AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **41** (2008), Pages 107–114

On the number of tilings of the rectangular board with T-tetrominoes

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Abstract

The classical combinatorial problem of counting domino tilings of a $2n \times 2m$ rectangle was solved by P.W. Kasteleyn and also by H.N.V. Temperley and M.E. Fisher in 1961. We shall consider the similar problem for T-tetrominoes, that is, pieces formed by 4 unit squares in the shape of a T. We give explicit formulae for the number of tilings with T-tetrominoes for the $4n \times 4m$ rectangle when $n = 1, 2, 3$ and 4, and a computational method for values of n up to 8.

1 Introduction

One of the classical results in enumerative combinatorics is the formula for the number of domino tilings of a $2n \times 2m$ chess-board. The formula is the following:

$$4^{nm} \prod_{s=1}^{n} \prod_{t=1}^{m} \left(\cos^2 \frac{s\,\pi}{2n+1} + \cos^2 \frac{t\,\pi}{2m+1} \right).$$

This was obtained originally by Kasteleyn [7], and also by Temperley and Fisher [14]. Later Lieb [10] and more recently V. Strehl [13] gave different methods to obtained the same formula.

In order to state our problem we give some definitions first. A *tetromino* is a two-dimensional shape made by connecting 4 unit squares along their edges. There are 5 possible tetrominoes (up to rotations and reflections) of which one has a T-shape, and we called it *T-tetromino*. A *tiling* of a plane region R is a covering of R using a given set of tiles, completely and without any overlap. We have the following:

* Supported by CONACYT of México, Proyecto 43098.

Theorem 1 (D.W. Walkup 1965). *An $r \times s$ rectangular board has a tiling with T-tetrominoes as tiles if and only if r and s are multiples of* 4.

Our problem is to count the number of tilings of a $4n \times 4m$ rectangular board with T-tetrominoes or *T-tilings*, which we denote by $f(n,m)$. We use $f(n)$ for the number of T-tilings of the square board.

The rectangular lattice $L_{n,m}$ is a well-known graph (see Biggs [2]), but for reference we include the definition here. The graph $L_{n,m}$ has as vertices the set $\{0,\dots,n-1\} \times \{0,\dots,m-1\}$ where two vertices (i,j) and (i',j') are adjacent if $|i-i'|+|j-j'|=1$. This definition also gives a natural planar embedding of $L_{n,m}$, in which each bounded face is a unit square. We consider the region defined by this planar embedding of $L_{n+1,m+1}$ as the $n \times m$ rectangular board. For the basic graph theory required in this paper the reader is referred to the book of Bondy and Murty [3].

2 T-tilings and The Tutte polynomial

For a graph $G=(V,E)$, we identify a subset of edges $A \subseteq E$ of G with the subgraph of G that A spans, that is, the spanning subgraph (V,A). Then the number of connected components of A is denoted by $k(A)$ and the *rank* of A is defined by $r(A)=|V|-k(A)$. When we have a planar embedding of G, any subgraph of G has an induced planar embedding. In this case, we defined the number $f(A)$, as the number of bounded faces of A.

The *Tutte polynomial* $T(G;x,y)$ of a graph $G=(V,E)$ is a two variable polynomial and has the following expansion

$$T(G;x,y)=\sum_{A\subseteq E}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \tag{2.1}$$

If G is a connected plane graph, $r(E)-r(A)=k(A)-1$ and $|A|-r(A)=f(A)$. The last equality is obtained by applying Euler's formula to each connected component of A.

When evaluating the Tutte polynomial along different curves and points we get several interesting invariants of graphs. Among them we have the chromatic and flow polynomials of a graph, the all terminal reliability probability of a network and the partition function of the Q-state Potts model. Further details of many of the invariants can be found in the survey article of Brylawski and Oxley [4] and in the book of Welsh [17].

