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Ring-magic labelings of graphs

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Abstract

In this paper, a generalization of a group-magic graph is introduced and studied. Let R be a commutative ring with unity 1. A graph $G = (V, E)$ is called R-ring-magic if there exists a labeling $f : E \to R - \{0\}$ such that the induced vertex labelings $f^+ : V \to R$, defined by $f^+(v) = \Sigma f(u, v)$ where $(u, v) \in E$, and $f^\times : V \to R$, defined by $f^\times(v) = \Pi f(u, v)$ where $(u, v) \in E$, are constant maps. General algebraic results for R-ring-magic graphs are established. In addition, $\mathbb{Z}_n$-ring-magic graphs and, in particular, trees are examined.

1 Introduction and notation

Let $G = (V, E)$ be a connected simple graph. For any non-trivial abelian group A (written additively), let $A^* = A - \{0\}$. A function $f : E \to A^*$ is called a labeling of G. Any such labeling induces a map $f^+ : V \to A$, defined by $f^+(v) = \Sigma f(u, v)$, where the sum is over all $(u, v) \in E$. If there exists a labeling f whose induced map on V is a constant map, we say that f is an A-magic labeling of G and that G is an A-magic graph.

Doob [1, 2, 3] and others [7, 9, 15, 16, 22] have studied A-magic graphs, and $\mathbb{Z}_k$-magic graphs were investigated in [4, 6, 8, 10, 11, 12, 13, 14, 17, 23]. $\mathbb{Z}$-magic graphs were considered by Stanley [24, 25], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations.

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A-magic graph is due to J. Sedláček [19, 20], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [5] had introduced yet another definition of a magic graph. Over the years, there has been great research interest in graph labeling problems. The interested reader is directed to Wallis' [26] recent monograph on magic graphs.

In this paper, we introduce a very natural generalization of group-magic graphs, namely the concept of a *ring-magic* graph. We assume all graphs are finite and simple. Let R be a commutative ring with unity 1. A graph $G = (V, E)$ is called *R-ring-magic* if there exists a labeling $f : E \to R - \{0\}$ such that the induced vertex labelings $f^+ : V \to R$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$, and $f^\times : V \to R$, defined by $f^\times(v) = \prod_{uv \in E} f(uv)$, are constant maps. In this case, the labeling f is called an *R-ring-magic labeling* of G. The values of f^+ and $f^\times$ are called the *additive* and *multiplicative R-magic values* of the R-ring-magic labeling f, respectively. If there is no ambiguity, we will omit the R.

Let $U(R)$ denote the multiplicative group of units in ring R. For $v \in V$, $d(v)$ will denote the degree of v in G.

2 Some observations

Theorem 2.1. *A regular graph G is R-ring-magic, for any ring R.*

Proof. Label each edge of G with an element $x \in R - \{0\}$. Then, the induced vertex labelings f^+ and $f^\times$ are constant maps. □

Theorem 2.2. *Let A be the (additive) abelian group associated with ring R. If G is not A-magic, then G is not R-ring-magic.*

Proof. This is immediately clear. □

Theorem 2.3. *G is $\mathbb{Z}_2$-ring-magic $\iff$ the degree of each vertex is of the same parity.*

Proof. If G is $\mathbb{Z}_2$-ring-magic, then all of the edges must be labeled 1. Furthermore, the additive $\mathbb{Z}_2$-magic value is constant. Thus, the degree of each vertex is of the same parity. Conversely, if the degree of v_i is of the same parity for all $v_i \in V(G)$, then label every edge of G with 1. This gives a $\mathbb{Z}_2$-ring-magic labeling of G. □

3 General results

Theorem 3.1. *Let R be an integral domain. Then, the multiplicative R-magic value is nonzero.*

Theorem 3.2. *Let R be a ring and $G = (V, E)$ be an R-ring-magic graph of order p. Let h and k be the additive and multiplicative R-magic values of an R-ring-magic labeling f. Then, $hp = 2a$ and $k^p = b^2$ for some $a, b \in R$.*

Proof. From the definition, we have

$$hp = \sum_{v\in V} f^+(v) = \sum_{v\in V}\sum_{uv\in E} f(uv) = 2\sum_{e\in E} f(e) = 2a,$$

where $a = \sum\limits_{e\in E} f(e)$, and

$$k^p = \prod_{v\in V} f^\times(v) = \prod_{v\in V}\prod_{uv\in E} f(uv) = \left(\prod_{e\in E} f(e)\right)^2 = b^2,$$

where $b = \prod\limits_{e\in E} f(e)$. □

Theorem 3.3. *Let R_1 be a ring, which contains a subring isomorphic to ring R_2. If graph G is R_2-ring-magic, then G is R_1-ring-magic.*

