since $z+3 \leq y+1 = 2z$ for all $z \geq 3$. The rest of Y can be labeled using a z-ext $\mathcal{S}_{z-2}$ since $2(z-2) = y-3$ and $z \equiv 3 \pmod 4$.

c) $y + z \equiv 10 \pmod{12}$, $x \geq n-1$ and $y \geq z \geq \frac{y+5}{2}$

Take $d = \frac{y+z+2}{3}$; use

$\mathcal{L}_d^{n+1-d}$ for X, the pivot and $(x+1, z), \ldots, (x+1, z-d+1)$;

the label from $(x+1, z-1)$ for $(x+3, z+1)$;

$d-1$ for $(x+1, z-1)$ and $(x+1, z-d)$; plus

$y \pmod 8$	end of Z	Y	$y + 4(y-z) \geq$
3,5	$\mathcal{S}_{d-\frac{y+3}{2}}$	$h\mathcal{L}_{d-\frac{y+3}{2}+1}^{\frac{y-1}{2}}$	0
7	1-near $\mathcal{S}_{d-\frac{y+3}{2}+1}$	$h\mathcal{L}_{d-\frac{y+3}{2}+2}^{\frac{y-3}{2}}, 11$	17
1	2-near $\mathcal{S}_{d-\frac{y+3}{2}+1}$	$2-2$ hooked into $\mathcal{L}_{d-\frac{y+3}{2}+2}^{\frac{y-3}{2}}$	17

To use this construction, $z-d\geq 1$, so $z\geq\frac{y+5}{2}$ and $d-\frac{y+3}{2}\geq 0$. If $y\equiv 5 \pmod 8$, $d-\frac{y+3}{2}+1\geq 2$, so $z\geq\frac{y+11}{2}$. Since $y+z\equiv 10 \pmod{12}$, there is only one case for z, $\frac{y+11}{2}>z>\frac{y}{2}$, which is not covered by the above construction: $z=\frac{y+5}{2}$ and $y\equiv 5$ (mod 8). In this case, $d=\frac{2z-5+z+2}{3}=z-1$, so instead of the third line of the table we use $h\mathcal{S}_{d-2}$.

4.2.7 Special constructions for $n\equiv 2$ or 3 (mod 4)

a) Let $x=n$. Then $n\equiv 3 \pmod 4$. The following labelings can be used.

$z\equiv$ (mod 8)	n in positions	Z-vane	XY path	constraints
1,3	$(2,z+1),(x+1,z)$	$\mathcal{L}_3^{\frac{z-1}{2}}$	2-ext $\mathcal{S}_2\mathcal{L}_{\frac{z+5}{2}}^{n-\left(\frac{z+5}{2}\right)}$	$z\geq 11$
5	$(3,z+1),(x+1,z-1)$	$h\mathcal{L}_2^{\frac{z-1}{2}}$	$\mathcal{S}_1 h\mathcal{L}_{\frac{z+3}{2}}^{n-\left(\frac{z+3}{2}\right)}$	$z\geq 7$
7	$(2,z+1),(x+1,z)$	$\mathcal{L}_2^{\frac{z-1}{2}}$	$h\mathcal{L}_{\frac{z+3}{2}}^{n-\left(\frac{z+3}{2}\right)}\mathcal{S}_1$	$z\geq 7$

The only remaining cases are: $z=1,3,5,9$.

$W(n:n,n-2,1)$: put n in the sole vertex of Z and the second vertex (2,2) X; fill the XY-path with a $h\mathcal{S}_{n-1}$.

$W(n:n,n-4,3)$: put n in (2,4) and $(x+1,3)$; fill Z with $\mathcal{S}_1$ and the XY-path with $h\mathcal{L}_2^{n-2}$.

$W(n:n,n-6,5)$: put n in (3,6) and $(x+1,4)$; fill Z with $h\mathcal{L}_2^2$ (i.e., 2 3 2 0 3) and the XY-path with $\mathcal{S}_1$ and $\mathcal{L}_4^{n-4}$. Note that $5=z\leq\frac{n-1}{2}$, so $n\geq 11$ and $\mathcal{L}_4^{n-4}$ exists.

