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C_4-factorizations with two associate classes

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Abstract

Let $K = K(a, p; \lambda_1, \lambda_2)$ be the multigraph with: the number of vertices in each part equal to a; the number of parts equal to p; the number of edges joining any two vertices of the same part equal to λ_1; and the number of edges joining any two vertices of different parts equal to λ_2. This graph was of interest to Bose and Shimamoto in their study of group divisible designs with two associate classes [*J. Amer. Stat. Assoc.* 47 (1952), 151–184]. Necessary and sufficient conditions for the existence of z-cycle decompositions of this graph have been found when $z \in \{3, 4\}$ [Fu, Rodger and Sarvate, *Ars Combin.* 54 (2000), 33–50; Fu and Rodger, *Combin. Probab. Comput.* 10 (2001), 317–343]. The existence of resolvable 4-cycle decompositions of K has been settled when a is even [Billington and Rodger, *Discrete Math.* doi:10.1016/j.disc.2006.11.043 (to appear)], but the odd case is much more difficult. In this paper, necessary and sufficient conditions for the existence of a C_4-factorization of $K(a, p; \lambda_1, \lambda_2)$ are found when $a \equiv 1 \pmod 4$ and λ_1 is even, and substantial progress is made in the case where λ_1 is odd.

1 Introduction

In this paper, graphs usually contain multiple edges. In particular, if G is a simple graph then for any $\lambda \geq 1$, let λG denote the multigraph formed by replacing each edge in G with λ edges. Throughout this paper we allow sets to contain repeated elements. Let C_z denote a cycle of length z.

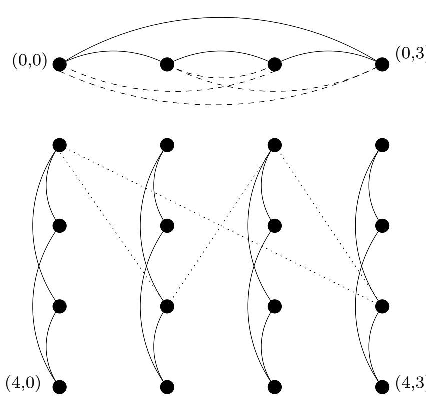
Let $K = K(a, p; \lambda_1, \lambda_2)$ denote the graph formed from p vertex-disjoint copies of the multigraph $\lambda_1 K_a$ by joining each pair of vertices in different copies with λ_2 edges (so naturally, λ_1, λ_2 are non-negative integers). The vertex set, $V(K(a, p; \lambda_1, \lambda_2))$, is always chosen to be $\mathbb{Z}_a \times \mathbb{Z}_p$, with parts $\mathbb{Z}_a \times \{j\}$ for each $j \in \mathbb{Z}_p$; naturally, each part induces a copy of $\lambda_1 K_a$.We say the vertex (i,j) is on *level* i and in *part* j. An edge is said to be a *mixed edge* if it joins vertices in different parts, and is said to be a *pure edge* (in part j) if it joins two vertices in the jth part.

A 2-*factor* of a graph G is a spanning 2-*regular* subgraph of G. A 2-*factorization* of G is a set of edge-disjoint 2-factors, the edges of which partition $E(G)$. A C_z-factorization is a 2-factorization such that each component of each 2-factor is a cycle of length z; each 2-factor of a C_z-factorization is known as a C_z-factor. A G-decomposition of a graph H is a partition of $E(H)$, each element of which induces a copy of G. C_z-factorizations are also known as *resolvable* C_z*-decompositions*.