Proof. Let $S \leq R_1$. Suppose that $\phi : R_2 \to S$ is a ring isomorphism and that f is an R_2-ring-magic labeling of G. Let h and k be the additive and multiplicative R_2-magic values, respectively, of f. Now, consider an arbitrary vertex $v \in G$ and let e_i denote the number of edges labeled i, which are incident to v. Then, $h = \Sigma i e_i$ and $k = \Pi i^{e_i}$, where i varies through all the elements of $R_2 - \{0\}$. We now apply ϕ to all of the labeled edges of G. Under this new labeling, we see the following relationships from v:

$$\phi(h) = \phi[\Sigma i e_i] = \Sigma\phi(i)e_i, \text{ and}$$

$$\phi(k) = \phi[\Pi i^{e_i}] = \Pi[\phi(i)]^{e_i}.$$

Since $i \in R_2 - \{0\}$ and ϕ is a ring isomorphism, no edge is labeled 0_{R_1}. Hence, we have an R_1-ring-magic labeling of G. □

It should be noted that the converse of Theorem 3.3 is not true. In the next section, we give a counter-example.

Theorem 3.4. *Let R be an integral domain. Suppose that f is an R-ring-magic labeling of graph G and $u \in U(R)$, where $o(u)$ is the order of u in $U(R)$. Then, uf is an R-ring-magic labeling of $G \iff o(u) | [d(v_i) - d(v_j)]$, for all v_i, $v_j \in V(G)$.*

Proof. Suppose that f is an R-ring-magic labeling of G with additive and multiplicative R-magic values h and k, respectively. Then for each vertex v, $(uf)^{+}(v) = uh$. Furthermore, uf does not label any edge of G with the zero element in R. Also, for each vertex v, $(uf)^{\times}(v) = u^{d(v)}k$. Thus, $(uf)^{\times}$ is a constant map if and only if for any two vertices v_i, $v_j \in V(G)$,

$$\begin{aligned} u^{d(v_i)}k = u^{d(v_j)}k &\iff u^{d(v_i)}k - u^{d(v_j)}k = 0 \\ &\iff (u^{d(v_i)} - u^{d(v_j)})k = 0 \\ &\iff u^{d(v_i)} - u^{d(v_j)} = 0 \\ &\iff u^{d(v_i)} = u^{d(v_j)} \\ &\iff u^{d(v_i)-d(v_j)} = 1 \\ &\iff o(u) | [d(v_i) - d(v_j)]. \end{aligned}$$

□

4 $\mathbb{Z}_n$-ring-magic graphs

In this section (and future sections) of the paper, we will use some number theory to further analyze ring-magic labelings. We now recall some definitions and facts [18].

Definition. If m is a positive integer, we say that the integer a is a *quadratic residue of* m if $\gcd(a, m) = 1$ and the congruence $x^2 \equiv a \pmod{m}$ has a solution. If the congruence $x^2 \equiv a \pmod{m}$ has no solution, we say that a is a *quadratic nonresidue of* m.

Definition. Let n be an odd prime and a an integer not divisible by n. Then, the *Legendre symbol* $\left(\frac{a}{n}\right)$ is defined by

$$\left(\frac{a}{n}\right) = \begin{cases} 0, & \text{if } n \text{ is a divisor of } a; \\ 1, & \text{if } a \text{ is a quadratic residue of } n; \\ -1, & \text{if } a \text{ is a quadratic nonresidue of } n. \end{cases}$$

Theorem A. *Let n be an odd prime and a and b integers not divisible by n. Then,*

- *if* $a \equiv b \pmod{n}$, *then* $\left(\frac{a}{n}\right) = \left(\frac{b}{n}\right)$.
- $\left(\frac{a}{n}\right) \cdot \left(\frac{b}{n}\right) = \left(\frac{ab}{n}\right)$.
- $\left(\frac{a^2}{n}\right) = 1$.

Theorem B. *Let n be an odd prime. Then, there are exactly $(n-1)/2$ quadratic residues of n and $(n-1)/2$ quadratic nonresidues of n among the integers $1, 2, \ldots, n-1$.*

The following results are now established.