$W(n:n,n-10,9)$: put n in positions (2,10) and $(x+1,3)$, $\mathcal{S}_4$ on Z and $h\mathcal{L}_5^{n-5}$ on the XY-path. Note that $9=z\leq\frac{n-1}{2}$, so $n\geq 19$ and $h\mathcal{L}_5^{n-5}$ exists.

b) Let $x=n+1$. Then $n\equiv 2 \pmod 4$. The following labelings can be used.

z (mod 8)	n in positions	Z-vane	XY path	constraints
1,7	$(3,z+1),(x+1,z)$	$\mathcal{L}_2^{\frac{z-1}{2}}$	$\mathcal{S}_1\mathcal{L}_{\frac{z+3}{2}}^{n-\left(\frac{z+3}{2}\right)}$	$z\geq 7$
3	$(3,z+1),(x+1,z)$	$\mathcal{L}_3^{\frac{z-1}{2}}$	$\mathcal{S}_1$2-2$h\mathcal{L}_{\frac{z+5}{2}}^{n-\left(\frac{z+5}{2}\right)}$	$z\geq 11$
5	$(4,z+1),(x+1,z-1)$	$h\mathcal{L}_2^{\frac{z-1}{2}}$	$\mathcal{S}_1 h\mathcal{L}_{\frac{z+3}{2}}^{n-\left(\frac{z+3}{2}\right)}$	$z\geq 7$

The only remaining cases are: $z=1,3,5$.

$W(n:n+1,n-3,1)$: put n in the sole vertex of Z and in $(3,z+1)$; fill with a 3-ext $\mathcal{S}_{n-1}$.

$W(n:n+1,n-5,3)$: put n in positions $(3,z+1)$ of X and $(x+1,z)$ of Z; $\mathcal{S}_1$ in the remaining positions of Z and use a 3-ext $\mathcal{L}_2^{n-2}$ to fill the XY-path.

$W(n : n+1, n-7, 5)$: put n in positions $(4, z+1)$ of X and $(x+1, z-1)$ of Z; $h\mathcal{S}_2$ on Z and 4-ext $\mathcal{L}_3^{n-3}$ on the XY-path.

c) Let $x = n+2$. Then $n \equiv 3(mod\ 4)$ and the labelings are given in the table below.

z (mod 8)	n in positions	Z-vane	$X - Y$ path	constraints
1,3	$(4, z+1), (x+1, z)$	$\mathcal{L}_3^{\frac{z-1}{2}}$	4-ext $\mathcal{S}_2\mathcal{L}_{\frac{z+5}{2}}^{n-\left(\frac{z+5}{2}\right)}$	$z \geq 11$
5	$(5, z+1), (x+1, z-1)$	$h\mathcal{L}_5^{\frac{z-1}{2}}$	5-ext $\mathcal{S}_4\mathcal{L}_{\frac{z+9}{2}}^{n-\left(\frac{z+9}{2}\right)}$	$z \geq 19$
7	$(4, z+1), (x+1, z)$	$\mathcal{L}_4^{\frac{z-1}{2}}$	4-ext $\mathcal{S}_3\mathcal{L}_{\frac{z+7}{2}}^{n-\left(\frac{z+7}{2}\right)}$	$z \geq 15$

The only remaining cases are: $z = 1, 3, 5, 7, 9, 13$. We provide labelings for these cases below.

$W(n : n+2, n-4, 1)$: Put n in positions (4,2) of X and $(x+1, 1)$ of Z; fill the XY-path with a 4-ext $\mathcal{S}_{n-1}$.

$W(n : n+2, n-6, 3)$: Note that $n - 6 \geq 3$ and $n \equiv 3(mod\ 4)$, so $n \geq 11$. Put 2 in positions $(x, 4)$ of X and $(x+1, 3)$ of Z; $\mathcal{S}_1$ in the remaining positions of Z; fill the rest of the XY-path with an $(n+2)$-ext $\mathcal{L}_3^{n-2}$.