There has been considerable interest over the past 20 years in C_z-decompositions of various graphs, such as complete graphs and complete mutipartite graphs. In the resolvable case, these results are collectively known as addressing the Oberwolfach problem. More recently, the existence problem for C_z-decompositions of $K(a,p;\lambda_1,\lambda_2)$ for $z=\{3,4\}$ has been solved [4, 5]. Such decompositions are known as C_z*-group-divisible designs with two associate classes*, following the notation of Bose and Shimamoto who considered the existence problem for K_z-group divisible designs. The reason for this name is that the structure can be thought of as partitioning ap symbols, or vertices, into p sets of size a in such a way that symbols that are in the same set in the partition occur together in λ_1 blocks, and are known as *first associates*, whereas symbols that are in different sets in the partition occur together in λ_2 blocks, and are known as *second associates*.

Resolvable C_z-decompositions of G have also been of interest [6]. Recently the existence of a C_4-factorization of $K(a,p;\lambda_1,\lambda_2)$ has been completely settled when a is even [2], but the case where a is odd is proving to be considerably more difficult. In this paper, we consider the case where $a \equiv 1 \pmod 4$, completely settling the case where λ_1 is even and making substantial progress on the case where λ_1 is odd.

Example 1 The following examples of C_4-factors of $K(5,4;4,2)$ give good insight into the constructions used in Sections 3 and 4:



For each $r \in \mathbb{Z}_5$, let $\pi_r^-(k) = \{(r+1,k),(r+2,k),(r+4,k),(r+3,k)\}$ be a near C_4-factor (i.e. includes all except one of the vertices) in the kth part. Then

$\bigcup_{0\le k\le 3}\pi_r^-(k)\cup\{(r,0),(r,1),(r,2),(r,3)\}$ is a C_4-factor of K (see the solid edges) for the case when $r=0$. Notice that $\bigcup_{0\le k\le 3}\pi_r^-(k)\cup\{(r,0),(r,2),(r,1),(r,3)\}$ is also a C_4-factor that could be used if λ_1 is large (see the dashed mixed edges). Finally, observe that mixed edges can easily be used in C_4-factors of the form $P(s,j)=\{((i,0),(i+j,1),(i,2),(i+j,3))\mid i\in\mathbb{Z}_5\}$ (see the dotted lines for one component when $j=2$).

2 Preliminary Results

We begin by finding some necessary conditions in the next two lemmas.

Lemma 2.1 *Let a be odd. If there exists a C_4-factorization of $K(a,p;\lambda_1,\lambda_2)$, then:*

1. $p\equiv 0 \pmod 4$, *and*
2. $\lambda_2>0$ *and is even.*

Proof Since the number of 4-cycles in each C_4-factor is the number of vertices divided by four, four must divide ap, and since a is odd, $p\equiv 0 \pmod 4$. Similarly, if $\lambda_2=0$ then the number of vertices in each part, namely a, would be divisible by 4, contradicting a being odd.

Each vertex is joined with λ_1 edges to each of the $(a-1)$ other vertices in its own part and with λ_2 edges to each of the $a(p-1)$ vertices in the other parts; so the degree of each vertex is:

$$d_K(v)=\lambda_1(a-1)+\lambda_2 a(p-1).$$

Clearly, since K has a C_4-factorization, it is regular of even degree. Since a is odd, $(a-1)$ is even so the first term in $d_K(v)$ is even. The second term must therefore be even, so since both a and $(p-1)$ are odd, λ_2 must be even. ■

Lemma 2.2 *Let $a\equiv 1 \pmod 4$. If there exists a C_4-factorization of $K(a,p;\lambda_1,\lambda_2)$, then $\lambda_1\le\lambda_2 a(p-1)$.*