Theorem 4.1. *Suppose G is a $\mathbb{Z}_n$-ring-magic graph of odd order. Then, the multiplicative magic value k satisfies $\left(\frac{k}{r}\right) = 1$ or 0, for each odd prime factor r of n.*

Proof. Let G be of odd order p. By Theorem 3.2, there exists $b \in \mathbb{Z}_n$ such that $\left(\frac{k}{r}\right) = \left(\frac{k}{r}\right)^p = \left(\frac{k^p}{r}\right) = \left(\frac{b^2}{r}\right) = \left(\frac{b}{r}\right)^2$. Hence, $\left(\frac{k}{r}\right)$ is either 1 or 0. □

Corollary 4.2. *Suppose G is a $\mathbb{Z}_n$-ring-magic graph of odd order, where n is an odd prime. Then, the multiplicative magic value k must be a square.*

Corollary 4.3. *Suppose G is a $\mathbb{Z}_3$-ring-magic graph of odd order. Then, the multiplicative magic value must be 1.*

Corollary 4.4. *Suppose G is a $\mathbb{Z}_5$-ring-magic graph of odd order. Then, the multiplicative magic value must be 1 or 4.*

Corollary 4.5. *Let $k|n$. If G is $\mathbb{Z}_k$-ring-magic, then G is $\mathbb{Z}_n$-ring-magic.*

Proof. This follows immediately from Theorem 3.3. □

As we alluded to previously, the converse of Corollary 4.5 is not true. The graph G illustrated in Figure 1 provides a counter-example. In this case, G is Eulerian. Thus by Theorem 2.3, G is $\mathbb{Z}_2$-ring-magic. By Corollary 4.5, G is $\mathbb{Z}_6$-ring-magic. However, it is straight-forward to show (using an exhaustive case analysis) that G is not $\mathbb{Z}_3$-group-magic and hence, cannot be $\mathbb{Z}_3$-ring-magic.

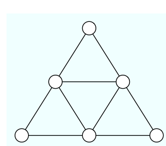

Figure 1: G is $\mathbb{Z}_6$-ring-magic, but is not $\mathbb{Z}_3$-ring-magic.

At this point, it is very natural for the reader to ask the following question. For a given n, are there graphs which are $\mathbb{Z}_n$-group-magic but not $\mathbb{Z}_n$-ring-magic? As we will see, the answer to this question is yes. First, we establish the following technical lemma.

Lemma 4.6. *Let n be an odd prime. Then, there exists $y \geq 1$ such that the following hold:*

- $y \equiv -1 \ (mod\ n-1)$.
- $\left(\frac{y^2-2y-3}{n}\right) = -1$, *where $\left(\frac{a}{b}\right)$ is the Legendre symbol.*

Proof. Clearly, $y \equiv -1 \pmod{n-1}$ is equivalent to $y = -1 + k(n-1)$. This yields

$$\left(\frac{y^2 - 2y - 3}{n}\right) = \left(\frac{(-1-k)^2 - 2(-1-k) - 3}{n}\right) = \left(\frac{k(k+4)}{n}\right).$$

Since $\left(\frac{1}{n}\right) = 1$, Theorem B implies the existence of an integer w $(1 \le w \le n-1)$ such that $\left(\frac{w}{n}\right) = 1$ and $\left(\frac{w+1}{n}\right) = -1$. Thus, $\left(\frac{4w}{n}\right)\left(\frac{4(w+1)}{n}\right) = -1$. Hence, $\left(\frac{k(k+4)}{n}\right) = -1$, where $k = 4w$. □

Let $C_m(y)$ denote the m-cycle with y pendants attached to each vertex.

Theorem 4.7. *Let n be an odd prime. Then, there exists an integer y_0 such that $C_4(y_0)$ is $\mathbb{Z}_n$-group-magic, but not $\mathbb{Z}_n$-ring-magic.*

Proof. Consider the graph $C_4(y_0)$, where y_0 satisfies the two conditions in Lemma 4.6. Note that $C_4(y_0)$ is $\mathbb{Z}_n$-group-magic, as all of the pendants can be labeled with 1 and the edges in the cycle labeled a, $1-a-y_0$, a, and $1-a-y_0$ respectively. We now claim that $C_4(y_0)$ is not $\mathbb{Z}_n$-ring-magic. Assume that $C_4(y_0)$ has a $\mathbb{Z}_n$-ring-magic labeling. Because of Theorem 3.4 and the first condition in Lemma 4.6, we can assume without loss of generality that the pendants are labeled 1. Let a and b be the labels of the two non-pendant edges incident to a vertex of degree $y_0 + 2$ in $C_4(y_0)$. Then, the following two relationships must hold:

- $a + b + y_0 \equiv 1 \pmod{n}$.
- $ab \equiv 1 \pmod{n}$.