$W(n : n+2, n-8, 5)$: Here $n \geq 15$. Put $\mathcal{A}_3^5 + (n-7)$ along Z and in positions $(6, 6), \ldots, (11, 6)$ of X; $2 - 2\ 1\ 1$ in positions $(1, 6), \ldots (5, 6)$ of X; $n - 5$ in (2,6) and $(n-3, 6)$ of X. The remaining vertices of the XY-path are labeled using an $(n-13)$-ext $\mathcal{L}_3^{n-8}$.

$W(n : n+2, n-10, 7)$: Here $n \geq 19$. For $n \geq 23$, put $\mathcal{A}_4^7 + (n-10)$ on Z and in positions $(7, 8), \ldots, (13, 8)$ of X; $\mathcal{L}_2^3$ in $(1, 8), \ldots, (6, 8)$ of X; $\mathcal{S}_1$ in (14,8), (15,8) of X and fill the rest of the XY-path with $\mathcal{L}_5^{n-11}$. $W(19 : 21, 9, 7)$ can be labeled using 4.2.4 because $6(21) + 7(7) + 7 = 182 \leq 190$.

$W(n : n+2, n-12, 9)$: Here $n \geq 23$. Put $\mathcal{S}_1$ in $(x+1, 1), (x+1, 2)$ of Z; $\mathcal{A}_4^7 + (n-10)$ in the remaining positions of Z and positions $(7, 10), \ldots, (13, 10)$ of X; $\mathcal{L}_2^3$ in $(1, 10), \ldots, (6, 10)$ of X and fill the rest of the XY-path with $\mathcal{L}_5^{n-11}$.

$W(n : n+2, n-16, 13)$: So $n \geq 31$. Put $\mathcal{S}_1$ in $(x+1, 1), (x+1, 2)$ of Z; $\mathcal{A}_6^{11} + (n-16)$ in the rest of Z and positions $(9, 14), \ldots, (19, 14)$ of X; a 1-near $\mathcal{S}_5$ in $(1, 14), \ldots, (8, 14)$ of X and fill the rest of the XY-path with $\mathcal{L}_6^{n-16}$.

5 Skolem labeling $g3$-windmills

Theorem 7 *Every $g3$-windmill that satisfies the Skolem parity condition can be Skolem-labeled.*

Proof: Let $G = W(n : x, y, z)$ be a $g3$-windmill which satisfies the Skolem parity condition. First, we note that if $x < n$, the graph can be pruned, so we need only consider graphs with $x \geq n$. Then $y + z + 1 = 2n - x \leq 2n - n = n$, so $z \leq \frac{n-1}{2}$.

Case 1. $n \equiv 0, 1 \pmod 4$.

i) Suppose first that z is even. If $n \geq 13$, then $z \leq \frac{n-1}{2} \leq \frac{2n-8}{3}$, so construction 4.2.1 can be used. If $n < 13$, then $z < 6$, so z is either 2 or 4. For $z = 2$, 4.2.1 can be used for all n. If $z = 4$, then $n \geq 9$; so 4.2.1 can be used for all $n \geq 10$. This leaves only $W(9 : 9, 4, 4)$ to label:

$$\begin{array}{cccccccccccccc}
9 & 7 & 5 & 3 & 1 & 1 & 3 & 5 & 7 & 9 & 2 & 4 & 6 & 8 \\
 & & & & & & & & & 2 & & & & \\
 & & & & & & & & & 4 & & & & \\
 & & & & & & & & & 6 & & & & \\
 & & & & & & & & & 8 & & & &
\end{array}$$

ii) Now suppose that z is odd. Since $n \equiv 0, 1 (mod\ 4)$, x and y must be even. Construction 4.2.2 can always be used if $y \leq (2n - 8)/3$, so we need only consider $y > \frac{2n-8}{3}$.

In general, 4.2.4 can be used whenever $6x+7z \leq 10n-19$. Since $x \geq n$, $x+y > \frac{5n-8}{3}$, so $z = 2n - 1 - x - y < \frac{n+5}{3}$. Therefore,

$$\begin{aligned}
6x + 7z &= 6(x + z) + z \\
&= 6(2n - 1 - y) + z \\
&< \frac{25n + 35}{3}.
\end{aligned}$$

This is less than $10n - 19$ whenever $19 \leq n$, so 4.2.4 can be used in all these cases. In addition, 4.2.4 can also be used for some smaller values of n.