Theorem 2 (Korn and Pak, 2003). *The number of T-tilings of a $4n \times 4m$ board satisfy*

$$f(n,m)=2\,T(L_{n,m};3,3). \tag{2.2}$$

Sketch. We defined the *top* of a T-tetromino as the line segment of length 3 on its boundary and its *bottom* as the unit line segment on the boundary opposite to the top.

Consider the lattice induced by the points (a,b) such that $a, b \equiv 2 \pmod 4$ in a $4n \times 4m$ board. We obtain a graph isomorphic to $L_{n,m}$, where each edge has length 4. Now, let us take any subset of edges A in $L_{n,m}$ and let C be the set of boundary points of a bounded (unbounded) face F in A. Then, let R be the region inside (outside) F made of by all the points at Chebyshev distance at most 2 from C. Here the Chebyshev distance is the one induced by the l^{∞}-norm. The left-hand side of Figure 1 shows a set A of edges of $L_{6,5}$ with 4 bounded faces and 3 unbounded faces.

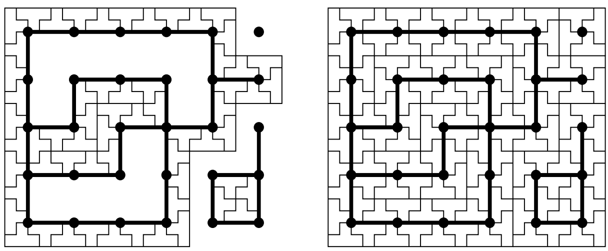

Figure 1: The figure on the left shows a partial T-tiling of a 24 by 20 rectangular board. The figure on the right shows a complete T-tiling of the board.

In general, the region R can be tiled with T-tetrominoes in many different ways. But there are exactly two T-tilings of R where the top and bottom of the tiles alternatively intersect the set C. An example is shown on the left-hand side of Figure 1.

For a fixed A, the union of these regions is the $4n \times 4m$ rectangular board. Thus, to each subset A of edges of the lattice $L_{n,m}$ we have associated $2^{f(A)+k(A)}$ T-tilings.

Clearly, from different edge-sets we get different tilings. Also, each tiling can be obtained b this procedure from an edge-set A and a choice of T-tilings o each region. Summing up these quantities for all the subsets of edges in $L_{n,m}$ we get that the number of T-tilings is

$$\begin{aligned} f(n,m) &= \sum_{A \subseteq E(L_{n,m})} 2^{k(A)+f(A)} \\ &= 2 \sum_{A \subseteq E(L_{n,m})} 2^{k(A)-1}\, 2^{f(A)} \\ &= 2 \sum_{A \subseteq E(L_{n,m})} 2^{r(E)-r(A)}\, 2^{|A|-r(A)} = 2\,T(L_{n,m}; 3, 3). \end{aligned}$$

□

For example, the Tutte polynomial of the square lattice $L_{3,3}$ is

$$\begin{aligned} T(L_{3,3}; x, y) = {} & x^8 + 4x^7 + 10x^6 + 16x^5 + 19x^4 + 16x^3 + 10x^2 + 3x \\ & + 4x^5y + 12x^4y + 20x^3y + 20x^2y + 13xy + 3y \\ & + 4x^3y^2 + 10x^2y^2 + 12xy^2 + 6y^2 + 4xy^3 + 4y^3 + y^4; \end{aligned}$$

and so, the number of T-tilings of a 12-by-12 board is 78696.

3 The number of T-tilings

Using Equation 2.2, we can easily compute the number of T-tilings for the $4 \times 4m$ board, as the corresponding lattice $L_{1,m}$ is a path of length $m-1$, whose Tutte polynomial is just x^{m-1}. Thus, for the 4x4 board we get $f(1,1) = 2$ and, in general, we have that

$$f(1,m) = 2(3)^{m-1}.$$

This sequence is already known as A008776 [11] and could have been obtained more easily by direct counting. However, direct counting is not so straightforward for the $8 \times 4m$ board. But the corresponding lattice in Theorem 2 is a ladder, and it is easy to find the recurrence relation $f(2,m) = 16f(2,m-1) - 27f(2,m-2)$, for $m \geq 3$, with initial conditions $f(2,1) = 6$ and $f(2,2) = 84$. After solving we get the following formula:

$$f(2,m) = \left(3 - \frac{18}{\sqrt{37}}\right)(8-\sqrt{37})^{m-1} + \left(3 + \frac{18}{\sqrt{37}}\right)(8+\sqrt{37})^{m-1}. \tag{3.1}$$

