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Trees whose domination subdivision number is one

H. Karami

Department of Mathematics
Sharif University of Technology
P.O. Box 11365-9415, Tehran
I.R. Iran

S.M. Sheikholeslami*

Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz
I.R. Iran
s.m.sheikholeslami@azaruniv.edu

Abstract

A set S of vertices of a graph $G = (V, E)$ is a *dominating set* if every vertex of $V(G)\setminus S$ is adjacent to some vertex in S. *The domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. *The domination subdivision number* $\mathrm{sd}_\gamma(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the domination number. Velammal in his Ph.D. thesis [Manonmaniam Sundaranar University, Tirunelveli, 1997] showed that for any tree T of order at least 3, $1 \le \mathrm{sd}_\gamma(T) \le 3$. Furthermore, Aram, Favaron and Sheikholeslami, recently, in their paper entitled "Trees with domination subdivision number three," gave two characterizations of trees whose domination subdivision number is three. In this paper we characterize all trees whose domination subdivision number is one.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use [16] for terminology and notation which are not defined here. For every vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *open neighborhood*

* Corresponding author

of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$. A set S of vertices is a *dominating set* if $(V \setminus S) \subseteq N(S)$, or equivalently, every vertex in $V \setminus S$ has a neighbor in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G, and a dominating set of minimum cardinality is called a γ-set.

The *domination subdivision number* $\mathrm{sd}_\gamma(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the domination number. (An edge $uv \in E(G)$ is *subdivided* if the edge uv is deleted, but a new vertex w is added, along with two new edges uw and vw.) Since the domination number of the graph K_2 does not change when its only edge is subdivided, we assume that the graph is of order $n \geq 3$. Similar definitions exist for the connected domination number $\gamma_c(G)$ and the connected domination subdivision number $\mathrm{sd}_{\gamma_c}(G)$ if G is connected and, when G has no isolated vertex, for the double domination number $\mathrm{dd}(G)$ and the double domination subdivision number $\mathrm{sd}_{\mathrm{dd}}(G)$ and for the total domination number $\gamma_t(G)$ and the total domination subdivision number $\mathrm{sd}_{\gamma_t}(G)$. The domination subdivision number was first introduced in Velammal's thesis [15] and since then many results have also been obtained on the parameters sd_γ, $\mathrm{sd}_{\mathrm{dd}}$, sd_{γ_c}, and sd_{γ_t} (see for instance [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14]). In particular, Velamma [15] showed that:

Theorem A *For any tree T of order $n \geq 3$, $1 \leq \mathrm{sd}_\gamma(T) \leq 3$.*

Similarly, Haynes et al. [10] showed that:

Theorem B *For any tree T of order $n \geq 3$, $1 \leq \mathrm{sd}_{\gamma_t}(T) \leq 3$.*

Hence, trees can be classified as Class 1, Class 2, or Class 3 depending on whether their (total) domination subdivision numbers are 1, 2, or 3, respectively. Haynes et al. [10], posed the following questions.

Question 2 of [10] Characterize the trees achieving the lower (respectively, upper) bound of Theorem A.

Question 3 of [10] Characterize the trees whose total domination subdivision number is i for $i = 1, 2, 3$.

Haynes et al. [11] give a constructive characterization of trees whose total domination subdivision number is 3. Karami et al. [14] characterized the trees whose total domination subdivision number is one. Aram et al. [1] give a constructive characterization and a structural one of trees whose domination subdivision number is 3. Our purpose in this paper is to characterize all trees T with $\mathrm{sd}_\gamma(T) = 1$.

2 Trees with domination subdivision number 1

In this section we characterize trees of order $n \geq 3$ whose domination subdivision number is 1 which gives a solution to Question 2 for the lower bound. For a tree

T define $\mathcal{L}(T) = \{v \in V(T) \mid \deg(v) = 1\}$ (the leaves) and $\mathcal{L}'(T) = \{u \in V(T) \mid T + uw$ has a γ-set containing $w\}$, where uw is a pendant edge added at u. It is useful to partition the vertices of T in two ways according to how deleting a vertex or adding a pendant edge affects $\gamma(G)$. Define

$$\begin{aligned} V_0(T) &= \{v \in V(T) \mid \gamma(T - v) = \gamma(T)\}; \\ V_+(T) &= \{v \in V(T) \mid \gamma(T - v) > \gamma(T)\}; \\ V_-(T) &= \{v \in V(T) \mid \gamma(T - v) < \gamma(T)\}; \\ W_0(T) &= \{v \in V(T) \mid \gamma(T + vw) = \gamma(T)\}; \\ W_+(T) &= \{v \in V(T) \mid \gamma(T + vw) > \gamma(T)\}; \end{aligned}$$

where vw is a pendant edge at v. Let T_1 and T_2 be two trees, one of which is of order at least two and $u_i \in V(T_i)$ for $i = 1, 2$. Let $\mathcal{B}$ be the collection of trees T of order at least 3, such that each $T \in \mathcal{B}$ satisfies one of the following properties:

Property 1: $T = T_1 \cup T_2 + \{u_1u_2\}$, where $u_i \in (V_0(T_i) \cup V_+(T_i)) \cap W_+(T_i)$ for $i = 1, 2$;

Property 2: $T = T_1 \cup T_2 + \{u_1u_2\}$, where $u_1 \notin \mathcal{L}'(T_1)$, $u_1 \in W_0(T_1)$ and $u_2 \in V_-(T_2)$;

Property 3: There exists a vertex u in $\mathcal{L}(T)$ such that u is a bad vertex; that is, there is no γ-set of T containing u.