Consider $n = 17$. Then $8.7 < y$ and $17 \leq x$; however, both x and y are even so $10 \leq y$ and $18 \leq x$. Therefore, $z = 2n - 1 - x - y \leq 5$. Then

$$\begin{aligned}
6x + 7z &= 6(2n - 1 - y - z) + 7z \\
&= 12n - 6 - 6y + z \\
&\leq 143 \\
&\leq 10n - 19.
\end{aligned}$$

So 4.2.4 can be used in all the remaining cases with $n = 17$. A similar discussion applies when $n = 16$ or 13.

The only remaining windmills are:

$W(12{:}12, 6, 5), W(12{:}12, 8, 3), W(12{:}12, 10, 1), W(12{:}14, 6, 3), W(12{:}14, 8, 1)$;

$W(9 : 10, 4, 3), W(9 : 10, 6, 1), W(9 : 12, 4, 1)$;

$W(8 : 8, 4, 3), W(8 : 8, 6, 1), W(5 : 6, 2, 1), W(4 : 4, 2, 1)$.

All of these, with 3 exceptions, can be labeled using the specific techniques of 4.2.4. For $W(4:4,2,1)$, use

$$\begin{array}{ccccccc} 4 & 1 & 1 & 3 & 4 & 2 & 3 \\ & & & & 2 & & \end{array}$$

For $W(9:12,4,1)$, put a 5-extended Skolem sequence of order 8 along X, the pivot and Y and use 9 to label the remaining 2 vertices. For $W(12:14,6,3)$, put a 14-extended Langford sequence with $d=3$ and $m=10$ along X, the pivot and Y, then use a hooked Skolem sequence of order 2 to label the remaining vertices.

Case 2. Let $n \equiv 2,3 \pmod 4$. Then x, y and z are all odd. If $x \geq (4n-1)/3$, 4.2.3 can be used and if $x = n, n+1$ or $n+2$, 4.2.7 can be used, so it suffices to consider $n+2 < x < (4n-1)/3$.

i) First, consider those remaining windmills with relatively short Z-vanes: $z < \frac{y}{2}$. Let $x = n+k$. Then $3 \leq k \leq \frac{n-2}{3}$ and $y+z = n-1-k$.

If $z \leq \frac{y-3}{2}$, then $3z \leq n-4-k$ and

$$\begin{aligned} 10n - 6x - 7z - 19 &\geq (5n - 11k - 29)/3 \\ &\geq (4n - 65)/9 \end{aligned}$$

which is nonnegative if $n \geq 17$, so 4.2.4 can be used to label these windmills.

Similarly, if $z = \frac{y-1}{2}$, then

$$10n - 6x - 7z - 19 \geq (4n - 107)/9,$$

so 4.2.4 can be used if $n \geq 27$. Note that since $z = \frac{y-1}{2}$, $n-1-k = y+z = 3z+1$, so $\frac{n-2-k}{3} \in \mathbf{Z}^+$ and the only remaining windmills with $17 < n < 27$ are $W(26:29,15,7), W(23:29,11,5)$ and $W(22:27,11,5)$, all of which can be labeled using 4.2.4.

Now suppose that $n \leq 15$ and $z < \frac{y}{2}$, then the only windmills are: $W(15:19,7,3), W(15:19,9,1), W(14:17,7,3)$ and $W(14:17,9,1)$. The last three can be labeled using 4.2.4. For $W(15:19,7,3)$, use $h\mathcal{S}_2$ to label Z and vertex $(x+2, z+1)$ of Y (note that the hook would fall on the pivot) and 7-ext $\mathcal{L}_3^{13}$ for the remaining vertices.

ii) Now consider the remaining windmills. Then $n+3 \leq x < (4n-1)/3$ and $z > \frac{y}{2}$. Each of these can be labeled using 4.2.5 or 4.2.6 unless $y + 4(y-z)$ is too small. In general, 4.2.5 and 4.2.6 can always be used whenever $y + 4(y-z) \geq 29$; however, the constant is actually smaller in many cases. First we identify the remaining cases and then we provide labelings for them.

