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On graphs admitting codes identifying sets of vertices

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Abstract

Let $G = (V, E)$ be a graph and $N[X]$ denote the *closed neighbourhood* of $X \subseteq V$, that is to say, the union of X with the set of vertices which are adjacent to X. Given an integer $t \geq 1$, a subset of vertices $C \subseteq V$ is said to be a *code identifying sets of at most t vertices* of G—or, for short, a *t-set-ID code* of G—if the sets $N[X] \cap C$ are all distinct, when X runs through subsets of at most t vertices of V. A graph G admits a t-set-ID code if and only if $N[X] \neq N[Y]$ for all pairs X and Y which are distinct subsets of at most t vertices of V.

Graphs admitting identifying codes is a recent topic. In this paper, we show that for G_1 admitting a t_1-set-ID code, and G_2 admitting a t_2-set-ID code, the cartesian product $G_1 \Box G_2$ admits a $\max\{t_1, t_2\}$-set-ID code, and we show that this result is the best possible. We also study the extremal question of minimizing the number of vertices of a graph admitting a t-set-ID code. Asymptotically, this number is $\Omega(t^2)$, and we give an explicit construction of an infinite family of *t-regular* graphs attaining this bound. The construction uses so-called distance-regular graphs.

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1 Codes identifying sets of vertices

Let $G = (V, E)$ be an undirected and connected graph and $t \geq 1$ an integer. Denote by $d(u, v)$ the *graphic distance* between the vertices u and v, *i.e.*, the number of edges in any shortest path between u and v. We say that a vertex u *covers* a vertex v if $d(u, v) \leq 1$. Let us denote by $N[X]$ the *closed neighbourhood* of $X \subseteq V$, that is, the union of X with the set of vertices which are adjacent to an element of X. A subset of vertices $C \subseteq V$ is said to be a *code identifying sets of at most t vertices* of G (or, for short, a *t-set-ID code*) if the sets $N[X] \cap C$ are distinct for all $X \subseteq V$ with $|X| \leq t$.

A graph may not admit a t-set-ID code (that is, there does not exist any $C \subseteq V$ such that all of the sets $N[X] \cap C$ were different), for example, in the complete graph K_n on $n \geq 2$ vertices we have $N[x] = N[y]$ for any two vertices $x \neq y$, so K_n does not admit a t-set-ID code for any $t \geq 1$. It is easy to see (by choosing $C = V$) that G admits a t-set-ID code if and only if $N[X] \neq N[Y]$ for all pairs of distinct subsets X and Y of at most t vertices of V.

The notion of identifying codes was introduced in [10] to model a fault-detection problem in multiprocessor systems. For another application to sensor networks consult [16]. Identifying codes are closely related to other types of codes, like covering codes (which are frequently used to construct identifying codes in Hamming spaces, see *e.g.* [1, 2, 10]) or superimposed codes (see [6]). There is a large and fast-growing bibliography on identifying codes, which can be found from Antoine Lobstein's webpage [17].

In [4], structural properties of graphs admitting t-set-ID codes are derived for $t = 1$, but very little is known on these graphs for the general case $t \geq 1$. In particular, only few constructions of graphs admitting t-set-ID codes are known (see [8, 14] for some constructions). In the next section, we show that if G_i admits a t_i-set-ID code, for $i = 1, 2$, then the cartesian product of G_1 and G_2 admits a $\max\{t_1, t_2\}$-set-ID code. We also prove that this result is the best possible. It should be noticed that, for example, multiprocessor systems such as binary hypercubes and certain grids are obtained using the cartesian product.

A question (posed in [8]) of finding the minimum number of vertices of a graph admitting a t-set-ID code is studied in Section 3. Asymptotically, the number of vertices of a graph admitting a t-set-ID code is $\Omega(t^2)$ (for the usual notations $\Omega(.)$, $O(.)$ and $\Theta(.)$, we refer to [5]). In this paper, we give an infinite family of graphs attaining this bound. This family has the additional property that all its graphs are t-regular (that is, they satisfy an extremal degree property as well), which improves a construction given in [11].

