The Geometric Points of Coverings

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1. Introduction.

We consider v elements labelled by the integers from 1 to v and ask for the cardinality of a minimal family of k-sets with the property that each pair occurs at least once (one can also, more generally consider the case when each pair occurs at least λ times). The cardinality of such a minimal family is denoted by C(2,k,v), and satisfies an inequality analogous to the bk = rv relationship for Balanced Incomplete Block Designs, namely,

$$k\,C(2,k,v) \geq v\,C(1,k-1,v-1).$$

Indeed, if the covering is exact and there are no surplus pairs, then this inequality becomes the usual relationship bk = rv for BIBDs.

For example, let k = 3, v = 6. It is easy to show that the covering design is unique and can be generated cyclically as the blocks 124, 235, 346, 451, 562, 613; if we form a graph (all our graphs are loopless, but they usually will have multiple edges) from the pairs appearing in this covering, we see that it consists of the complete graph K_6, together with an excess graph E. In this example, E consists of the 3 links 14, 25, and 36. On the other hand, the unique (2,3,5) covering is given by 123, 145, 245, 345; here the graph of the covering is K_5 + E, where E consists of the three isolated points 1,2,3, and the edge 45 taken with multiplicity 2.

If we proceed to the case k = 3, v = 7, then the covering design is the usual Fano Geometry generated cyclically, modulo 7, from the initial block 124. This covering has no excess pairs and so the excess graph consists of 7 isolated points (in general, any BIBD will have an excess graph consisting only of isolated points).

2. Geometric Points of a Covering.

For a general covering, the graph formed from all pairs occurring in the covering can be written in the form $K_v + E$, where E is the excess graph. Then the *geometric points* of the covering are defined to be the isolated points of the excess graph. In general, with a design on v elements, the valencies of the points in the excess graph are given by the expression

$$(k - 1)\lceil (v-1)/(k-1)\rceil - (v - 1) + j(k - 1)$$

where j = 0, 1, 2, ... In order for the excess graph to possess isolated points, we must have $(v - 1)/(k - 1) = i$, $j = 0$.

We shall restrict ourselves to those covers between the values $i = 1$ (that is, $C(2,k,k) = 1$) and $i = k$; in this latter case,

$$C(2, k, k^2 - k + 1) \leq k^2 - k + 1,$$

with equality in the case that a projective geometry exists. In both cases (that is, the case of a single block and the case of a finite geometry), all points in the covering are geometric points.

If we think of the covering array as comprising all b blocks of k elements, then we can count the number of elements in the array and obtain bk. However, there are v elements and each element must appear a minimum of $\lceil (v-1)/(k-1)\rceil$ times. Consequently, we have the usual Fisher-Yates counting bound on b as a result of the inequality

$$bk \geq v \lceil (v-1)/(k-1)\rceil.$$

We now recall the weight function of a block B (cf. [3]). This is a non-negative function that can be defined as follows. Suppose that the covering contains b blocks and that the elements in block B have frequencies r_i, where i ranges from 1 to k. Let the total number of occurrences of the pair (ij) from B be designated by λ_{ij}, where $\lambda_{ij} \geq 1$. Then we have

$$w(B) = (b - 1) - \Sigma (r_i - 1) + \Sigma (\lambda_{ij} - 1)$$

$$= (b - 1) - \Sigma (r_i - 1) + \Sigma e_{ij},$$

where the summation is taken over all elements from B or all pairs from B, and where e_{ij} is the number of times that the edge (ij) occurs in the excess graph E.

It is straightforward to show that w(B) can also be written in the form

$$w(B) = x_0 + x_3 + 3x_4 + 6x_5 + \ldots$$

where x_i denotes the total number of blocks C in the covering that have the property that $|C \cap B| = i$. We should remark that other definitions of the function w are possible; cf [3].

The use of the weight function is most effective for values of b that are small relative to v (that is, the difference $v - b > 0$; the larger $v - b$ is, the more information is given by the quantity w).