Since $z \geq 1$, $y > y - z$. Then $y + 4(y-z) - 29 > 5(y-z) - 29$ which would be greater than 0 whenever $y - z \geq 6$. Note that $y - z$ is even since both y, z are odd, so we need only consider $y - z = 0, 2, 4$.

Suppose $y - z = 4$. Then $\frac{y}{2} < z = y - 4$, so $8 < y$. If $y \geq 13$, then $y + 4(y - z) \geq 13 + 16 = 29$, so they can all be labeled by 4.2.5 or 4.2.6. If $y = 11 \equiv 3 \pmod 8$, then $y + 4(y - z) = 11 + 16 = 27$, so 4.2.5 or 4.2.6 can be used. If $y = 9 \equiv 1$ (mod 8), then $y + 4(y - z) = 9 + 16 = 25$, but $y + z = 9 + 5 = 14 \equiv 2 \pmod{12}$, so 4.2.5 b) can be used.

Now suppose $y - z = 2$. Then $\frac{y}{2} < z = y - 2$, so $4 < y$. If $y \geq 21$, then $21 + 4(2) = 29$, so 4.2.5 or 4.2.6 can be used. 4.2.5 and 4.2.6 can also be used in the following cases:

if $y = 19$, then $y + z = 36 \equiv 0 \pmod{12}$ and $y + 4(y - z) = 27$;

if $y = 17$, then $y + z = 32 \equiv 8 \pmod{12}$ and $y + 4(y - z) = 25 \geq 7$;

if $y = 9$, then $y + z = 16 \equiv 4 \pmod{12}$ and $y + 4(y - z) = 17 \geq 11$.

The remaining values of y are: $15, 13, 11, 7, 5$. Since $2y - 2 = y + z = 2n - 1 - x$ and $n + 3 \leq x < (4n - 1)/3$, we have

$$2(n - 1)/3 < 2y - 2 \leq n - 4 \text{ or } 2y + 2 \leq n < 3y - 2.$$

Since $x = 2n - 1 - y - z = 2n - 1 - 2y + 2 = 2n - 2y + 1$, the only (n, x) pairs left to label are:

for $y = 15, (34, 39), (35, 41), (38, 47), (39, 49), (42, 55)$;

for $y = 13, (30, 35), (31, 37), (34, 43), (35, 45)$;

for $y = 11, (26, 31), (27, 33), (30, 39)$

for $y = 7, (18, 23)$.

4.2.4 can be used for $W(38 : 47, 15, 13)$, $W(26 : 31, 11, 9)$ and $W(18 : 23, 7, 5)$. For the others, see the Appendix.

Finally, suppose that $y = z$ which implies that $y + z = 2y \equiv 2, 6$ or $10 \pmod{12}$ so only 3 of the cases in 4.2.5 and 4.2.6 are applicable. If $y \geq 25$, then $y + 4(y - z) \geq 25$, so 4.2.5 or 4.2.6 can be used. If $y = 23, 21, 17, 11, 9, 7, 5$ or 3, the appropriate labeling from 4.2.5 or 4.2.6 can also be used. The only remaining cases are: $y = z = 19, 15, 13, 1$.

Since $2y = y + z = 2n - 1 - x$ and $n + 3 \leq x < (4n - 1)/3$, we have

$$2y + 4 \leq n < 3y + 1.$$

Therefore, since $n \equiv 2, 3 \pmod 4$, $x = 2n - 2y - 1$ and $n + 3 \leq x < \frac{4n-1}{3}$, the only (n, x) pairs left to label are:

for $y = 19, (42, 45), (43, 47), (46, 53), (47, 55), (50, 61), (51, 63), (54, 69), (55, 71)$;

for $y = 15, (34, 37), (35, 39), (38, 45), (39, 47), (42, 53), (43, 55)$;

for $y = 13, (30, 33), (31, 35), (34, 41), (35, 43), (38, 49), (39, 51)$.