2 Structural properties and the cartesian product

First we derive two lemmas dealing with structural properties of graphs admitting t-set-ID codes, which will be useful in the sequel. Let $N(x)$ denote the set of vertices adjacent to $x \in V$.

Lemma 1 *Let G be a graph admitting a t-set-ID code, and let x be a vertex of G. Then $N(x)$ contains a stable set of cardinality t, i.e., a set of cardinality t without edges between vertices belonging to the set.*

Proof: Let $k \leq t-1$, and let K be a subset of $N(x)$ of cardinality k. Observe that, since G admits a t-set-ID code, then K and $K \cup \{x\}$ have distinct closed neighbourhoods, hence there exists $y \in N(x) \smallsetminus N[K]$. Now, K is a stable set if and only if $K \cup \{y\}$ is a stable set, and the desired result is obtained by inductively applying the above observation, starting with K reduced to any single point of $N(x)$. □

Lemma 2 *Let G be a graph admitting a t-set-ID code, and let x be a vertex of G. Let us denote $N_2(x)$ the set of vertices which are at distance 2 of x. Let K be a subset of $N(x) \cup N_2(x)$, and let K' be any subset of vertices of $V \setminus (N[x] \cup N_2(x))$. Then the number of vertices in $N(x)$ left uncovered by $K \cup K'$ satisfies*

$$|N(x) \smallsetminus N[K \cup K']| \;\geq\; t - |K|.$$

Proof: First observe that, by definition, $N(x) \smallsetminus N[K'] = \emptyset$; hence only K contributes to covering vertices of $N(x)$, that is

$$N(x) \smallsetminus N[K \cup K'] \;=\; N(x) \smallsetminus N[K].$$

Now, by way of contradiction, if $|N(x) \smallsetminus N[K]| \;<\; t - |K|$, then the sets

$$K \cup (N(x) \smallsetminus N[K])$$

and

$$K \cup (N(x) \smallsetminus N[K]) \cup \{x\}$$

would have the same closed neigbourhood, which is a contradiction to the fact that G admits a t-set-ID code, since the larger set $K \cup (N(x) \smallsetminus N[K]) \cup \{x\}$ has cardinality at most t. □

Notice that this implies that the minimum degree of G is at least t (take $K = K' = \emptyset$ in the lemma), which was already noticed in [13].

Now we focus on the cartesian product: Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *cartesian product* of G_1 and G_2, denoted by $G_1 \Box G_2$, is the graph on vertex set $V_1 \times V_2$ such that

- $(v_1, v_2)(v_1, v_2') \in E(G_1 \Box G_2)$ if and only if $v_2 v_2' \in E(G_2)$,
- $(v_1, v_2)(v_1', v_2) \in E(G_1 \Box G_2)$ if and only if $v_1 v_1' \in E(G_1)$,
- and $(v_1, v_2)(v_1', v_2') \notin E(G_1 \Box G_2)$ if $v_1 \neq v_1'$ and $v_2 \neq v_2'$.

We use a convenient notation $v_{i,j}$ to denote the vertex (v_i, v_j) of $G_1 \Box G_2$. One can visualize the cartesian product $G_1 \Box G_2$ as a set of $|V(G_2)|$ copies of G_1, such that each copy of G_1 corresponds to a vertex of G_2, and such that there are edges between two copies of G_1 if and only if these copies correspond to adjacent vertices in G_2. The edge set which run between adjacent copies is a perfect matching, which connects any vertex to its "copy". Since the definition is symmetric, one can reverse the role of G_1 and G_2.

The cartesian product is a useful operation in the theory of graphs in general (see [9]), and, on the other hand, well-known multiprocessor architectures, such as binary hypercubes and multidimensional grids, can be obtained using cartesian products. The next theorem shows that using the cartesian product, we can guarantee that the product graph admits a t-set-ID code for at least as large t as the *better* of the initial graphs (notice that this is not true, for example, for the usual *join product* [9], in which case the product graph never admits a t-set-ID code for $t \geq 2$ no matter how good the initial graphs are).

