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Pan-orientable block designs

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Abstract

A *balanced incomplete block design* BIBD(v, k, λ) is a pair $(V, \mathcal{B})$ where V is a v-set (points) and $\mathcal{B}$ is a collection of k-subsets of V (blocks) such that each pair of elements of V occurs in exactly λ blocks. A *k-tournament* is a directed graph on k vertices in which there is exactly one arc between any two distinct vertices.

Given a k-tournament T, we call a BIBD$(v, k, 2)$ *T-orientable* if it is possible to replace each block B by a copy of T on the set B such that every ordered pair of distinct points appears in exactly one of the tournaments. We call a BIBD$(v, k, 2)$ *pan-orientable* if it is T-orientable for every possible k-tournament T.

There is an extensive literature on oriented triple systems. In this paper, we investigate the case $k = 4$. We prove that pan-orientable BIBD$(v, 4, 2)$s exist for any admissible order v with a finite number of possible exceptions and show for each admissible order v except $v = 7$ the existence of a BIBD$(v, 4, 2)$ which is not pan-orientable. Moreover, we discuss the asymptotic existence of pan-orientable designs for general k, and study the repeated block problem.

1 Introduction

Let k, v and λ be positive integers. A *balanced incomplete block design* BIBD(v, k, λ) is a pair $(V, \mathcal{B})$ where V is a v-set (points) and $\mathcal{B}$ is a collection of k-subsets of V

(blocks) such that each pair of distinct elements of V occurs in exactly λ blocks. A *k-tournament* is a directed graph on k vertices in which there is exactly one arc between any two distinct vertices. A tournament is said to be *transitive* if whenever (a, b) and (b, c) are arcs of the tournament, then (a, c) is also an arc.

Given a k-tournament T, we call a BIBD$(v, k, 2)$ *T-orientable* if it is possible to replace each block B by a copy of T on the set B such that every ordered pair of distinct points appears in exactly one of the tournaments. Clearly, this provides a decomposition of the arc set of the complete directed graph D_v on v vertices into subgraphs each isomorphic to T. In the other direction, by replacing each subgraph in such a decomposition by a block containing all the vertices of the subgraph a BIBD$(v, k, 2)$ is obtained, the *underlying* design. We call a BIBD$(v, k, 2)$ *pan-orientable* if it is T-orientable for every possible k-tournament T.

There is an extensive literature on oriented triple systems; for a survey with original references and proofs see Colbourn and Rosa [8]. For $k = 3$ there are two possible choices for T, the cyclically directed triangle C and the transitively directed triangle R, giving rise to Mendelsohn triple systems and directed triple systems, respectively. Mendelsohn triple systems and directed triple systems exist whenever $v \equiv 0, 1 \bmod 3$ except when $v = 6$ (in which case there is no Mendelsohn triple system); see Huang and Mendelsohn [13], and Mendelsohn [16]. Furthermore, Colbourn and Colbourn [7] proved that every BIBD$(v, 3, 2)$ is R-orientable. This implies that BIBD$(v, 3, 2)$s which are both C-orientable and R-orientable exist for every $v \equiv 0, 1 \bmod 3$, $v \neq 6$. Note that not every BIBD$(v, 3, 2)$ can be C-oriented as shown by Bennett and Mendelsohn [3] who found a non-C-orientable BIBD$(v, 3, 2)$ for every order $v \equiv 0, 1 \bmod 3$.

A generalisation of block designs (allowing blocks of different size) called pairwise balanced designs have been used to produce orientable triple systems [8] and similarly we shall in this paper construct pan-orientable block designs using these structures. Therefore, we continue with a definition of pairwise balanced designs and introduce related concepts. Let K be a set of positive integers, and let v and λ be positive integers. A *pairwise balanced design* PBD with index λ is a pair $(V, \mathcal{B})$ where V is a v-set (points) and $\mathcal{B}$ is a collection of subsets of V (blocks) such that each pair of distinct points occurs in exactly λ blocks. A PBD(v, K, λ) is a pairwise balanced design in which each block has size from the set K. In the case where $\lambda = 1$ we also write PBD(v, K), for short. A *group divisible design* GDD with index λ is a triple $(V, \mathcal{G}, \mathcal{B})$ where $\mathcal{G}$ is a partition of V into groups and $(V, \mathcal{G} \cup \mathcal{B})$ is a PBD with index λ. A (K, λ)-GDD of *type* $g_1^{t_1} g_2^{t_2} \ldots g_r^{t_r}$ is a group divisible design in which each block has size from the set K and in which there are precisely t_i groups of size g_i, $i = 1, 2, \ldots, r$.