4.2.4 can be used for $W(42 : 45, 19, 19)$, $W(47 : 55, 19, 19)$ and $W(30 : 33, 13, 13)$. For the rest, see the Appendix.

Remark 4 The most difficult part of this proof was keeping track of which $g3$-windmills had been labeled by the various constructions. While we were creating the constructions, we made use of a computer program which determined how many of the windmills of a particular size were labeled by the techniques to-date. The final version of this is available at: http://www.math.mun.ca/∼manzer/.

6 Strong Skolem labelings

Unfortunately, not all of the labelings used above are strong. The use of sequences as building blocks clarifies the constructions; however, it often results in the introduction of non-essential edges. The problem is somewhat ameliorated when pruning is used or when a near sequence forms part of the labeling. Pruning makes all the edges of the Y- and Z-vanes essential. If a near sequence is used, the omitted labels are inserted elsewhere and help to tie the windmill together.

Conjecture 2 *Every $g3$-windmill that meets the Skolem parity condition can be strongly Skolem labeled.*

Conjecture 3 *Every gk-windmill that meets the Skolem parity and nondegeneracy conditions can be strongly Skolem-labeled.*

In [5] and [6], Mendelsohn and Shalaby also introduce the notion of *[strong] hooked Skolem-labelings* in which they permit some vertices, the *hooks*, to be labeled 0. These hooks may be in any position. Such a labeling with as few hooks as possible is called a *minimum hooked Skolem-labeling*. They then show that any path, cycle [5] or k-windmill, $k \geq 3$, that satisfies their degeneracy condition [6] has a [strong] Skolem or minimum hooked Skolem-labeling with the exception of the 3-windmills with vanes of length 2 or vanes of length 3 and the 4-windmills with vanes of length 1 or 2.

While the problem of minimum hooked labelings for $g3$-windmills is left for future work, we do expect similar results to hold. Here we will consider weak hooked labelings. As we have shown that every $g3$-windmill which meets the Skolem parity condition can be [weakly] Skolem-labeled, weak hooked Skolem-labelings will only be of interest in $g3$-windmills which do not meet the Skolem parity condition or which have an odd number of vertices. We mention the following partial result, but suspect that there is a minimum hooked Skolem labeling with at most 2 hooks in all cases.

Theorem 8 *Any $g3$-windmill, W, on v vertices, which cannot be Skolem-labeled has a weak hooked Skolem-labeling with at most 3 hooks.*

Proof. Suppose first that W has $v = 2n$ vertices. If $n \equiv 0 \pmod 4$ with 3 odd-length vanes or if $n \equiv 2 \pmod 4$ with one odd-length vane, label the last two vertices on the longest vane 0. If $n \equiv 1 \pmod 4$, then W has 3 odd-length vanes. Label the last vertex on each of the two longest vanes 0. If $n \equiv 3 \pmod 4$, W has one odd-length vane. Label the last vertex on each of the even-length vanes 0. In each case, except when $n = 2$, the remaining vertices form a $g3$-windmill which can be Skolem-labeled. If $n = 2$, the remaining 2 vertices can be labeled 1.

Now suppose that W has an odd number of vertices, say $v = 2n + 1$. Then W has 0 or 2 odd-length vanes. If $n \equiv 0, 1 \pmod 4$ with no odd-length vanes or if $n \equiv 2, 3 \pmod 4$ with 2 odd-length vanes, then label the last vertex on any even-length vane 0. If $n \equiv 0, 1 \pmod 4$ with 2 odd-length vanes, label the last vertex on the longest odd-length vane 0. Finally, suppose that W has no odd-length vanes. If $n \equiv 3 \pmod 4$, label the last vertex on each vane 0. If $n \equiv 2 \pmod 4$, label the 3 last vertices on the longest vane 0; note that this implies that the longest vane contains at least 3 (actually 4 since vane lengths are even) vertices, so the case of a windmill with 3 vanes of length 2 is not covered. The remaining vertices in all cases form a $g3$-windmill which can be Skolem-labeled except when W has 2 vanes of length 1 and $n \equiv 0, 1 \pmod 4$. In that case, the remaining $2n$ vertices form a path which can be Skolem-labeled.