Theorem 1 *Let $t_1 \geq 1$ and let $t_2 \geq 1$, and let G_1 be a connected graph on at least 2 vertices admitting a t_1-set-ID code and G_2 be a connected graph on at least 2 vertices admitting a t_2-set-ID code. Then the cartesian product $G_1 \Box G_2$ admits a $\max\{t_1, t_2\}$-set-ID code.*

Proof: Let us denote $v_{i,j}$ the vertices of $G_1 \Box G_2$ (here i runs through vertices of G_1 and j runs through vertices of G_2). Let X and Y be two distinct subsets of vertices of $G_1 \Box G_2$ such that $N[X] = N[Y]$. We show that this implies $|X| > \max\{t_1, t_2\}$ or $|Y| > \max\{t_1, t_2\}$. Without loss of generality, we can assume that there exists $v_{i_0,j_0} \in X \smallsetminus Y$, and that $t_1 \geq t_2$. Let us denote

$$K_1 := \{v_{i,j} \in Y \mid j = j_0\},$$

$$K_2 := \{v_{i,j} \in Y \mid i = i_0\},$$

and, all the others of Y, by

$$K := Y \smallsetminus (K_1 \cup K_2).$$

Note that K_1 and K_2 are disjoint, since $v_{i_0,j_0} \notin Y$. Denote (see Figure 1)

$$N_1 := \{v_{i,j} \in N(v_{i_0,j_0}) \mid j = j_0\},$$

and

$$N_2 := \{v_{i,j} \in N(v_{i_0,j_0}) \mid i = i_0\}.$$

By Lemma 2, we know that

$$|N_1 \smallsetminus N[K_1]| \geq t_1 - |K_1|, \tag{1}$$

since G_1 admits a t_1-set-ID code. Similarly, we have

$$|N_2 \smallsetminus N[K_2]| \geq t_2 - |K_2|.$$

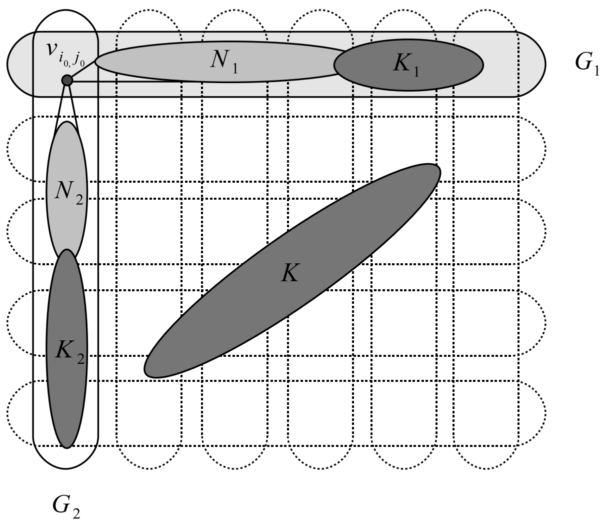

Figure 1: Illustration of the notations used in the proof of Theorem 1.

Note that, by the definition of the cartesian product, each vertex of K has at most one neighbour in N_1, and at most one neighbour in N_2. Since K must cover all the neighbours of v_{i_0,j_0} which are not covered by $K_1 \cup K_2$ (in order to have $N[X] = N[Y]$), then K must have at least

$$\max\{t_1 - |K_1|, t_2 - |K_2|\}$$

vertices. Hence we have

$$|Y| = |K_1| + |K_2| + |K| \geq |K_1| + |K_2| + \max\{t_1 - |K_1|, t_2 - |K_2|\} \geq t_1 + |K_2|.$$

If K_2 is nonempty, then we are done, since we assumed that $t_1 = \max\{t_1, t_2\}$. Now let us assume that $|K_2| = 0$. In this case, we have $|Y| = |K_1| + |K| \geq t_1$, and we are done if the inequality is strict. Let us then assume that $|K_1| + |K| = t_1$. To summarize, we may now assume that

$$|K_1| = t_1 - |K|,\ |K_2| = 0, \text{ and } |K| \geq t_2. \tag{2}$$

If $t_1 = t_2$, then this implies that K_1 is empty, and v_{i_0,j_0} can not be covered by $Y = K$, a contradiction. Hence it suffices to consider $t_1 \geq t_2 + 1$.