Proof Since $a\equiv 1 \pmod 4$, each C_4-factor contains at most $(a-1)$ pure edges in each part. So each C_4-factor contains at most $(a-1)p$ pure edges. Since there are $\lambda_1\binom{a}{2}p$ pure edges, the number of C_4-factors in any C_4-factorization is at least:

$$\frac{\lambda_1\binom{a}{2}p}{(a-1)p}=\frac{\lambda_1 a}{2}.$$

Each C_4-factor has ap edges, of which at most $(a-1)p=ap-p$ are pure, so there are at least p mixed edges in any C_4-factor. Then the number of mixed edges in any C_4-factorization is at least:

$$\frac{\lambda_1 ap}{2}.$$

Therefore, this number must be at most the number of mixed edges, $\lambda_2\binom{p}{2}a^2$, in K:

$$\frac{\lambda_1 ap}{2} \leq \lambda_2 \binom{p}{2} a^2,$$

so

$$\lambda_1 \leq \lambda_2 a \left(p-1\right).$$

■

A set of 4-cycles is said to be a *near* C_4-factor of G if it contains $|V(G)|/4$ 4-cycles, which are vertex-disjoint; the vertex in $V(G)$ that is in none of these cycles is called the *deficient* vertex of the *near* C_4-factor. We will use the following known results in considering C_4-factorizations of $K(a,p;\lambda_1,\lambda_2)$.

Lemma 2.3 [3] *Suppose $a \equiv 1 \pmod 4$. Then near C_4-factorizations of λK_a exist for all even λ.*

Lemma 2.4 [7] *Suppose $p \equiv 0 \pmod 4$. Then C_4-factorizations of λK_p exist for all even λ.*

3 The main result: λ_1 is even

Theorem 3.1 *Suppose $a \equiv 1 \pmod 4$, and λ_1 is even. There exists a C_4-factorization of $K(a,p;\lambda_1,\lambda_2)$ if and only if:*

1. $p \equiv 0 \pmod 4$,
2. $\lambda_2 > 0$ *and is even, and*
3. $\lambda_1 \leq \lambda_2 a (p-1)$.

Proof The necessity of these conditions follows from Lemmas 2.1 and 2.2. So now assume that K satisfies conditions (1–3).

Using Lemma 2.4, let

$$\pi = \{\pi_s \mid s \in \mathbb{Z}_{\frac{\lambda_2(p-1)}{2}}, \pi_s \text{ is the } s^{th}\ C_4\text{-factor of a } C_4\text{-factorization of } \lambda_2 K_p\}.$$

For each $s \in \mathbb{Z}_{\frac{\lambda_2(p-1)}{2}}$, $j \in \mathbb{Z}_a$, and $i \in \mathbb{Z}_a$, let

$$P(s,j,i) = \{((i,w),(i+j,x),(i,y),(i+j,z)) \mid (w,x,y,z) \in \pi, w < x,y,z\}.$$

Then for each $s \in \mathbb{Z}_{\frac{\lambda_2(p-1)}{2}}$ and for each $j \in \mathbb{Z}_a$, define the following C_4-factor of $K(a,p;\lambda_1,\lambda_2)$ that consists entirely of mixed edges:

$$P(s,j) = \bigcup_{i \in \mathbb{Z}_a} P(s,j,i).$$

Notice that it is easy to see that these C_4-factors can be used to produce a C_4-factorization of $K(a,p;0,\lambda_2)$, namely:

$$\bigcup_{s\in\mathbb{Z}_{\frac{\lambda_2(p-1)}{2}}}\ \bigcup_{j\in\mathbb{Z}_a} P(s,j)\,.$$

However, we may have pure edges to use too, which is accomplished by spreading the 4-cycles in $P(s,j)$ among a C_4-factors, each of which contains $P(s,j,i)$ for some $i\in\mathbb{Z}_a$ together with a pure *near* C_4-factor in each part. More specifically, for each $r\in\mathbb{Z}_a$ and $k\in\mathbb{Z}_p$, using Lemma 2.3, let $\pi_r^-(k)$ be the *near* C_4-factor of a *near* C_4-factorization of $2K_a$ on the vertex set $\mathbb{Z}_a\times\{k\}$ with deficient vertex (r,k).