The 3-windmill with vanes of length 2 does not have a one-hook strong Skolem-lableling [6]; however, it does have a weak labeling with one hook. Label the two vertices of a single vane 1 and the remaining 5 vertices with a 1-near hooked Skolem sequence of order 3.

Tying this altogether, we conclude with a final conjecture.

Conjecture 4 *All $g3$-windmills can either be strongly Skolem-labeled or have a minimum hooked Skolem-labeling with at most 2 hooks with the exception of the 3-windmills with vanes of length 2 or vanes of length 3.*

7 Appendix

7.1 For the following windmills, take $d = \frac{z+1}{2}$ and use $\mathcal{A}_d^{2d-1} + (n - 3d + 2)$ to label Z and the corresponding vertices of X. Then there is a path, B, of $x - n + d$ unlabeled vertices, $(1, z+1), \ldots, (x-n+d, z+1)$, at the end of X and a path, C, of $y+n-3d+2$ unlabeled vertices, $(x - n + 3d - 1, z + 1), \ldots, (x + y + 1, z + 1)$, along X and Y.

The largest unused label is $n-2d+1 = n-z$, which is used to label vertices $(a, z+1)$ and $(a + n - z, z + 1)$ in B and C respectively where a is given in the table below. The remaining vertices are also labeled as below.

parameters	$n-z$ in (-, $z+1$)	B	C
(27: 33,11,9)	3	3-ext $\mathcal{S}_5$	$\mathcal{L}_6^{12}$
(30: 35,13,11)	5	5-ext $\mathcal{S}_5$	$h\mathcal{L}_6^{13}$
(31: 35,13,13)	7	7-ext $\mathcal{S}_5$	$\mathcal{L}_6^{12}$
(34: 43,13,11)	4	4-ext $\mathcal{S}_7$	$\mathcal{L}_8^{15}$
(34: 37,15,15)	9	9-ext $\mathcal{S}_5$	$h\mathcal{L}_6^{13}$
(35: 41,15,13)	6	6-ext $\mathcal{S}_6$	$h\mathcal{L}_7^{15}$
(38: 45,15,15)	8	8-ext $\mathcal{S}_7$	$\mathcal{L}_8^{15}$
(39: 49,15,13)	5	5-ext $\mathcal{S}_8$	$\mathcal{L}_9^{17}$
(46: 53,19,19)	11	11-ext $\mathcal{S}_8$	$h\mathcal{L}_9^{18}$

7.2 We can modify this method slightly by placing the two labels $n-z$ and $n-z-1$ before labeling the rest of B and C.

parameters	$n-z$	$n-z-1$	B	C
(39: 47,15,15)	$(8, z+1)$	$(10, z+1)$	1 1 3 5 6 3 7 - 5 - 6 4 2 7 2 4	$\mathcal{L}_7^{16}$
(43: 47,19,19)	$(10, z+1)$	$(12, z+1)$	3 1 1 3 2 5 2 6 4 - 5 - 4 6	$\mathcal{L}_7^{16}$

7.3 For the windmills listed below, put

i) $(n-j)$ in $(2+2j, z+1)$ and $(n+2+j, z+1)$, for $j = 0, \dots, x-n-1$;

ii) $(n-j)$ in $(2+2j, z+1)$ and $(x+1, z-n+x-j)$, for $j = x-n, \dots, \lfloor \frac{n-3}{2} \rfloor$;

iii) a doubled $\mathcal{S}_{\lfloor \frac{n+2}{4} \rfloor}$ in vertices $(1+2j, z+1)$, for $j = 0, \dots, \lfloor \frac{n}{2} \rfloor$;

iv) $n - \lfloor \frac{n+1}{2} \rfloor$ in vertices $(2+2\lfloor \frac{n-1}{2} \rfloor, z+1)$ and $(n+1+\lfloor \frac{n-1}{2} \rfloor, z+1)$.