Since, by (1), we have $|N_1 \smallsetminus N[K_1]| \geq t_1 - |K_1| = |K|$, and because, by the definition of the cartesian product, each vertex of K covers at most one vertex of N_1, then we actually have $|N_1 \smallsetminus N[K_1]| = |K|$, and each vertex of K covers exactly one vertex of N_1. Hence,

$$\text{the edges running between } K \text{ and } N_1 \smallsetminus N[K_1] \text{ is a perfect matching.} \tag{3}$$

Moreover, this implies that the set of copies of G_1 which contain vertices of Y corresponds *exactly* to the closed neighbourhood of v_{j_0} in G_2, that is to say

$$\{j \mid \exists i, v_{i,j} \in Y\} \;=\; \{j_0\} \cup \{j \mid v_{i_0,j_0} v_{i_0,j} \in E(G_1 \square G_2)\}. \tag{4}$$

Note that if $Y \subset X$, then we are done since $|X| \geq |Y| + 1 \geq t_1 + 1$. So we can assume that there exists $v_{i',j'} \in Y \setminus X$. Using similar notations as for v_{i_0,j_0}, let us denote $N_1' := \{v_{i,j} \in N(v_{i',j'}) \mid j = j'\}$, $N_2' := \{v_{i,j} \in N(v_{i',j'}) \mid i = i'\}$, $K_1' := \{v_{i,j} \in X \mid j = j'\}$, $K_2' := \{v_{i,j} \in X \mid i = i'\}$, and $K' := X \setminus (K_1' \cup K_2')$.

Analogously, we are done, unless $K_2' = \emptyset$, $|K_1'| = t_1 - |K'|$ and $|K'| \geq t_2$ (see (2)). Now we examine separately the following two cases depending on whether $j_0 = j'$ or not.

Case 1 : $j_0 = j'$.
In this case, we have

$$K \cap K' = \emptyset. \tag{5}$$

Indeed, the set of neighbours of v_{i_0,j_0} which are not covered by K_1 is disjoint from the set of neighbours of $v_{i',j'}$ which are not covered by K_1' (since $v_{i_0,j_0} \in K_1'$ and $v_{i',j'} \in K_1$), and each vertex of $K \cup K'$ has one and only one neighbour in $N_1 \cup N_1'$ (see (3)).

Now, take a vertex $v_{i_1,j_1} \in K'$ (note that this is possible since $|K'| \geq t_2$, and that this implies $j_1 \neq j_0$). From (5) we know that $v_{i_1,j_1} \notin Y$. Hence, by considering v_{i_1,j_1} in the role of v_{i_0,j_0}, we can get for v_{i_1,j_1} the following equality, which is analogous to the one we obtained for v_{i_0,j_0} (see (4)):

$$\{j \mid \exists i, v_{i,j} \in Y\} \;=\; \{j_1\} \cup \{j \mid v_{i_1,j_1} v_{i_1,j} \in E(G_1 \square G_2)\}.$$

Putting this together with (4), we must have

$$\{j_0\} \cup \{j \mid v_{i_0,j_0} v_{i_0,j} \in E(G_1 \square G_2)\} \;=\; \{j_1\} \cup \{j \mid v_{i_1,j_1} v_{i_1,j} \in E(G_1 \square G_2)\}.$$

Since $j_0 \neq j_1$, this equality implies the existence of two distinct vertices a and b in G_2 which have the same closed neighbourhood in G_2, which contradicts the fact that G_2 admits a t_2-set-ID code (take a as the projection of v_{i_0,j_0} onto G_2, and b as the projection of v_{i_1,j_1} onto G_2). Hence we are done in the case where $j_0 = j'$.