For the board of width 12 we have to evaluate $T(L_{3,m}; 3, 3)$ that could still be done by finding a recurrence relation and solving it. This method is going to give a formula for the boards of fixed width n for small n; but it is bound to fail for the general case. There is another way to compute $T(L_{n,m}; x, y)$ for a fixed width n at point (x,y) which is described in Calkin et al. [6] and it is an application of the transfer-matrix method. In this case we have the same restriction as before, a fixed width n for small values of n, but it has the advantage of being easily automatized.

Theorem 3 (Calkin et al. 2001)**.** *For real values x and y with $x \neq 0$ and integers $n, m \geq 2$, we have*

$$T(L_{n,m}; x+1, y+1) = x^{nm-1} X_n^t \cdot (\Lambda_n)^{m-1} \cdot \vec{1},$$

where X_n, a vector of length c_n, and Λ_n, a $c_n \times c_n$ matrix, depend on x,y and n but not m. And $\vec{1}$ is the vector of length c_n with all entries equal to 1.

The quantity c_n is the n-Catalan number, so the method is just practical for small values of n. Computing the vectors X_n and the matrix Λ_n can be easily done in a computer. Following the example in Calkin et al. [6] we get

$$T(L_{2,m}; 3, 3) = 2^{2m-1} \begin{pmatrix} 1, & 1/2 \end{pmatrix} \begin{pmatrix} 9/4 & 9/8 \\ 2 & 7/2 \end{pmatrix}^{m-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In the above equation we can get integer entries by multiplying by an appropriate power of 2. Thus, by using Equation 2.2 we get the formula

$$f(2,m) = \frac{1}{2^{m-2}} \begin{pmatrix} 2, & 1 \end{pmatrix} \begin{pmatrix} 18 & 9 \\ 16 & 14 \end{pmatrix}^{m-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The method applied to the rectangular lattice $L_{3,m}$ gives

$$f(3,m) = \frac{1}{2^{2m-4}} \begin{pmatrix} 4, & 2, & 0, & 2, & 1 \end{pmatrix} \begin{pmatrix} 108 & 54 & 0 & 54 & 27 \\ 96 & 84 & 0 & 48 & 42 \\ 96 & 48 & 12 & 48 & 48 \\ 96 & 48 & 0 & 84 & 42 \\ 80 & 64 & 8 & 64 & 96 \end{pmatrix}^{m-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The first 5 terms of the sequence are 18, 1182, 78696, 5253822, 350950482.

In general, we have the formula

$$f(n,m) = \frac{1}{2^{(n-1)(m-1)}} \vec{v}_n A_n^{m-1} \vec{1},$$

where $A_m = 2^{2m-1}\Lambda_n$ and $\vec{v}_n = 2^{m-1}X_n$. For $n = 4$ the matrix A_4 is the 14-by-14 matrix