For each $r\in\mathbb{Z}_a,\ s\in\mathbb{Z}_{\frac{\lambda_2(p-1)}{2}}$, and $j\in\mathbb{Z}_a$, let

$$P^-(s,j,r)=$$
$$P(s,j,r)\cup\left(\bigcup_{\substack{(w,x,y,z)\in\pi_s\\ w<x,y,z}}\left(\pi_r^-(w)\cup\pi^-_{(r+j)\pmod{a}}(x)\cup\pi_r^-(y)\cup\pi^-_{(r+j)\pmod{a}}(z)\right)\right).$$

Notice that in parts w and y, $P(s,j,r)$ contains the vertex only on level r, and in parts x and z, it contains the vertex only on level $(r+j)\pmod{a}$; in each case this vertex is the *deficient* vertex in the *near* C_4-factor being used. So, then $P^-(s,j,r)$ is a C_4-factor of K that contains exactly p mixed edges and p *near* C_4-factors of K_a. Furthermore,

$$\bigcup_{r\in\mathbb{Z}_a} P^-(s,j,r)$$

contains:

(a) each pure edge twice, and

(b) precisely the mixed edges in $P(s,j)$.

Let $S=\{(s,j)\,|\,s\in\mathbb{Z}_{\frac{\lambda_2(p-1)}{2}},\, j\in\mathbb{Z}_a\}$. Let $S_1\subseteq S$ have size $\frac{\lambda_1}{2}$. Notice that by condition 3 of the theorem, $\lambda_1\le\lambda_2 a(p-1)$, so $|S_1|=\frac{\lambda_1}{2}\le\frac{\lambda_2 a(p-1)}{2}=|S|$, so such a set $|S_1|$ exists. Then

$$\bigcup_{\substack{r\in\mathbb{Z}_a\\ (s,j)\in S_1}} P^-(s,j,r)$$

is a set of $\frac{\lambda_1 a}{2}$ C_4-factors that contains each pure edge $2|S_1|=\lambda_1$ times by (a), and uses precisely the mixed edges in

$$\bigcup_{(s,j)\in S_1} P(s,j)$$

by (b). Therefore, the required C_4-factorization of K is defined by

$$P = \left(\bigcup_{\substack{r \in \mathbb{Z}_a \\ (s,j) \in S_1}} P^- (s,j,r) \right) \cup \left(\bigcup_{(s,j) \in S \setminus S_1} P(s,j) \right).$$

Notice that

$$\begin{aligned} |P| &= a|S_1| + |S \setminus S_1| \\ &= \frac{\lambda_1 a}{2} + \frac{\lambda_2 a (p-1)}{2} - \frac{\lambda_1}{2} \\ &= \frac{\lambda_1 (a-1)}{2} + \frac{\lambda_2 a (p-1)}{2} \end{aligned}$$

as required. ■

4 λ_1 is odd

We now turn our attention to the case where λ_1 is odd. The main difficulty now is that there is no near C_4-factorization of $\lambda_1 K_a$, and so some C_4-factors cannot look like $P^- (s,j,r)$ in the previous section. Instead, they must use a higher proportion of mixed edges. So we need a tool that provides an efficient use of the pure edges in forming C_4-factors.

Let P_2 denote a path of length 2. We begin with a special cyclic P_2 *− decomposition* of K_a. Let $V(K_n) = \mathbb{Z}_n$, and define the *difference* of the edge $\{x, y\} \in E(K_n)$, with $x < y$, to be $d(x, y) = \min\{y - x, n - (x - y)\}$. If B is a set of paths of length 2, let $V(B)$ and $E(B)$ denote the set of vertices and edges in the paths in B respectively, and let $d(B)$ be the multiset of differences of the edges in $E(B)$. For $j \in \mathbb{Z}_n$, let $B_j = \{(x + j, y + j, z + j) \mid (x, y, z) \in B\}$, reducing the sums modulo n. It is well known that if $d(B) = \{1, 2, \ldots, \frac{n-1}{2}\}$, then $\bigcup_{j \in \mathbb{Z}_n} B_j$ is a cyclic P_2-decomposition of K_n. Each 2-path in B is known as a *base path*.