In a sequence of three papers Wilson [18, 19, 20] developed a theory of PBD-closed sets. A set S of positive integers is said to be *PBD-closed* if the existence of a PBD(v, S) implies that v belongs to S. Let K be a set of positive integers and let $B(K) = \{v \mid \exists \text{ PBD}(v, K)\}$. Then $B(K)$ is a PBD-closed set called the *PBD-closure* of K. Concerning the structure of PBD-closed sets Wilson showed that if S is a PBD-closed set, then S is *eventually periodic* with period $\beta(S)$; that is, there exists a constant $v_0(S)$ such that for every $k \in S$, $\{v \mid v \geq v_0(S), v \equiv k \bmod \beta(S)\} \subseteq S$. The theory of PBD-closed sets is a powerful tool for investigating combinatorial structures: a finite number of known examples of objects with a certain property can establish the existence of an infinite set of these objects.

Unfortunately, the constant $v_0(B(K))$ is not known in general. Therefore, one attempts to determine $B(K)$ for given K as accurately as possible. In particular, sets K with at least one 'small' element have been widely investigated. For a survey see [1, Tables 3.17, 3.18]. We further cite a result from Greig, Grüttmüller, Hartmann [9] that will be used in Section 3.

Theorem 1.1 *If* $v \equiv 1 \bmod 6$ *and* $v \notin Q_{\{7,13,19,25,31,37,43\}}$ *(see Table 3), then* $v \in B(\{7, 13, 19, 25, 31, 37, 43\})$.

In Section 2 we will prove an asymptotic existence result. In Section 3 we investigate the case $k = 4$. We prove that pan-orientable BIBD$(v, 4, 2)$s exist for all admissible orders v with at most 244 possible exceptions, and for each admissible order v except $v = 7$ we demonstrate the existence of a BIBD$(v, 4, 2)$ which is not pan-orientable. Finally, in Section 4 we discuss some enumeration results.

2 Asymptotic Existence Results

Theorem 2.1 *The set of orders for which a pan-orientable BIBD*$(v, k, 2)$ *exists is PBD-closed.*

Proof. Let S be the set of orders v for which a pan-orientable BIBD$(v, k, 2)$ exists. Let v be a positive integer such that a PBD(v, S), say $(V, \mathcal{B})$ exists. The size of any block $B \in \mathcal{B}$ is from S and, therefore, a pan-orientable BIBD$(|B|, k, 2)$ on the elements of B exists with block set $\mathcal{B}_B$. Let $\mathcal{U}$ be the union of all sets $\mathcal{B}_B$ as B ranges over the block set $\mathcal{B}$. It is easy to check that $(V, \mathcal{U})$ is a BIBD$(v, k, 2)$. Since $(B, \mathcal{B}_B)$ is pan-orientable for each block $B \in \mathcal{B}$ we find that $(V, \mathcal{U})$ is pan-orientable, too. This proves $v \in S$. □

Though constructions for block designs do not forbid repeated blocks in general, designs without repeated blocks have been widely studied in the literature. These designs are said to be *simple*. For details on the *repeated block problem* in design theory, the interested reader is referred to [6, 12]. Bennett and Mendelsohn [2] also studied Mendelsohn triple systems whose underlying block design is simple. In such a Mendelsohn triple system any two triangles have distinct vertex sets. This motivates us to ask for *simple* pan-orientable BIBD$(v, k, 2)$s. Even more, for $k \geq 4$, it is interesting to ask for *super-simple* pan-orientable BIBD$(v, k, 2)$s where any two blocks have at most two elements in common. The notion super-simple was introduced by Gronau and Mullin [10].

It is easy to see that the proof of Theorem 2.1 extends to simple or super-simple pan-orientable designs.