$$\begin{pmatrix}
648 & 324 & 0 & 0 & 324 & 162 & 0 & 0 & 0 & 324 & 162 & 0 & 162 & 81 \\
576 & 504 & 0 & 0 & 288 & 252 & 0 & 0 & 0 & 288 & 252 & 0 & 144 & 126 \\
576 & 288 & 72 & 0 & 288 & 288 & 0 & 0 & 0 & 288 & 144 & 36 & 144 & 144 \\
576 & 288 & 0 & 72 & 288 & 144 & 36 & 0 & 36 & 288 & 144 & 36 & 144 & 162 \\
576 & 288 & 0 & 0 & 504 & 252 & 0 & 0 & 0 & 288 & 144 & 0 & 252 & 126 \\
480 & 384 & 48 & 0 & 384 & 576 & 0 & 0 & 0 & 240 & 192 & 24 & 192 & 288 \\
512 & 256 & 0 & 64 & 448 & 224 & 56 & 0 & 32 & 256 & 128 & 32 & 224 & 240 \\
576 & 288 & 0 & 0 & 288 & 144 & 0 & 72 & 36 & 288 & 144 & 0 & 288 & 144 \\
480 & 384 & 0 & 48 & 240 & 192 & 24 & 48 & 120 & 240 & 192 & 24 & 216 & 348 \\
576 & 288 & 0 & 0 & 288 & 144 & 0 & 0 & 0 & 504 & 252 & 0 & 252 & 126 \\
512 & 448 & 0 & 0 & 256 & 224 & 0 & 0 & 0 & 448 & 392 & 0 & 224 & 196 \\
480 & 240 & 48 & 48 & 240 & 216 & 24 & 0 & 24 & 384 & 192 & 120 & 192 & 348 \\
480 & 240 & 0 & 0 & 384 & 192 & 0 & 48 & 24 & 384 & 192 & 0 & 576 & 288 \\
384 & 288 & 32 & 32 & 288 & 400 & 16 & 32 & 80 & 288 & 192 & 80 & 400 & 776
\end{pmatrix}$$

and the vector $\vec{v}_4$ is $(8,4,0,0,4,2,0,0,0,4,2,0,2,1)$. The first 5 terms of the sequence are 54, 16644, 5253822, 1668091536, 530454033510.

The vectors and matrices for n up to 8 can be found in `http://www.matem.unam.mx/~merino/e_publications.html`. All the matrices are nonnegative real matrices, in fact all are nonnegative primitive matrices. So in principle, we can get a formula similar to Equation 3.1 that involves the eigenvalues of the matrix. Also, the asymptotic behaviour can be obtained by computing the spectral radius of the aforementioned matrix Λ_n and then taking the n-root. Thus, for $n = 2, 3$ and 4 we have

$$\begin{aligned}
\lim_{n\to\infty} f(2,m)^{\frac{1}{2m}} &= \sqrt{8+\sqrt{37}} = 3.75270\ldots, \\
\lim_{n\to\infty} f(3,m)^{\frac{1}{3m}} &= 4.05769\ldots, \\
\lim_{n\to\infty} f(4,m)^{\frac{1}{4m}} &= 4.22351\ldots.
\end{aligned}$$

The values of the limits for n from 5 to 8 are respectively 4.32788..., 4.39966..., 4.45208... and 4.49199....

4 The square board

To compute $f(n)$ we used the above method and obtained the values for n up to 8, that is, for the square board of side 32. There are also computer programs which compute the Tutte polynomial of moderated size graphs. One is `TuLiC` by Rodolfo Conde which is freely available at `http://ada.fciencias.unam.mx/~rconde/tulic/`. Using this program we computed the Tutte polynomial for all the square lattices from $L_{2,2}$ to $L_{12,12}$. The values for the number of T-tilings are given in the following list:

$$
\begin{aligned}
f(1) &= 2\\
f(2) &= 84\\
f(3) &= 78696\\
f(4) &= 1668091536\\
f(5) &= 804175873700640\\
f(6) &= 884088950284453704480\\
f(7) &= 2219885416449546846322852561536\\
f(8) &= 12743498392347171159734108119436194009344\\
f(9) &= 1673655934365810075982323780364346176451059139240448\\
f(10) &= 5031230898942160933982250013114536314591579141675092922832\\
&\quad 491520\\
f(11) &= 3463131467175532420109700259258995693571631028572354110874\\
&\quad 7291827187099401625 6\\
f(12) &= 5459704314008539845990741829270093470430155282377357168707\\
&\quad 98525852987071373589503117526536192
\end{aligned}
$$

The values up to $n = 8$ were checked with both methods. All these numbers have the form $2^n\, q$, for an odd integer q. This is true in general.