This is equivalent to saying that $x(1 - y_0 - x) \equiv 1 \pmod{n}$ has a solution, which implies that $x^2 + (y_0 - 1)x + 1 \equiv 0 \pmod{n}$ has a solution. However, this equation has discriminant $D = (y_0 - 1)^2 - 4(1) = y_0^2 - 2y_0 - 3$. Since y_0 was chosen as to satisfy the conditions of Lemma 4.6, D is not a square in $\mathbb{Z}_n$. Thus, $x^2 + (y_0 - 1)x + 1 \equiv 0$ has no solutions in $\mathbb{Z}_n$. Hence, we reach a desired contradiction. □

Let us consider an example which illustrates Theorem 4.7. In Figure 2, the graph $C_4(5)$ is $\mathbb{Z}_7$-group-magic, but not $\mathbb{Z}_7$-ring-magic.

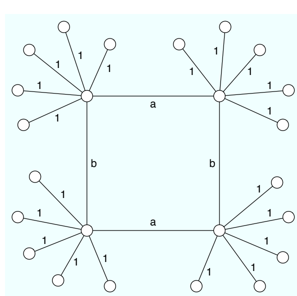

Figure 2: The system $a + b \equiv 3 \pmod{7}$, $ab \equiv 1 \pmod{7}$ has no solutions.

5 Ring-magicness of trees

Suppose that, as before, R is a commutative ring with unity 1 and that T is a tree. If f is an R-ring-magic labeling of T, then the additive and multiplicative magic values of f are the same. We call this value the *R-ring-magic value* of T.

To prove the first lemma in this section of the paper, we need slightly more general definitions of "group-magic" [1, 2, 3] and "ring-magic" graphs, as well as a result (Theorem C) from [21].

Definition. Let A be a non-trivial abelian group. A graph $G = (V, E)$ is called *A'-group-magic* if there exists a labeling $f : E \to A$ such that the induced vertex labeling $f^+ : V \to A$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$, is a constant map.

The A'-group-magic concept defined above corresponds to Doob [1, 2, 3] and Stanley's [24, 25] definition of "group-magic", which allows edges of a graph to be labeled 0.

Definition. Let R be a commutative ring with unity. A graph $G = (V, E)$ is called *R'-ring-magic* if there exists a labeling $f : E \to R$ such that the induced vertex labelings $f^+ : V \to R$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$, and $f^\times : V \to R$, defined by $f^\times(v) = \prod_{uv \in E} f(uv)$, are constant maps.

Theorem C. *Let T be a tree and suppose that f is a $\mathbb{Z}'_n$-group-magic labeling of T. If there is an edge e which is incident to a leaf of T and $f(e) = 0$, then $f = 0$.*

We now continue the analysis of the ring-magic property for trees.

Lemma 5.1. *Let T be a tree. Then, T has at most one $\mathbb{Z}_n$-ring-magic labeling with $\mathbb{Z}_n$-ring-magic value k.*

Proof. Suppose that f and g are two $\mathbb{Z}_n$-ring-magic labelings of T, with $\mathbb{Z}_n$-ring-magic value k. In particular, f and g are $\mathbb{Z}_n$-group-magic labelings of T. Now, consider the labeling $F = f - g$, which is a $\mathbb{Z}'_n$-group-magic labeling of T. By Theorem C, $F = 0$. Thus, $f = g$. □

Lemma 5.2. *Let T be a tree. Suppose that f is a $\mathbb{Z}_n$-ring-magic labeling of T, with $\mathbb{Z}_n$-ring-magic value* 1. *Then, $f = 1$ (i.e., all the values of f are* 1*).*

Proof. Suppose that T is a rooted tree and let T have a $\mathbb{Z}_n$-ring-magic labeling with $\mathbb{Z}_n$-ring-magic value 1. Clearly, every edge of T which is adjacent to a leaf must be labeled 1. Also, note that every vertex of T which is adjacent to a leaf must be of degree congruent to 1 (mod n). We induct on the number of vertices in T. Clearly, the lemma is true for $T = P_2$. Assume that the lemma holds for all trees of order less than p. Now, let T be a tree of order p and suppose that it has a $\mathbb{Z}_n$-ring-magic labeling f with $\mathbb{Z}_n$-ring-magic value 1. Choose a vertex u at the highest level (where the root is at the 0-level) of rooted tree T. Let v be parent of u and x be a parent of v. Note that each child w_i of v (including u) is a leaf of T and that there are kn children of v. Furthermore, the edges w_iv, edge uv and edge vx are all labeled 1. Using f, tree T' obtained by deleting all of the children of v has a $\mathbb{Z}_n$-ring-magic labeling with $\mathbb{Z}_n$-ring-magic value 1. By the induction hypothesis, $f = 1$ on T'. Adding back the deleted children of v and edges w_iv labeled 1, edge uv labeled 1 to T', establishes the claim. □