Theorem 2.2 *The set of orders for which a super-simple (or simple, respectively) pan-orientable BIBD$(v,k,2)$ exists is PBD-closed.*

Let $T_1,\dots,T_h$ be given k-tournaments. If we place them on a k-element vertex set U, we obtain a *vector* of k-tournaments. For $h \geq 2$, there are in general various ways to place the given k-tournaments on U resulting in various non-isomorphic vectors of k-tournaments.

For a given vector $\underline{T}$ of k-tournaments, we define the *degree-vector* $\Delta_{\underline{T}}(x)$ of a vertex $x \in U$ to be the $2h$-integer vector $(\mathrm{out}_{T_1}(x), \mathrm{in}_{T_1}(x), \dots, \mathrm{out}_{T_h}(x), \mathrm{in}_{T_h}(x))$, where $\mathrm{out}_{T_i}(x)$ denotes the out-degree and $\mathrm{in}_{T_i}(x)$ the in-degree of vertex x in the tournament T_i, with $i = 1,\dots,h$.

Next, let $\mathcal{T}$ be a collection of vectors of k-tournaments, and let $\alpha(\mathcal{T})$ denote the greatest common divisor of the integers z where the $2h$-integer vector $(z,\dots,z)$ may be written as an integral linear combination of the degree-vectors $\Delta_{\underline{T}}(x)$, with $\underline{T}$ ranging over $\mathcal{T}$, and x ranging over U. If $h = 1$ and $\mathcal{T}$ consists of a single k-tournament T only, we also write $\alpha(T)$ instead of $\alpha(\mathcal{T})$.

Let $(V,\mathcal{B})$ be a BIBD$(v,k,2)$ that is T_i*-orientable* for $i = 1,\dots,h$. Each block $B \in \mathcal{B}$ induces a vector $\underline{T}^B$ of k-tournaments. Lamken and Wilson [14] studied the asymptotic existence of decompositions of complete directed multigraphs into subgraphs each isomorphic to vectors of directed graphs.

From their results, one immediately derives that $(V,\mathcal{B})$ satisfies the conditions

$$v - 1 \equiv 0 \bmod \alpha(\mathcal{T}) \tag{1}$$
$$v(v-1) \equiv 0 \bmod k(k-1)/2 \tag{2}$$

if for every block B the vector $\underline{T}^B$ is isomorphic to a vector in $\mathcal{T}$. Even more, we may conclude from [14, Thm. 1.2] that for almost all positive integers v satisfying the conditions (1) and (2) there exists a BIBD$(v,k,2)$ $(V,\mathcal{B})$ that is T_i*-orientable* for $i = 1,\dots,h$ and where for every block B, the vector $\underline{T}^B$ is isomorphic to a vector in $\mathcal{T}$. For simple and super-simple BIBD$(v,k,2)$s, the same conclusions may be drawn from Hartmann [12, Thm. 2.2].

When looking for a pan-orientable BIBD$(v,k,2)$, we consider all possible k-tournaments $T_1,\dots,T_h$, and let $\mathcal{T}$ consist of all possible non-isomorphic vectors of k-tournaments. We denote $\alpha(\mathcal{T})$ by α_k, for short.

Theorem 2.3 *The following conditions are necessary and asymptotically sufficient for the existence of a (simple, super-simple) pan-orientable BIBD$(v,k,2)$:*

$$v - 1 \equiv 0 \bmod \alpha_k \tag{3}$$
$$v(v-1) \equiv 0 \bmod k(k-1)/2 \tag{4}$$

We do not go into further details here since the results in [12, 14] are more general. However, we refer to the proof of Theorem 3.1 for the special case of 4-tournaments which we included into this paper for the sake of convenience.

Out-degree vectors of k-tournaments written as a non-decreasing sequence are known as *score sequences.* A complete characterisation of score sequences is due to Landau [15].