Case 2 : $j_0 \neq j'$.
In this case, we may assume that there is no $v_{i,j}, v_{k,j}$ such that $v_{i,j} \in X \setminus Y$ and $v_{k,j} \in Y \setminus X$, because else we could apply instead Case 1 to these vertices. Since $j_0 \neq j'$, we must have

$$\{j_0\} \cup \{j \mid v_{i_0,j_0} v_{i_0,j} \in E(G_1 \square G_2)\} \;\neq\; \{j'\} \cup \{j \mid v_{i',j'} v_{i',j} \in E(G_1 \square G_2)\},$$

else it would contradict the fact that G_2 admits a t_2-set-ID code (more precisely, again as above, the projection of v_{i_0,j_0} onto G_2 would have the same closed neighbourhood as the projection of v_{i_1,j_1} onto G_2). Since

$$\{j_0\} \cup \{j \mid v_{i_0,j_0} v_{i_0,j} \in E(G_1 \square G_2)\} \;=\; \{j \mid \exists i, v_{i,j} \in Y\}$$

and

$$\{j'\} \cup \{j \mid v_{i',j'}v_{i',j} \in E(G_1 \square G_2)\} \;=\; \{j \mid \exists i, v_{i,j} \in X\}$$

(see (4)), we may assume that there exists $v_{i_1,j_1} \in X \setminus Y$ such that $j_1 \notin \{j \mid \exists i, v_{i,j} \in Y\}$. But then v_{i_1,j_1} can only be covered by some vertex $v_{i_1,j} \in Y$, which contradicts the condition $K_2 = \emptyset$ applied to v_{i_1,j_1} (see (2)), which plays the same role as v_{i_0,j_0}. □

Actually, the result of this theorem is, in general, the best possible according to the following observation.

Theorem 2 *Let integers $t_1 \geq 1$, $t_2 \geq 1$, and δ_2 be such that $\delta_2 \geq t_2 + 1$ and $t_1 \geq \delta_2 + 1$. Then for every connected graphs G_1 and G_2 on at least 2 vertices such that G_1 has minimum degree t_1 and admits a t_1-set-ID code, G_2 has minimum degree δ_2 and admits a t_2-set-ID code, $G_1 \square G_2$ does not admit a (t_1+1)-set-ID code. Moreover, there exist graphs which satisfy the conditions.*

Proof: Let v_{i_0,j_0} be a vertex having exactly t_1 neighbours in $\{v_{i,j} \mid j = j_0\}$. Such a vertex exists since G_1 has minimum degree t_1. Let $\{v_{i_1,j_0}, \ldots, v_{i_{t_1},j_0}\}$ be the set of neighbours of v_{i_0,j_0} in $\{v_{i,j} \mid j = j_0\}$, and let $\{v_{i_0,j_1}, \ldots, v_{i_0,j_{\delta_2}}\}$ be the set of neighbours of v_{i_0,j_0} in $\{v_{i,j} \mid i = i_0\}$. Now, set

$$K := \{v_{i_1,j_1}, \ldots, v_{i_{\delta_2},j_{\delta_2}}\}$$

and

$$K_1 := \{v_{i_{\delta_2+1},j_0}, \ldots, v_{i_{t_1},j_0}\}.$$

Since $t_1 \geq \delta_2 + 1$, we obtain $K_1 \neq \emptyset$, and hence

$$N[K \cup K_1] = N[K \cup K_1 \cup \{v_{i_0,j_0}\}].$$

Thus, $G_1 \square G_2$ does not admit a (t_1+1)-set-ID code, since $|K| + |K_1| + 1 = t_1 + 1$.