Lemma 4.1 *Let* $a \equiv 1 \pmod 4$*. There exists a cyclic* P_2*-decomposition of* K_a *with set of base paths* $B = \{b_k \mid k \in \mathbb{Z}_{\frac{a-1}{4}}\}$*, for which:*

1. *the base paths* b_k *for each* $k \in \mathbb{Z}_{\frac{a-1}{4}}$ *are vertex disjoint, and*
2. *there exists a function,* f*, such that:*
   (a) $f : B \to \mathbb{Z}_a \setminus V(B)$*, and*
   (b) $N(B) = \{N(b_k, x) = (a - f(b_k) - x) \mid k \in \mathbb{Z}_{\frac{a-1}{4}}, x$ *is an end vertex of* $b_k\}$ $\subseteq \mathbb{Z}_a$ *(reducing calculations modulo* a*) has size* $\frac{a-1}{2}$ *(i.e. contains no repetitions).*

Remark 4.1 *Let $f(B) = \{f(b_k) \,|b_k \in B\}$. Notice that since $|V(B)| = (3a-3)/4$, $|f(B)| = |B| = (a-1)/4$, and since the range of f ensures that $V(B) \cap f(B) = \emptyset$, it follows that $V(B) \cup f(B) = \mathbb{Z}_a \setminus \{v\}$ for some $v \in \mathbb{Z}_a$. This vertex v is named the deficient vertex of B. For B_j, $j \in \mathbb{Z}_a$, we can choose the deficient vertex to be j; so in particular, 0 is the deficient vertex of $B = B_0$.*

Proof The set of base paths, B, and function, f, are produced as follows, considering two cases in turn:

Case 1: $n = 8m + 1$. Define

$$\alpha = \{b_k = (4m-1-3k, 1+k, 4m-2-3k) \,|1 \le k < m\},$$

$$\beta = \{b_k = (8m-3k, 4m+k, 8m-1-3k) \,|0 \le k < m\},$$

$$\gamma = \{(4m-1, 1, 4m-2)\}, \textit{and}$$

$$B = \alpha \cup \beta \cup \gamma.$$

For each $b_k \in \alpha, f(b_k) = 4m - 3k$; for each $b_k \in \beta, f(b_k) = 8m - 2 - 3k$; and for $\gamma, f(b) = 5m$.

To see that B is a set of base paths, note that:

(i) if $b_k \in \alpha$, then b_k contains edges of differences $4m - 2 - 4k$ and $4m - 3 - 4k$ for $1 \le k < m$;

(ii) if $b_k \in \beta$, then b_k contains edges of differences $4m - 4k$ and $4m - 1 - 4k$ for $0 \le k < m$; and

(iii) the path in γ contains edges of differences $4m - 2$ and $4m - 3$.

So $D(B) = \{1, 2, \ldots, 4m\}$ as required.

To see that f satisfies condition (2a), notice that:

(i) $V(\alpha \cup \gamma) \subseteq \{1, 2, \ldots, 4m-1\}$, and if $v \in V(\alpha \cup \gamma)$ with $v \ge m+3$, then $v \equiv 4m - 1$ or $4m - 2 \pmod 3$, and

(ii) $V(\beta) \subseteq \{4m, 4m+1, \ldots, 8m\}$, and if $v \in V(\beta)$ with $v \ge 5m$, then $v \equiv 8m$ or $8m - 1 \pmod 3$.