Theorem 2.4 *A sequence of integers* $0 \leq out_1 \leq out_2 \leq \cdots \leq out_k \leq k-1$ *is a score sequence of a* k*-tournament if and only if*

$$\binom{m}{2} \leq \sum_{i=1}^{m} out_i \tag{5}$$

for $m = 1, 2, \ldots, k$ *with equality holding for* $m = k$*.*

It is easy to see that the parameter α_k used in Theorem 2.3 is the least common multiple of the values $\alpha(T_i)$, with $i = 1, \ldots, h$. Further, $\alpha(T_i)$ is a divisor of $k(k-1)/2$ for each $i = 1, \ldots, h$, and so is α_k. From Landau's theorem we may conclude an explicit formula for α_k. For small k we present α_k in Table 1.

k	3	5	7	9	11	13	15	17	19	21
α_k	1	2	3	12	5	6	105	8	9	210

Table 1: Values α_k for odd $k = 3, 5, \ldots, 21$

Corollary 2.5

$$\alpha_k = \begin{cases} p^{\beta-1}(k-1)/2 & \textit{if } k \textit{ is an odd prime power } k = p^\beta, \\ k(k-1)/2 & \textit{otherwise.} \end{cases}$$

Proof. For even values of k, it is easy to construct a k-tournament T with out-degree vector $(0, k/2, \ldots, k/2)$. For this k-tournament, we find $\alpha(T) = k(k-1)/2$ which immediately yields $\alpha_k = k(k-1)/2$.

For odd values of k, the situation is more complicated. Obviously, $(k-1)/2$ is a divisor of α_k since the sum of out-degree and in-degree of an arbitrary vertex is $k-1$ and, therefore, in any linear combination (z, z) of degree-vectors $\Delta_{\underline{T}}(x)$ we have that $2z$ is a multiple of $k-1$. Now, if $k = d \cdot k'$ is an odd composite integer with $d, k' \geq 3$, then the sequence defined by

$$\mathrm{out}_i = \begin{cases} (k-d)/2 & \text{if } 1 \leq i \leq (k-k')/2, \\ (k+d)/2 & \text{if } (k-k')/2 < i \leq k, \end{cases}$$

satisfies (5) and is therefore the score sequence of a k-tournament T_d. Each vertex of T_d has an out-degree which is a multiple of d. Hence, $\alpha(T_d)$ is a multiple of $d(k-1)$.

This in turn implies that α_k is a multiple of $\mathrm{lcm}\{\alpha(T_d) : d|k \text{ with } 3 \leq d < k\}$. Therefore, $\alpha_k = k(k-1)/2$ holds for odd composite k which are not a prime power.

For an odd prime power $k = p^\beta$ the same argument implies that α_k is a multiple of $p^{\beta-1}(k-1)/2$. To show that α_k is precisely $p^{\beta-1}(k-1)/2$ it suffices to find for any k-tournament T a linear combination (z, z) of degree-vectors $\Delta_T(x)$ where z is not a multiple of $k(k-1)/2$. Let T be a k-tournament and consider its out-degree sequence. If we have $\mathrm{out}_1 = (k-1)/2$, or $\mathrm{out}_1 = 0$ and $\mathrm{out}_k = (k-1)$, then it is easy to obtain such a linear combination for $z = (k-1)/2$. Otherwise, put $a = \mathrm{in}_1 - \mathrm{out}_1$ and $b = \mathrm{out}_k - \mathrm{in}_k$. Both a and b are even positive integers smaller than k. Moreover, put $c = \mathrm{lcm}\{a, b\}$, and $z = c(a+b)/(ab) \cdot (k-1)/2$. Note that $z < k(k-1)/2$ and

$$(z, z) = c/a \cdot (\mathrm{out}_1, \mathrm{in}_1) + c/b \cdot (\mathrm{out}_k, \mathrm{in}_k)$$

hold, that is, we have found the desired linear combination. □

Corollary 2.6 *Let S be the set of orders v for which a pan-orientable BIBD$(v, k, 2)$ exists. Then S is eventually periodic with period*

$$\beta(S) = k(k-1)/2$$

and a necessary and asymptotically sufficient condition for v to be an element of S is

$$v \equiv \begin{cases} 1 \text{ or } k \bmod k(k-1)/2 & \text{if } k \text{ is an odd prime,} \\ 1 \bmod k(k-1)/2 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $v \in S$. It suffices to check that v fulfills condition (3) and (4) in Theorem 2.3 with the α_k determined in Corollary 2.5. We consider three cases. First, let k be an even integer or an odd non prime power, then $\alpha_k = k(k-1)/2$ and, therefore, (3) requires $v \equiv 1 \bmod k(k-1)/2$. Clearly, then also (4) is satisfied.