Lemma 1 *If $T \in \mathcal{B}$, then* $\mathrm{sd}_\gamma(T) = 1$.

Proof. If T has Property 3, then obviously $\mathrm{sd}_\gamma(T) = 1$. Now let T satisfy one of the Properties 1, 2. Then $\gamma(T) \le \gamma(T_1) + \gamma(T_2)$. Let $T' = (T - u_1u_2) + \{u_1w, u_2w\}$, where $w \notin V(T)$; that is, T' is the graph obtained by subdividing the edge u_1u_2. We show that $\gamma(T') > \gamma(T)$, which implies that $\mathrm{sd}_\gamma(T) = 1$. Let D be a γ-set of T'. Consider two cases.

Case 1 T has Property 1. We consider two subcases.

Subcase 1.1 $w \in D$. If $u_1 \in D$ or $u_2 \in D$, then $D \setminus \{w\}$ is a dominating set of T which implies $\gamma(T') > \gamma(T)$. Now let $u_1, u_2 \notin D$. Then $D \cap V(T_1)$ and $D \cap V(T_2)$ are dominating sets for $T_1 - u_1$ and $T_2 - u_2$, respectively. Thus, by assumption,

$$|D| \ge \gamma(T_1 - u_1) + \gamma(T_2 - u_2) + 1 > \gamma(T_1) + \gamma(T_2) \ge \gamma(T).$$

Subcase 1.2 $w \notin D$. Then $u_1 \in D$ or $u_2 \in D$. Let $u_1 \in D$ (the case $u_2 \in D$ is similar). Then $D \cap V(T_1)$ is a dominating set of $T_1 + u_1w$ and $D \cap V(T_2)$ is a dominating set of T_2. Hence, by assumption,

$$|D| \ge \gamma(T_1 + u_1w) + \gamma(T_2) > \gamma(T_1) + \gamma(T_2) \ge \gamma(T).$$

Case 2 T has Property 2. First we show that $\gamma(T) \leq \gamma(T_1) + \gamma(T_2 - u_2)$. Let D_1 and D_2 be γ-sets of $T_1 + u_1w$ and $T_2 - u_2$, respectively. Obviously, $u_1 \in D_1$ and by assumption $\gamma(T_1) = \gamma(T_1 + u_1w)$. This implies that $D_1 \cup D_2$ is a dominating set of T which implies that $\gamma(T) \leq \gamma(T_1) + \gamma(T_2 - u_2)$. We consider two subcases.

Subcase 2.1 $w \in D$. If $u_2 \in D$, then $D \setminus \{w\}$ is a dominating set of T which implies $\gamma(T') > \gamma(T)$. Now let $u_2 \notin D$. Then $D \cap V(T_1 + u_1w)$ is a dominating set of $T_1 + u_1w$ containing w and $D \cap V(T_2)$ is a dominating set of $T_2 - u_2$. Since $u_1 \notin \mathcal{L}'(T_1)$, $|D \cap V(T_1 + u_1w)| > \gamma(T_1)$. Now it follows that

$$\gamma(T') = |D| > \gamma(T_1) + \gamma(T_2 - u_2) \geq \gamma(T).$$

Subcase 2.2 $w \notin D$. Then $D \cap V(T_1)$ is a dominating set of T_1 and $D \cap V(T_2)$ is a dominating set of T_2. This implies that

$$\gamma(T') = |D| \geq \gamma(T_1) + \gamma(T_2) > \gamma(T_1) + \gamma(T_2 - u_2) \geq \gamma(T).$$

□

Now we are ready to prove the main theorem of this paper.

Theorem 1 *Let T be a tree of order $n \geq 3$. Then* $\mathrm{sd}_\gamma(T) = 1$ *if and only if* $T \in \mathcal{B}$.

Proof. If $T \in \mathcal{B}$, then $\mathrm{sd}_\gamma(T) = 1$ by Lemma 1. Now let $\mathrm{sd}_\gamma(T) = 1$. Then there exists an edge $e = u_1u_2$ such that subdividing e increases the domination number of T. Let $T' = (T - e) + \{u_1w, u_2w\}$ be obtained from T by subdividing e. First let e be a pendant edge and $deg(u_1) = 1$. We claim that u_1 is a bad vertex. Let, to the contrary, D be a γ-set of T containing u_1. Then $(D \setminus \{u_1\}) \cup \{w\}$ is a dominating set of T' of size $\gamma(T)$, a contradiction. Therefore, $u_1 \in \mathcal{L}(T)$ is a bad vertex and, hence, T has Property 3.