So, since $f(b_k) \equiv 4m \pmod 3$ for each $b_k \in \alpha$, $f(b_k) \equiv 8m + 1 \pmod 3$ for each $b_k \in \beta$, and $f(b) = 5m$ for $\gamma \notin V(B)$, f satisfies condition (2a). To see that f satisfies condition (2b), notice that:

(i) if $b_k \in \alpha$, then $N(b_k) = \{n - (4m-3k) - (4m-1-3k), n - (4m-3k) - (4m-2-3k)\} = \{6k+2, 6k+3\}$ for $1 \le k < m$;

(ii) if $b_k \in \beta$, then $N(b_k) = \{n - (8m-2-3k) - (8m-3k), n - (8m-2-3k) - (8m-1-3k)\} = \{6k+4, 6k+5\}$ for $1 \le k < m$; and

(iii) if $b \in \gamma$, then $N(b) = \{n - 5m - (4m-1), n - 5m - (4m-2)\} = \{7m + 3, 7m+4\}$.

Since clearly no element of $\mathbb{Z}_n$ occurs in two of the above sets, f satisfies condition (2b).

Case 2: $n = 8m+5$. Define

$$\alpha = \{b_k = (4m+4-3k, k, 4m+2-3k) \,|1 \le k \le m\},$$

$$\beta = \{b_k = (8m+5-3k, 4m+2+k, 8m+3-3k) \,|1 \le k \le m\},$$

$$\gamma = \{(8m+3, 4m+2, 8m+4)\}, \textit{and}$$

$$B = \alpha \cup \beta \cup \gamma.$$

For each $b_k \in \alpha$, $f(b_k) = 4m+3-3k$; for each $b_k \in \beta$, $f(b_i) = 8m+4-3k$; and for γ, $f(b) = m+1$.

To see that B is a set of base paths, note that:

(i) if $b_k \in \alpha$, then b_k contains edges of differences $4m+4-4k$ and $4m+2-4k$ for $1 \le k < m$;

(ii) if $b_k \in \beta$, then b_k contains edges of differences $4m+3-4k$ and $4m+1-4k$ for $1 \le k < m$; and

(iii) the path in γ contains edges of differences $4m+1$ and $4m+2$.

So $D(B) = \{1, 2, \dots, 4m+2\}$ as required.

To see that f satisfies condition (2a), notice that:

(i) $V(\alpha) \subseteq \{1, 2, \dots, 4m+1\}$, and if $v \in V(\alpha)$ with $v \ge m+1$, then $v \equiv 4m+4$ or $4m+2 \pmod 3$,

(ii) $V(\beta) \subseteq \{4m+3, 4m+4, \dots, 8m+2\}$, and if $v \in V(\beta)$ with $v \ge 5m+3$, then $v \equiv 8m$ or $8m+5 \pmod 3$, and

(iii) $V(\gamma) \subseteq \{4m+2, 8m+3, 8m+4\}$, and if $v \in V(\gamma)$, then $v \equiv 4m+2, 8m$, or $8m+4 \pmod 3$.

So, since $f(b_k) \equiv 4m+3 \pmod 3$ for each $b_k \in \alpha$, $f(b_k) \equiv 8m+4 \pmod 3$ for each $b_k \in \beta$, and $f(b) = m+1$ for γ, f satisfies condition (2a). To see that f satisfies condition (2b), notice that:

(i) if $b_k \in \alpha$, then $N(b_k) = \{n - (4m+3-3k) - (4m+4-3k), n - (4m+3-3k) - (4m22-3k)\} = \{6k-2, 6k\}$ for $1 \le k \le m$;

(ii) if $b_k \in \beta$, then $N(b_k) = \{n - (8m+4-3k) - (8m+5-3k), n - (8m+4-3k) - (8m+3-3k)\} = \{6k+1, 6k+3\}$ for $1 \le k \le m$; and

(iii) if $b \in \gamma$, then $N(b) = \{n - (m+1) - (8m+3), n - (m+1) - (8m+4)\} = \{7m+5, 7m+6\}$.

Since clearly no element of $\mathbb{Z}_n$ occurs in two of the above sets, f satisfies condition (2b). ■

We now see how to use the base paths found in Lemma 4.1, finding C_4-factors in K that use each pure edge once and only $\frac{a(a+1)p}{2}$ mixed edges.