Second, let $k = p^\beta$ be an odd prime power with $\beta \geq 2$, then $\alpha_k = p^{\beta-1}(k-1)/2$. Now (3) is satisfied if and only if $v = m \cdot p^{\beta-1}(k-1)/2 + 1$ for some $m \in \mathbb{N}$. If $m \equiv 0 \bmod p$ (that is $v \equiv 1 \bmod k(k-1)/2$), then v satisfies immediately (4). If otherwise $m \not\equiv 0 \bmod p$, then in order to fulfill (4) we need that p divides v. But this means p divides 1, a contradiction. Hence $v \equiv 1 \bmod k(k-1)/2$ is the only possible solution.

Finally, let k be an odd prime, then $\alpha_k = (k-1)/2$. Again, (3) implies $v = m \cdot (k-1)/2 + 1$ for some $m \in \mathbb{N}$. Obviously, $v \equiv 1 \bmod k(k-1)/2$ satisfies (4). If $m \not\equiv 0 \bmod k$, then again (4) implies that k divides v. We consider two subcases. If $m = 2m'$ is even, then $v = m'k - m' + 1$. Thus k divides $1 - m'$ and this in turn implies $m' = m''k + 1$ for some $m'' \in \mathbb{N}$. So, every $v = 2m''k(k-1)/2 + (k-1) + 1 \equiv k \bmod k(k-1)/2$ is another solution. If $m = 2m' + 1$ is odd, then $v = m'k - m' + (k-1)/2 + 1$. Hence, k divides $-m' + (k-1)/2 + 1$. This implies $m' \equiv (k+1)/2 \bmod k$ or, equivalently, $2m' + 1 \equiv 2 \bmod k$. Therefore, we obtain the same residue class modulo $k(k-1)/2$ as before: $v = m''k(k-1)/2 + 2(k-1)/2 + 1 \equiv k \bmod k(k-1)/2$. □

3 Existence Results for $k = 4$

We will use the PBD-closure result together with Theorem 1.1 to establish the existence of pan-orientable block designs in the case $k = 4$ up to 244 possible exceptions. There are four non-isomorphic 4-tournaments $T_1, \ldots, T_4$ which are best characterised by their out-degree vectors:

$$\begin{aligned} T_1 &: \ (3,2,1,0) \\ T_2 &: \ (3,1,1,1) \\ T_3 &: \ (2,2,1,1) \\ T_4 &: \ (2,2,2,0) \end{aligned}$$

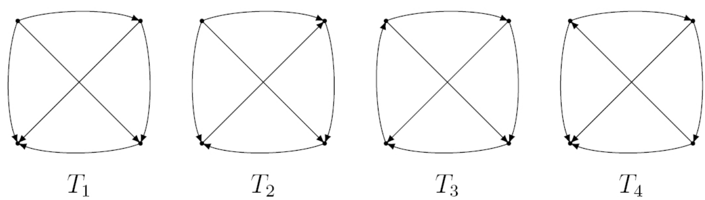

Figure 1: Drawings of the four non-isomorphic 4-tournaments $T_1, \ldots, T_4$

To begin with, we repeat the necessary conditions for the existence of a pan-orientable BIBD$(v, 4, 2)$ stated in Theorem 2.3. For the sake of convenience, we include a short proof of these necessary conditions which basically follows the approach taken by Wilson in [21].

Theorem 3.1 *A pan-orientable BIBD*$(v, 4, 2)$ *exists only if* $v \equiv 1 \bmod 6$.

Proof. Suppose there exists a T_i-decomposition of the complete directed graph D_v for $i = 1, \ldots, 4$. There are two main conditions which need to be satisfied. First, each tournament has 6 arcs, so the number of arcs $v(v-1)$ of D_v needs to be divisible by 6.

For the second condition, we study the *degree-vector* $\tau(x) = (\text{out}(x), \text{in}(x))$ of a vertex x in some directed graph. In the complete directed graph D_v, each vertex y has degree-vector $\tau(y) = (v-1, v-1)$. Hence, if a T_i-decomposition of D_v exists then the set of arcs incident with a vertex of D_v is partitioned by the isomorphic copies of T_i so that the vector $(v-1, v-1)$ is a non-negative integral linear combination of the degree-vectors $\tau(x)$, where x runs through the vertex set of the tournament T_i.