The following theorem is a generalization of a result found in [9]. Its proof mimics the one given by Lee, Saba, Salehi and Sun.

Lemma 5.3. *Let A be an abelian group of even order. Then, tree T has an A-group-magic labeling with additive magic value h, where $o(h) = \frac{|A|}{2} \iff T$ has no vertex of even degree.*

Proof. ($\Leftarrow$). Let all of the vertices of T be of odd degree. Then, label all the edges of T with h where $o(h) = \frac{|A|}{2}$ and $h \in A^*$. Such an element h exists since A is a nilpotent group.

($\Rightarrow$). Suppose that T has an A-group-magic labeling with additive magic value h, where $o(h) = \frac{|A|}{2}$. We show, by induction on $q = |E(T)|$, that all of the vertices of T are of odd degree. Clearly when $q = 1$, T has no vertex of even degree. If $q = 2$, then $T = P_3$ which is non-magic. If $q = 3$, then $T = P_4$ which is non-magic, or $T = K_{1,3}$ with all of its vertices of odd degree. Suppose that all of the vertices of T are of odd degree, for all non-trivial trees with at most $q - 1$ edges. Now, let T be a tree with $q = |E(T)| > 3$. Consider a longest path P in T, say that it joins the vertex v_1 to v_{k+1}. Note that $\deg(v_1) = 1$ and $\deg(v_2) > 2$. Also, all of the vertices adjacent to v_2 have degree one, except possibly for v_3. Let w_1 ($\neq v_3$) be another vertex adjacent to v_2. Now, if we remove the two edges v_1v_2 and v_2w_1 (along with v_1 and w_1), then $f^+(v_2)$ does not change nor is the parity of the degree of v_2 affected. By the induction hypothesis, this resulting tree T' has an A-group-magic labeling with additive magic value h, $o(h) = \frac{|A|}{2}$, and all of its vertices are of odd degree. Hence, all of the vertices of T are of odd degree. □

Let V_4 denote the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma 5.4. *A tree T is V_4-group-magic $\iff$ T has no vertex of even degree.*

Lemma 5.5. *Let T be a tree of odd order. Then, T is not V_4-ring-magic.*

Proof. Every graph has an even number of vertices of odd degree. Since T has an odd number of vertices, T must have at least one vertex of even degree. By Lemma 5.4, T is not V_4-group-magic and hence, is not V_4-ring-magic. □

Theorem 5.6. *Let T be a tree. Then, T is V_4-ring-magic $\iff d(v) \equiv 1 \pmod 2$, for all $v \in V(T)$. Moreover, for this case, suppose T is of order p. Then, $p \equiv 0 \pmod 2$.*

Proof. Suppose that T is V_4-ring-magic. Then in particular, T is V_4-group-magic. By Lemma 5.4, we have that $d(v) \equiv 1 \pmod 2$, for all $v \in V(T)$.

Conversely, suppose that $d(v) \equiv 1 \pmod 2$, for all $v \in V(T)$. Then, the constant map f which labels every edge of T with x, where $x \in V_4 - \{0\}$, is a V_4-ring-magic labeling of T. Here, $f^+ = f^\times = x$.

If T is V_4-ring-magic, then the contrapositive of Lemma 5.5 implies that T is of even order. □

Theorem 5.7. *Let T be a tree. Then, T is $\mathbb{Z}_3$-ring-magic with $\mathbb{Z}_3$-ring-magic value 1 $\iff d(v) \equiv 1 \pmod 3$, for all $v \in V(T)$. Moreover, for this case, suppose T is of order p. Then, $p \equiv 2 \pmod 3$.*

Proof. Suppose T has a $\mathbb{Z}_3$-ring-magic f with $\mathbb{Z}_3$-ring-magic value 1. By Lemma 5.2 $f = 1$. Hence, $d(v) = f^+(v) \equiv 1 \pmod 3$ for all $v \in V(T)$.