Proposition 4. *The number $f(n,m) = 2^{\gcd(n,m)}\, q$, where q is an odd integer.*

Proof. Las Vergnas [9] proved that if G is a plane graph with medial graph H, then $T(G;3,3) = 2^{c(H)-1}\, q$, where q is an odd integer. Given a connected plane graph G, its *medial graph* H is constructed by putting a vertex on each edge of G and two vertices of H are joined by an edge if the corresponding edges in G are neighbours in the cyclic order of edges around a vertex. Thus, H is an Eulerian 4-regular graph. The medial graph of $L_{9,3}$ is shown in Figure 2. The graph invariant $c(H)$ is the number of *crossing circuits* of H, see Las Vergnas [9], that is, the number of circuits in the Eulerian partition of the edges of H defined by choosing at each vertex opposite edges to be on the same circuit. An example of one crossing circuit for the medial graph of $L_{9,3}$ is shown in Figure 2.

In the case of $L_{n,m}$, a crossing circuit of its medial graph corresponds to the trajectory of a ball thrown at 45° in a billiard table of $2n$ by $2m$ from a point with integer coordinates $(0,j)$, for some odd integer j with $1 \leq j \leq 2m-1$. The $2n+2m$

points in the boundary with integer coordinates $(i,\ j)$ such that $i+j$ is an odd integer are partitioned by the trajectories. The points in the same equivalent class will be those that correspond to rebounds of the ball of a particular trajectory. The same method as for the classical billiard problem, see Steinhaus [12], gives that the number of rebounds in a trajectory is $2r+2s$, where r and s are the integers satisfying the equation $\mathrm{lcm}(n,m) = rm = sn$. Thus, the number of different trajectories is $\gcd(n,m)$. The proposition follows from Theorem 2. □

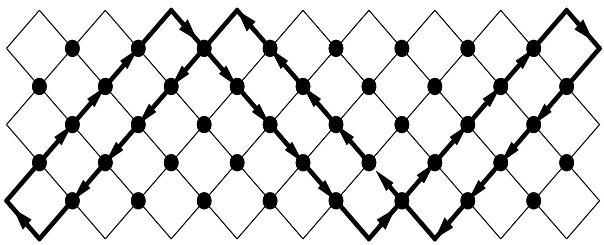

Figure 2: The medial graph of $L_{9,3}$ where the big dots are the vertices of the graph. Also one of its 3 crossing circuits is in thick line.

Corollary 5. *The number $f(n) = 2^n\, q$, where q is an odd integer.*

The asymptotic behaviour for the square lattice is known; the value corresponds to the free energy of the 4-Potts model and the result is due to Baxter [1]:

$$\lim_{n\to\infty} f(n)^{\frac{1}{n^2}} = \left(\frac{\Gamma(1/4)}{2\,\Gamma(3/4)}\right)^4 = 4.78926\ldots.$$

5 Conclusion

We used Korn and Pak's result [8] to obtain general formulae for the number of T-tilings of a rectangular board of $4n \times 4m$, when n is at most 4 and m is arbitrary. The same method gives formulae when n is at most 8. We also give some numerical values for $f(n)$, the number of T-tilings of a square board of side $4n$. A generalization of Theorem 2 is given by Jacobsen in [5]. There each tile is assigned a weight that depends on its orientation and position on the board and, for a particular choice of the weights, the generating function of weighted tilings is shown to be the evaluation of the multivariate Tutte polynomial $\mathrm{ZG}(Q,v)$.

For this approach, it is necessary to evaluate the Tutte polynomial of $L_{n,m}$ at the point $(3,3)$. Evaluating $T(G;3,3)$ is #P-hard, even for planar bipartite graphs, see Vertigan and Welsh [15], so better understanding about the Tutte polynomial of the lattice is needed to extend our results. However, it is possible that using a different method a similar formula to Kasteleyn's could be found.

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(Received 26 Mar 2007; revised 23 Dec 2007)