As before, let $\alpha(T_i)$ denote the greatest common divisor of the integers z where (z, z) is an integral linear combination of the degree-vectors $\tau(x)$ with x ranging through all vertices of T_i. Clearly, $\alpha(T_i)$ divides $v-1$. The degree-vectors of the 4-tournament T_1 are $(3,0)$, $(2,1)$, $(1,2)$, $(0,3)$, and thus $\alpha(T_1) = 3$. Similarly, we find $\alpha(T_2) = 6$,

$\alpha(T_3) = 3$, and $\alpha(T_4) = 6$. Therefore, $\alpha_4 = \text{lcm}\{\alpha(T_1), \ldots, \alpha(T_4)\} = 6$ divides $v - 1$. This implies $v - 1 \equiv 0 \bmod 6$. □

Note, that reversing the direction of all arcs in a tournament isomorphic to T_1 yields again a tournament isomorphic to T_1. Similarly, the reverse of a tournament isomorphic to T_3 is again a tournament isomorphic to T_3. Finally, the reverse of a tournament isomorphic to T_2 is a tournament isomorphic to T_4, and vice versa. The latter observation yields the following two lemmas.

Lemma 3.2 *A BIBD$(v, 4, 2)$ is T_2-orientable if and only if it is T_4-orientable.*

Lemma 3.3 *A BIBD$(v, 4, 2)$ containing a repeated block is not T_2-orientable.*

Proof. Suppose there is a block B that occurs twice. If the first copy of B is replaced by a tournament T isomorphic to T_2, then the second copy of B has to be replaced by the reverse of T which is isomorphic to T_4. □

Next, we observe that not every BIBD$(v, 4, 2)$ is pan-orientable.

Theorem 3.4 *There exists a BIBD$(v, 4, 2)$ for every order $v \equiv 1 \bmod 6, v > 7$, which is not pan-orientable.*

Proof. For every $v \equiv 1 \bmod 12$, $v > 1$, there exists a BIBD$(v, 4, 1)$ [4]. Adjoining a second copy of each block yields a BIBD$(v, 4, 2)$ which is not T_2-orientable by Lemma 3.3 and, therefore, not pan-orientable.

Otherwise, let $v \equiv 7 \bmod 12$. To begin with, we look for a BIBD$(v, 4, 2)$ with at least one repeated block. The existence of such a BIBD would settle the claim by Lemma 3.3 as above. Note that there exists a BIBD$(4, 4, 2)$ which consists of two copies of the block $\{0, 1, 2, 3\}$), and also a BIBD$(7, 4, 2)$ [11, Lemma 4.4]. Hence, it suffices to find a PBD$(v, \{4, 7\})$ with at least one block of size 4. By replacing the blocks of this PBD by the BIBDs of order 4 and 7, a BIBD$(v, 4, 2)$ with at least one repeated block can be obtained. In fact, for $v \equiv 7 \bmod 12, v > 19$ there exists such a PBD with exactly one block of size 7 and all the remaining blocks of size 4, a PBD$(v, \{4, 7^*\})$, as shown by Brouwer [4].

It remains to consider the case $v = 19$. We take a $(\{4\}, 2)$-GDD of type 3^6 (constructed explicitly by Brouwer, Schrijver and Hanani [5]), adjoin an infinite point, and replace each group and the infinite point by two copies of a block of size four to obtain a BIBD$(19, 4, 2)$ containing repeated blocks. This BIBD is not pan-orientable by Lemma 3.3, again. This completes the proof. □

To continue with, we give direct constructions for some small pan-orientable BIBDs. In particular, we show that the unique BIBD$(7, 4, 2)$ is pan-orientable.

Lemma 3.5 *There exists a pan-orientable BIBD$(v, 4, 2)$ for every order $v \in \{7, 13, 19, 25, 31, 37, 43\}$.*

Proof. Consider an ordered block (a, b, c, d). To obtain a 4-tournament from this block, we fix an orientation of the arcs as follows:

$$\begin{aligned} T_1 : &\quad ab, ac, ad, bc, bd, cd \\ T_2 : &\quad ab, ac, ad, bc, cd, db \\ T_3 : &\quad ab, ac, bc, bd, cd, da \end{aligned}$$

For each order v under inspection, it suffices to find for each $i = 1, 2, 3$ a collection of ordered blocks which, with the fixed orientation above, form a T_i-decomposition of the complete directed graph D_v, and which yield the same BIBD$(v, 4, 2)$ if the blocks are considered to be unordered. Note that in view of Lemma 3.2 we do not need to consider T_4.