The *mixed difference from x to y* of the mixed edge $\{(j,x),(k,y)\}$ in K is defined to be $\min\{k-j, a-k-j\}$.

Corollary 4.1 *Let $p \equiv 0 \pmod 4$ and $a \equiv 1 \pmod 4$. Let $P(s,j)$ be the C_4-factor of mixed edges in K defined in the proof of Theorem 1. There exists a set $S_1 \subseteq S = \{(s,j) \mid s \in \mathbb{Z}_{\frac{\lambda_2(p-1)}{2}}, j \in \mathbb{Z}_a\}$ with $|S_1| = \frac{(a+1)}{2}$ such that there exists a C_4-factorization of*

$$K(a,p;1,0) + \left(\bigcup_{(s,j)\in S_1} E(P(s,j)) \right)$$

containing a C_4-factors.

Proof Let $\pi = \{\pi_s \mid s \in \mathbb{Z}_{\frac{\lambda_2(p-1)}{2}}\}$ be a C_4-factorization of $\lambda_2 K_p$. Let B be the set of base 2-paths in K_a with associated function f found in Lemma 4.1. Let $B^- = \{b_k^- = (a-t, a-u, a-v) \mid b_k = (t,u,v) \in B\}$ (reducing the sums modulo a) be another set of base paths (think of these as “upside-down versions” of the paths in B), and let $f^-(b_k) = a - f(b) \ (mod\ a)$. Notice that for any fixed $s \in \mathbb{Z}_{\frac{\lambda_2(p-1)}{2}}$, a C_4-factor of K can be formed by:

$$\begin{aligned} C(s) = \\ &\{((t,w),(u,w),(v,w),(f^-(b_k),x)),((a-t,x),(a-u,x),(a-v,x),(f(b_k),y)),\\ &((t,y),(u,y),(v,y),(f^-(b_k),z)),((a-t,z),(a-u,z),(a-v,z),(f(b_k),w)),\\ &((0,w),(0,x),(0,y),(0,z)) \mid (w,x,y,z) \in \pi, w < x,y,z, b_k = (t,u,v) \in B\}; \end{aligned}$$

properties (1) and (2a) of Lemma 4.1 ensure that the 4-cycles are all vertex disjoint.

Next, let $C(s,i)$ be formed by adding $i \pmod a$ to the first coordinate in each vertex in each 4-cycle in $C(s)$. $C(s,i)$ is also a C_4-factor of K. Since B is a set of base paths, the pure edges in $\cup_{i\in\mathbb{Z}_a} C(s,i)$ are the edges in $K(a,p;1,0)$ (that is, one copy of each pure edge in K). Also, by Property (2b) of Lemma 4.1, for each $(w,x,y,z) \in \pi, w < x,y,z$, the mixed edges in $\cup_{i\in\mathbb{Z}_a} C(s,i)$ are precisely:

1. all the edges of mixed differences from w and x and from y and z in $N(B) \cup \{0\}$; and
2. all the edges of mixed differences from x and y and from z and w in $\{a - j \mid j \in N(B) \cup \{0\}\}$.

So setting $S_1 = \{(s,j) | s \in \mathbb{Z}_{\frac{\lambda_2(p-1)}{2}}, j \in N(B) \cup \{0\}\}$, this is precisely the set of edges in

$$\bigcup_{(s,j)\in S_1} P(s,j).$$

■

We now use Lemma 4.1 and Corollary 4.1 to construct a C_4-factorization of $K = K(a,p;\lambda_1,\lambda_2)$ when $a \equiv 1 \pmod 4$ and λ_1 is odd. We begin the construction by using the corollary to produce C_4-factors using each pure edge only once, thereby effectively reducing λ_1 by one. The construction from Theorem 3.1 is adapted to partition the remaining pure and mixed edges into C_4-factors, producing the required C_4-factorization.