There exists G_2 for all t_2 (for instance one could take G_2 as the binary hypercube of dimension $2t_2$, see [12]), and the existence of G_1 is ensured by Proposition 1 in [8] for all t_1. □

However, we know certain graphs G_1 and G_2 such that $G_1 \square G_2$ admits even a (t_1+t_2)-set-ID code. For example, let us take the hypercube of dimension n (n odd) for G_1 and G_2, which we know admits an $(n+1)/2$-set-ID code but does not admit an $(n+3)/2$-set-ID code [12]. Then the graph $G_1 \square G_2$ is the hypercube of dimension $2n$, which admits an $(n+1)$-set-ID code.

3 Minimizing the number of vertices of a graph admitting an identifying code

In this section we investigate the question of minimizing the number of vertices of a graph admitting a t-set-ID code. First we give a general result.

Theorem 3 *Let $(G_t)_{t\in\mathbb{N}}$ be a family of graphs such that G_t admits a t-set-ID code for all $t \in \mathbb{N}$. Let n_t denotes the number of vertices of G_t. Then we have $n_t = \Omega(t^2)$.*

Proof: From [6], we know that in any graph G_t on n_t vertices admitting a t-set-ID code C, we have

$$|C| \;=\; \Omega\left(\frac{t^2}{\log t}\log n_t\right).$$

By inserting $n_t \geq |C|$ into this inequality, we get

$$\frac{n_t}{\log n_t} \;=\; \Omega\left(\frac{t^2}{\log t}\right),$$

and by writing $n_t = tn'$, we get

$$n' \log t \;=\; \Omega\left(t \log(tn')\right),$$

which gives $n' = \Omega(t)$, hence $n_t = \Omega(t^2)$, which is the desired result. □

This bound is tight, as a construction given in [8] shows it with graphs whose minimum degree is *strictly greater* than t; minimum degree equal to t is, however, the bound one could hope to achieve for a graph admitting t-set-ID code, see [13]. In [8], it was asked whether there existed t-regular graphs admitting t-set-ID codes. It was shown in [11] that the answer is positive, providing a family of graphs $(G_q)_{q \text{ prime power}}$ such that G_q has $\Theta(q^3)$ vertices, admits a q-set-ID code, and is q-regular for all q. In the rest of this section we show how to improve this construction, by giving an infinite family of t-regular graphs achieving the bound $\Omega(t^2)$ of Theorem 3.

For a graph $G = (V, E)$, denote $S_i(v) = \{w \in V \mid d(w, v) = i\}$, $i \geq 0$. A connected graph G is called *distance-regular*, if there are integers b_i and c_i $(i \geq 0)$ such that for any two vertices v and u at distance $i = d(u, v)$, we have $c_i = |S_{i-1}(v) \cap N(u)|$ and $b_i = |S_{i+1}(v) \cap N(u)|$. Let D denote the diameter of G. It is known [3], that a distance regular graph is regular with degree equal to b_0. The sequence

$$\iota(G) = \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$$

is called the *intersection array* of G. The numbers c_i, b_i and a_i, where $a_i = |S_i(v) \cap N(u)| = b_0 - b_i - c_i$ $(i = 0, 1, \ldots, D)$, are called the *intersection numbers* of G. Obviously, $c_0 = 0$ and $c_1 = 1$.

Theorem 4 *If the intersection numbers of a distance-regular graph G satisfy $b_0 = t$, $b_1 = t - 1$, $b_2 = t - 1$, $b_3 \geq 1$ and $c_2 = 1$, then G admits a t-set-ID code.*