For $v = 7$, we take the ordered base block $(0, 2, 1, 5)$ to generate a cyclic T_1-decomposition of D_7. Similarly, we take the ordered base block $(2, 0, 1, 5)$ to produce a cyclic T_2-decomposition of D_7. Note, that every non-zero element of $\mathbb{Z}_7$ occurs exactly once as a difference $b - a$ for some arc ab. There is no cyclic T_3-decomposition of D_7, but the following ordered blocks provide a non-cyclic solution: $(0, 1, 2, 3)$, $(5, 4, 1, 0)$, $(4, 6, 2, 0)$, $(0, 6, 3, 5)$, $(5, 2, 6, 1)$, $(3, 1, 6, 4)$, $(3, 2, 4, 5)$, cf. Figure 2. Recall that there exists only one BIBD$(7, 4, 2)$. That is, the underlying BIBDs for $i = 1, 2, 3$ are the same as desired.

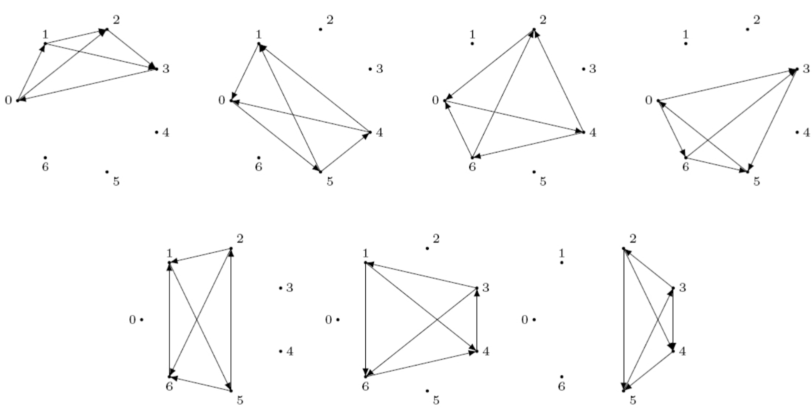

Figure 2: T_3-decomposition of D_7

For a cyclic T-decomposition of the complete directed graph D_v with $v \equiv 1 \bmod 6$ one needs $(v-1)/6$ ordered base blocks. These can be created from an ordered super base block by multiplying with the elements of a subgroup of index 6 of the multiplicative group $GF(v)^*$. Let ω be a generating element of $GF(v)^*$, define $\xi = \omega^6$, and consider the subgroup generated by ξ. In Table 2, we list for each order v a generator of $GF(v)^*$, the elements of the subgroup, and for each $i = 1, 2, 3$ an ordered super base block. It is easy to check from the table that if we multiply the differences for the arcs arising from the ordered super base block with the elements of the subgroup, then each element of $GF(v)^*$ occurs exactly once.

v	ω	subgroup $< \xi >$	T_1	T_2	T_3
13	2	$\{-1, 1\}$	$(0, 1, 4, 6)$		
19	2	$\{7, 11, 1\}$	$(1, 0, 6, 2)$	$(0, 1, 6, 2)$	$(0, 2, 1, 6)$
25	$*$	$\{\xi, -1, -\xi, 1\}$	$(0, 1, \omega + 2, 3\omega)$		
31	3	$\{16, 8, 4, 2, 1\}$	$(1, 0, 12, 9)$	$(0, 12, 9, 1)$	$(0, 9, 1, 12)$
37	2	$\{27, 26, -1, 10, 11, 1\}$	$(0, 1, 3, 24)$		
43	3	$\{41, 4, 35, 16, 11, 21, 1\}$	$(0, 1, 25, 28)$	$(0, 28, 25, 1)$	$(0, 28, 25, 1)$

Table 2: Parameters for the construction of small pan-orientable BIBDs; * indicates that the generating element ω is a root of the primitive polynomial $x^2 + x + 2$