The remaining vertices of Y and Z are labeled as in the table below (listed from the position closest to the pivot out).

parameters	rest of Y	rest of Z
(30: 39,11,9)	7 9 11 13 - 5 3 1 1 3 5	7 9 11 13
(31: 37,13,11)	11 13 9 7 1 1 3 5 - 3 7 9 5	11 13
(34: 39,15,13)	13 15 11 9 7 3 1 1 3 5 - 7 9 11 5	13 15
(35: 45,13,11)	9 11 13 15 7 5 - 1 1 3 5 7 3	9 11 13 15
(50: 61,19,19)	15 13 17 19 21 23 5 11 9 7 3 5 - 3 13 15 7 9 11	17 19 21 23 1 1
(51: 63,19,19)	15 13 17 19 21 23 5 11 9 7 3 5 - 3 13 15 7 9 11	17 19 21 23 1 1

7.4 For the following windmills, modify the above construction by using the indicated label for vertex $(2+2\lfloor \frac{n-1}{2} \rfloor, z+1)$ on X and the corresponding vertex on Y.

parameters	$(2+2\lfloor \frac{n-1}{2} \rfloor, z+1)$	Y	rest of Z
$(34: 41, 13, 13)$ $(35: 43, 13, 13)$	15	11 7 13 9 5 17 15 3 7 5 3 11 9	13 17 1 1
(35: 39,15,15)	13	15 17 11 9 7 1 1 3 13 5 3 7 9 11 5	15 17

7.5 The labelings for the following windmills are similar to those above except the long run of labels starts with the first vertex of X rather than the second. Use

i) $(n-j)$ in $(1+2j, z+1)$ and $(n+1+j, z+1)$, for $j = 0, 1, \ldots, \lfloor \frac{n-3}{2} \rfloor$;

ii) the double of the extended sequence (for brevity we use $k - \mathcal{S}_n$ for a k-ext $\mathcal{S}_n$) given in the table for the even positions on X;

iii) the label given in column iii of the table for vertex $(1+2\lfloor \frac{n-1}{2} \rfloor, z+1)$ on X and the hole in the extended sequence of ii.

parameters	even	iii	Z	Y
(38: 49,13,13)	9-$\mathcal{S}_9$	19	7 11 13 15 17 3 5 7 3 1 1 5 20	20 5 11 13 15 17 5
(39: 51,13,13)	9-$\mathcal{S}_9$	17	9 11 7 13 15 3 20 19 3 7 1 1 11	5 9 13 15 19 5 20
(42: 53,15,15)	10-$\mathcal{S}_{10}$	21	11 13 15 17 19 9 7 3 1 1 3 11 22 7 9	22 5 13 15 17 19 5
(43: 55,15,15)	12-$\mathcal{S}_{10}$	19	11 13 17 9 3 21 7 3 1 1 11 22 9 7	22 5 13 15 17 21 5
(54: 69,19,19)	13-$\mathcal{S}_{13}$	27	13 15 17 19 21 23 25 11 9 5 1 1 3 13 5 3 28 9 11	7 15 17 19 21 23 25 7
(55: 71,19,19)	15-$\mathcal{S}_{13}$	25	13 11 15 17 19 21 23 9 28 27 5 3 11 13 3 5 9 1 1	7 15 17 19 21 23 27 7 28

7.6 A variation of the last labeling can be used for W(42:55,15,13):

i) $(42-j)$ in $(1+2j, 14)$ and $(43+j, 14)$, for $j = 0, \ldots, 13$;

ii) $(42-j)$ in $(1+2j, 14)$ and $(55, 26-j)$, for $j = 14, \ldots, 19$;

iii) a doubled 10-ext $\mathcal{S}_{10}$ in vertices $(2+2j, 14)$, for $j = 0, \ldots, 20$;

iv) 21 in vertices $(20, 14)$ and $(41, 14)$.

Y and Z are filled (from the pivot out), respectively with 9 7 11 13 15 17 19 3 7 9 3 1 1 11 22 and 22 5 13 15 17 19 5.

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