Proof: Suppose that $G = (V, E)$ is such that $b_0 = t$, $b_1 = t - 1$, $b_2 = t - 1$, $b_3 \geq 1$ and $c_2 = 1$. Assume to the contrary, that $N[X] = N[Y]$ for two distinct sets $X, Y \subseteq V$ where $|X| \leq t$ and $|Y| \leq t$. Without loss of generality, we can take $x \in X \setminus Y$. Now x has exactly t neighbours, say $v_1, v_2, \ldots, v_t$, and since they belong to $N[X]$, they must also be in $N[Y]$ (otherwise we get a contradiction). To that end, each set $A_i := N[v_i] \setminus \{x\} \subseteq S_1(x) \cup S_2(x)$, $i = 1, 2, \ldots, t$, contains an element of Y. The assumption $b_1 = t - 1$ implies that $a_1 = 0$, which combined with $c_2 = 1$ yields that $A_i \cap A_j = \emptyset$ for any $i \neq j$. Consequently, each A_i has a unique element of Y and thus $|Y| = t$. The element of Y in A_i covers a vertex (at least one), say v_i', in $A_i \cap S_2(x)$. Hence $v_i' \in N[Y]$, and subsequently, we must also have $v_i' \in N[X]$ or we are done. Because there are t different vertices v_i' and only $t - 1$ members of X left to cover them (note that x itself cannot cover any v_i'), some $z \in X$ must cover at least two of them. Since $a_2 = t - b_2 - c_2 = 0$, the vertex z cannot be in $S_2(x)$ (from above we also know that z cannot be in $S_1(x)$ either). Therefore, $z \in S_3(x)$. However, since $b_3 \geq 1$, $N(z) \cap S_4(x)$ contains a vertex which is in $N[X]$ but not in $N[Y]$, a contradiction. This completes the proof. □

Theorem 5 *There exists a family of graphs (G_q), q a prime power, such that for all q the graph G_q admits a q-set-ID code and has n_q vertices, where $n_q = \Theta(q^2)$. Moreover, G_q is always q-regular.*

Proof: Due to Gardiner [7] (see also Brouwer-Cohen-Neumaier [3]), we know that for every prime power q, there exists a distance-regular graph G_q on $\Theta(q^2)$ vertices with

$$\iota(G_q) = \{q, q-1, q-1, 1; 1, 1, q-1, q\}.$$

By the previous theorem, G_q admits a q-set-ID code. □

Theorem 3 shows that the construction of Theorem 5 is optimal, but one can easily prove that, restricted to the class of graphs having minimum degree $t + O(1)$, the minimum number of vertices of a graph admitting a t-set-ID code is indeed $\Omega(t^2)$. As a complement of Theorem 3, we give here a simple proof of this result. As we shall see, its proof is elementary, and does not involve superimposed codes or deep results like the one of [15].

Theorem 6 *Let $(G_t)_{t \in \mathbb{N}}$ be a family of graphs such that G_t admits a t-set-ID code and has minimum degree δ_t for all $t \in \mathbb{N}$. Let n_t denotes the number of vertices of G_t. If $\delta_t = t + O(1)$, then we have $n_t = \Omega(t^2)$.*

Proof: Let x be a vertex of degree δ_t. By Lemma 1, we know that $N(x)$ contains a stable set T of cardinality t. Now let us denote $N_2(x)$ the set of vertices at distance 2 of x. Using Lemma 2, we know that any vertex of $N_2(x)$ can be adjacent only to a constant number of vertices of T. Now we are done since any vertex of T has at least $t - 1$ neighbours in $N_2(x)$, since the minimum degree is $\delta_t = t + O(1)$: $N_2(x)$ must contain at least $\Omega(t^2)$ vertices. □

4 Conclusion

In this paper we gave some structural properties of graphs admitting t-set-ID codes for the general case $t \geq 1$. The two main results are Theorems 1 and 5. Theorem 1 shows that the cartesian product of two graphs admits a t-set-ID code, given that at least one of the initial graphs admits a t-set-ID code. As shown in Theorem 2, this result is, in general, the best possible. However, it would be interesting to give conditions on the two graphs such that their cartesian product has a t-set-ID code with a t greater than the one in Theorem 1.

Theorem 5 answers in an optimal way a question of [8]. It shows that there exists a family of graphs G_t such that G_t has $\Theta(t^2)$ vertices, is t-regular, and admits a t-set-ID code. This also improves a previous result of [11].

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