Theorem 4.2 *Suppose $a \equiv 1 \pmod 4$ and λ_1 is odd. There exists a C_4-factorization of $K = K(a,p;\lambda_1,\lambda_2)$ if:*

1. $p \equiv 0 \pmod 4$,
2. λ_2 is even and greater than zero, and
3. $\lambda_1 \le \lambda_2 a(p-1) - a$.

Remark 4.2 *Conditions 1 and 2 are necessary, as is shown in Lemma 2.1.*

Proof Assume that K satisfies conditions (1–3). For each $s \in \mathbb{Z}_{\frac{\lambda_2(p-1)}{2}}$, $j \in \mathbb{Z}_a$, and $i \in \mathbb{Z}_a$, let π, S, $P(s,j,i)$, $P(s,j)$, and $P^-(s,j,r)$ be defined as in Theorem 3.1. Let S_1 be defined as in Corollary 4.1; so $S_1 \subseteq S$ with $|S_1| = \frac{(a+1)}{2}$.
By Corollary 4.1, there exists a C_4-factorization, C, of

$$K(a,p;1,0) + \left(\bigcup_{(s,j)\in S_1} E(P(s,j)) \right).$$

So it remains to partition the edges of the subgraph

$$K' = K(a,p;\lambda_1 - 1, 0) + \left(\bigcup_{(s,j)\in S\setminus S_1} E(P(s,j)) \right)$$

of K into C_4-factors.
Since $\lambda_1 - 1$ is even, it turns out that we can adapt the construction used in Theorem 3.1. By Condition 3, $\lambda_1 \le \lambda_2 a(p-1) - a$, so $\frac{\lambda_1 - 1}{2} \le \frac{\lambda_2 a(p-1)}{2} - \frac{a+1}{2} = |S| - |S_1|$. Therefore, we can choose a set $S_2 \subseteq S \setminus S_1$ with $|S_2| = \frac{\lambda_1 - 1}{2}$. Let $S_3 = S \setminus (S_1 \cup S_2)$. Then each element in

$$\{P^-(s,j,r) \mid (s,j) \in S_2, r \in \mathbb{Z}_a\}$$

induces a C_4-factor, and the union of the edges in all $\frac{a(\lambda_1-1)}{2}$ C_4-factors contains each pure edge $2|S_2| = \lambda_1 - 1$ times, and uses precisely the mixed edges in

$$\bigcup_{(s,j)\in S_2} P\,(s,j)\,.$$

Clearly the remaining edges can be partitioned into the following sets that induce the C_4-factors:

$$\{P\,(s,j) \mid (s,j) \in S_3\}.$$

So, the required C_4-factorization of K is defined by:

$$C \cup \{P^-(s,j,r) \mid (s,j) \in S_2, r \in \mathbb{Z}_a\} \cup \{P(s,j)|(s,j) \in S_3\}.$$

Notice that the number of C_4-factors is

$$a + a\frac{(\lambda_1-1)}{2} + \left(\frac{\lambda_2 a(p-1)}{2} - \frac{(a+1)}{2} - \frac{(\lambda_1-1)}{2}\right) = \frac{\lambda_2(p-1)}{2} + \frac{(\lambda_1-1)}{2}$$

as required. ■

5 Open Problems

When $a \equiv 1 \pmod 4$ there exists a gap in the known upper bound of λ_1 in Lemma 2.2 and the bound reached in Theorem 4.2. Either constructing C_4-factorizations of K when $\lambda_1 > \lambda_2 a(p-1) - a$ or proving that you cannot do so remains a priority. We conjecture that such constructions do exist.

Also while this paper concerns the case where $a \equiv 1 \pmod 4$, further research may be conducted when $a \equiv 3 \pmod 4$. As yet, there exist no tools that may be used to efficiently produce the required factorizations for this subsequent case; however, the first steps are being taken to build the tools needed.

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