For $v \equiv 1 \bmod 12$, we can use the same ordered super base block for each i since -1 is an element of the subgroup and this allows one to reverse the direction of any two opposite arcs independently from the direction of the other arcs. That is, a solution for $i = 1$ can be transformed to a solution for $i = 2$ and 3. Otherwise, for $v = 19, 31, 43$, we can still use the same elements in the super base block, only the ordering must be different. Thus, in both cases the underlying BIBDs for $i = 1, 2, 3$ are the same as desired. □

Theorem 3.6 *There exists a pan-orientable BIBD$(v, 4, 2)$ for all $v \equiv 1 \bmod 6$ with 244 possible exceptions, the largest being 6631, cf. Table 3.*

Proof. By Theorem 2.1 and the pan-orientable BIBD$(v, 4, 2)$s constructed in Lemma 3.5 we know that there is a pan-orientable BIBD$(v, 4, 2)$ for each $v \in B(\{7, 13, 19, 25, 31, 37, 43\})$. The claim now follows from Theorem 1.1. □

Theorem 3.7 *There exists a super-simple (or simple, respectively) pan-orientable BIBD$(v, 4, 2)$ for all $v \equiv 1 \bmod 6$ with 244 possible exceptions.*

Proof. It is not difficult to check that all underlying BIBDs constructed in Lemma 3.5 are super-simple (and thus simple). Therefore, Theorems 1.1 and 2.2 imply the claim. □

4 Enumeration Results

In this section, we report briefly on some enumeration results with respect to the property of being pan-orientable. We investigated all 2461 BIBD$(13, 4, 2)$s which we constructed using the program DESY implemented by Pietsch [17]. It is remarkable that all BIBD$(13, 4, 2)$s are T_1- and T_3-orientable. In view of the fact that all

55	61	67	73	79	97	103	109	115	121	127	139	145	157	163	181	193	199	205	211	229	235
241	265	271	277	283	289	313	319	331	349	355	367	373	391	397	409	415	433	439	445	451	
457	487	493	499	505	643	649	655	661	667	685	691	697	709	727	733	739	745	751	769	781	
787	793	799	805	811	853	859	865	871	877	937	943	949	955	979	985	991	997	1003	1063		
1069	1231	1237	1255	1315	1321	1327	1357	1363	1375	1381	1399	1405	1411	1417	1423						
1441	1447	1459	1465	1567	1579	1585	1609	1693	1711	1717	1819	1825	1831	1837	1843						
1861	1867	1879	1885	1903	1921	1927	1999	2005	2155	2161	2173	2257	2287	2299	2407						
2455	2461	2467	2473	2491	2497	2509	2515	2533	2551	2557	2701	2707	2725	2797	2803						
2827	2833	2839	2845	2851	2875	2881	2893	3001	3007	3013	3019	3037	3043	3049	3055						
3061	3079	3085	3091	3097	3121	3127	3139	3163	3337	3349	3373	3379	3391	3397	3415						
3421	3427	3433	3439	3457	3469	3475	3481	4471	4483	4507	4519	4531	4555	4573	4591						
4597	4615	4633	4639	4651	4867	5059	5065	5071	5077	5101	5107	5113	5119	5137	5143						
5149	5155	5179	5185	5191	5197	5203	5347	5353	5365	5371	5413	5431	5437	5449	5455						
5491	5497	5515	5521	5527	5533	5539	5581	5935	5941	5953	5995	6001	6613	6619	6631						

Table 3: $Q_{\{7,13,19,25,31,37,43\}}$

BIBD$(v,3,2)$s are R-orientable we like to ask the corresponding question for $k=4$, namely: Is it true that all BIBD$(v,4,2)$s are T_1- and T_3-orientable?

1576 of the BIBD$(13,4,2)$s are simple. 1529 of the simple BIBD$(13,4,2)$s are pan-orientable. That is, there are BIBD$(13,4,2)$s that are simple, but not pan-orientable. Consequently, there are reasons other than the one mentioned in Lemma 3.3 that cause a BIBD$(v,4,2)$ to be not T_2-orientable.

Appendix

For the sake of completeness, we list in Table 3 the set $Q_{\{7,13,19,25,31,37,43\}}$, that is, the set of those orders v for which the existence of a PBD$(v,\{7,13,19,25,31,37,43\})$ (see Theorem 1.1) and the existence of a pan-orientable BIBD$(v,4,2)$ (see Theorem 3.6) is unknown.

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