Now let e be a non-pendant edge. Let T_1 and T_2 be the components of $T - e$ containing u_1 and u_2, respectively. Obviously the order of T_1 or T_2 is greater than 1. Let D be a γ-set of T such that $|D \cap \{u_1, u_2\}|$ is minimum. If $|D \cap \{u_1, u_2\}| = 2$, then D is a dominating set of T' which is a contradiction. Now consider two cases.

Case 1 $|D \cap \{u_1, u_2\}| = 0$. It is easy to see that $\gamma(T) = \gamma(T_1) + \gamma(T_2)$. We claim that $\gamma(T_i + u_iw) > \gamma(T_i)$ for $i = 1, 2$. Let, to the contrary, $\gamma(T_1 + u_1w) = \gamma(T_1)$ (the case $\gamma(T_2 + u_2w) = \gamma(T_2)$ is similar). Let D_1 and D_2 be γ-sets of $T_1 + u_1w$ and T_2, respectively. Then $D_1 \cup D_2$ is a dominating set of T'. This leads to

$$\gamma(T') \leq |D_1 \cup D_2| \leq \gamma(T_1) + \gamma(T_2) = \gamma(T),$$

which is a contradiction. Hence, $\gamma(T_i + u_iw) > \gamma(T_i)$ for $i = 1, 2$. On the other hand, $D \cap V(T_i)$ is a γ-set of T_i and a dominating set of $T_i - u_i$ for $i = 1, 2$. This implies that $\gamma(T_i - u_i) \leq \gamma(T_i)$ for $i = 1, 2$. Now we claim that $\gamma(T_i - u_i) = \gamma(T_i)$ for $i = 1, 2$. Let, to the contrary, $\gamma(T_2 - u_2) < \gamma(T_2)$ (the case $\gamma(T_1 - u_1) < \gamma(T_1)$ is similar). If D_1 is a γ-set of $T_1 + u_1w$ containing w and D_2 is a γ-set of $T_2 - u_2$, then $D_1 \cup D_2$ is a dominating set of T' of size less than $\gamma(T')$, a contradiction. Therefore, $\gamma(T_i - u_i) = \gamma(T_i)$ for $i = 1, 2$. and, hence, T has Property 1.

Case 2 $|D \cap \{u_1, u_2\}| = 1$. Let $u_1 \in D$ and $u_2 \notin D$ (the case $u_1 \notin D$ and $u_2 \in D$ is similar). We claim that $\gamma(T) = \gamma(T_1 + u_1w) + \gamma(T_2 - u_2)$. Obviously $D \cap V(T_1)$ is a dominating set of $T_1 + u_1w$ and $D \cap V(T_2)$ is a dominating set of $T_2 - u_2$. It follows that $\gamma(T) \leq \gamma(T_1 + u_1w) + \gamma(T_2 - u_2)$. Now let D_1 be a γ-set of $T_1 + u_1w$ containing u_1 and let D_2 be a γ-set of $T_2 - u_2$. Then $D_1 \cup D_2$ is a dominating set of T which implies that $\gamma(T) = \gamma(T_1 + u_1w) + \gamma(T_2 - u_2)$. Now we have $D \cap N_{T_2}(u_2) = \emptyset$, for otherwise D is a γ-set of T', a contradiction. If $\gamma(T_2) \leq \gamma(T_2 - u_2)$, then for any γ-set of T_2, say S, $(D \cap V(T_1)) \cup S$ is a dominating set for T' of size at most $\gamma(T)$, which is a contradiction. Therefore, $\gamma(T_2) > \gamma(T_2 - u_2)$. We claim that $\gamma(T_1 + u_1w) = \gamma(T_1)$. Let, to the contrary, $\gamma(T_1 + u_1w) > \gamma(T_1)$. Let D_1 and D_2 be γ-sets of T_1 and $T_2 - u_2$, respectively, and $x \in N_{T_2}(u_2)$. Then obviously $u_1 \notin D_1$, and hence, $D' = D_1 \cup D_2 \cup \{x\}$ is a γ-set of T in which $D' \cap \{u_1, u_2\} = \emptyset$, which is a contradiction. Thus, $\gamma(T_1 + u_1w) = \gamma(T_1)$. Finally, we show that $u_1 \notin \mathcal{L}'(T_1)$. Let, to the contrary, $u_1 \in \mathcal{L}'(T_1)$. Let D_1 be a γ-set of $T_1 + u_1w$ containing w and let D_2 be a γ-set of $T_2 - u_2$. Then $D_1 \cup D_2$ is a dominating set of T' of size $\gamma(T)$, a contradiction. Hence, T has Property 2. This completes the proof. □

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