Conversely, if $d(v) \equiv 1 \pmod 3$, then let $g = 1$ be the constant mapping. Clearly, g is a $\mathbb{Z}_3$-ring-magic labeling of T with $g^\times = 1$.

Now, suppose T is of order p and has a $\mathbb{Z}_3$-ring-magic f with $\mathbb{Z}_3$-ring-magic value 1. Then by the proof of Theorem 3.2, we have

$$p = \sum_{v \in V} f^+(v) = 2(p-1) \pmod 3.$$

Hence, $p \equiv 2 \pmod 3$. □

Combining Theorem 5.7 with Corollary 4.3, we obtain:

Corollary 5.8. *A tree T of odd order p is $\mathbb{Z}_3$-ring-magic $\iff p \equiv 5 \pmod 6$ and $d(v) \equiv 1 \pmod 3$, for all $v \in V(T)$.*

Now, we direct our attention to trees of even order and having $\mathbb{Z}_3$-ring-magic value 2.

Theorem 5.9. *Suppose a tree T has a $\mathbb{Z}_3$-ring-magic labeling f with $\mathbb{Z}_3$-ring-magic value 2. Let v be a vertex of T which is adjacent to pendants. Then, T is of even order and $d(v) \equiv 1$ or $0 \pmod 6$.*

Proof. The order of T must be even follows immediately from the contrapositive of Corollary 4.3. Let a and b denote the number of 1 and 2 labeled to the edges incident with v, respectively. Clearly $a + b = d(v) = d$. Since the ring-magic value is 2, we have $2^b \equiv 2 \pmod 3$, and b is odd. Also, note that $a + 2b \equiv 2 \pmod 3$. Since $f(uv) = 2$ for all pendants u adjacent with v, this implies that $b = d$ or $b = d - 1$.

(CASE 1). If $b = d$, then $a = 0$ and $b \equiv 1 \pmod 3$. Since b is odd, $d = b \equiv 1 \pmod 6$.

(CASE 2). If $b = d - 1$, then $a = 1$ and $b \equiv 2 \pmod 3$. Since b is odd, $d = b + 1 \equiv 0 \pmod 6$. □

Note that the converse of Theorem 5.9 is not true in general. For example, the converse holds in the case where $H = K_{1,7}$. However, Figure 3 provides a counter-example which illustrates that the converse does not always hold. There, graph G is of order 18, $d(v_i) \equiv 1$ or $0 \pmod 6$, but G is not $\mathbb{Z}_3$-group-magic. Hence, it is not $\mathbb{Z}_3$-ring-magic.

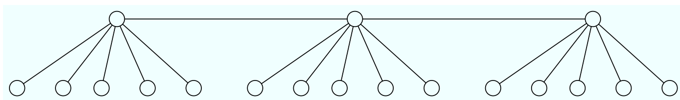

Figure 3: This caterpillar graph G is not $\mathbb{Z}_3$-ring-magic.

Using some of the results described above, we can construct an efficient algorithm for determining when a tree T is $\mathbb{Z}_3$-ring-magic. Suppose T is of order p. If p is odd, then Corollary 5.8 is used to determine if T is $\mathbb{Z}_3$-ring-magic. If p is even, the following procedure can be used. Here, T is viewed as a rooted tree.

Step 1. If $p \equiv 4 \pmod 6$ and $d(v) \equiv 1 \pmod 3$ for all $v \in V(T)$, then T is $\mathbb{Z}_3$-ring-magic (with ring-magic value 1).

Step 2. For each pendant p_i, check each father v_i and see if $d(v_i) \equiv 1$ or $0 \pmod 6$. If not, then T is not $\mathbb{Z}_3$-ring-magic. Otherwise, let $T^* = T$.

Step 3. If $T^* = K_2$, then T is $\mathbb{Z}_3$-ring-magic (with ring-magic value 2). Stop.

Step 4. Choose a pendant e and let w be its father.

Step 5. If $d(w) \equiv 1 \pmod 6$, then delete all pendants adjacent to w to obtain a smaller tree T', let $T^* = T'$ and go to Step 3.

Step 6. Let u be the father of w. Obtain a new tree T' by deleting all pendants adjacent to w from T^* and adding an extra vertex w' and an edge uw' to T'.

Step 7. If $d(u) \equiv 1$ or $0 \pmod 6$, then let $T^* = T'$ and go to Step 3. Otherwise, T is not $\mathbb{Z}_3$-ring-magic. Stop.

6 Acknowledgements

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