3. An Illustration: Designs on $k^2 - 2k + 2$ Points.

The Fisher-Yates bound for $v = k^2 - k + 1$ gives

$$C(2,k,k^2 - k + 1) \geq k^2 - k + 1,$$

and we know that this bound is achieved in the case of a finite geometry. The next lower value of v for which there can be geometric points in the covering design is $v = k^2 - 2k + 2$, and here the bound is found from the relation

$$k\ C(2,k,k^2 - k + 2) \geq (k^2 - k + 2)(k-1)$$

to be $C(2,k,k^2 - k + 2) \geq k^2 - 3k + 4$.

We shall mainly be concerned with this particular example in the case $k = 6$. However, for completeness, we first briefly look at the values of $k < 6$.

For k = 3, the bound is 4 and C(2,3,5) = 4 (the covering is unique). For k = 4, there are 10 points and the bound is 8; computation of w immediately shows that there is no covering in 8 blocks. The coverings in 9 blocks are easily found using w; there are at least 4 geometric points 1, 2, 3, 4. If they appear in one block 1234, then the other blocks all contain one of them; so there can be no further geometric points, and there is a unique completion 1abc, 1def, 2abd, 2cef, 3aef, 3 bcd, 4bef, 4acd. The weight of a block containing only 3 geometric points would be negative; so the other possibility is to have blocks 12ab, 13cd, 14ef, 23ef, 24cd, 34ab. This design can be completed by abcd, abef, cdef, or by abce, abdf, cdef, or by abce, acdf, bdef. Thus there are four non-isomorphic (2,4,10) covering designs.

For k = 5, there are 17 points and the bound is 14. There are at least 15 geometric points and the weight function shows that there can be no blocks containing 5 or 4 geometric points; hence there is no covering in 14 blocks. In a covering in 15 blocks, there would have to be at least 10 geometric points, and a similar argument shows that they can not occur either 5 to a block or 4 to a block. If they occur at most 3 to a block, then $4N \geq N(N-1)/2$, and this is not possible since $N \geq 10$. Hence there is likewise no covering in 15 blocks.

On the other hand, if we try for a covering in 16 blocks, we find that $N \geq 5$. If there are 5 geometric elements in a block, then the design is 12345, together with three blocks each of the form ixxxx, where i ranges from 1 to 5.

The weight function shows that there can not be 4 geometric elements in a block. If there at most 3 geometric elements in a block, the previous argument shows that $N \leq 9$.

The most interesting solutions are those that have the maximum number of geometric points; in a sense, they are "close" to being geometries. For N = 9, we find that all the geometric points lie in 12 subsets of 3, that is, they form the 9-point affine geometry. We represent this geometry in the usual manner and extend the lines to give 5-sets as ab123, ab456, ab789, cd147, cd258, cd369, ef159, ef267, ef348, gh357, gh249, gh168. It is then necessary to fit all missing "letter" pairs (that is, missing pairs involving the non-geometric points) into four blocks. There are various ways of doing this; one of the most obvious is provided by the four blocks abceg, abcfh, abdeh, abdfg.

4. The Design on 26 Points.

For k = 6, there does exist a geometry on 31 points; we now look at the value v = 26 which is the closest v-value beneath 31 for which there are geometric points.

The bound on C(2,6,26) is 22. If the bound is met, there are at least 24 geometric points (of frequency 5). Hence the design must contain a block of 6 geometric points. But such a block would have weight 21 - 24 = - 3.

If we try C(2,6,26) = 23, the weights of blocks with either 6 or 5 geometric points are negative. Hence there are at most 4 geometric points per block and therefore at most 23(6) = 138 geometric pairs. But there are at least 18 geometric points and therefore at least 153 geometric pairs. So we can reject the possibility that C(2,6,26) = 23.

If C(2,6,26) = 24, then the number of geometric points, N, is at least 12. On the other hand, if b_i denotes the number of blocks containing i geometric points, we have

$$b_1 + 2b_2 + 3b_3 + 4b_4 = 5N,$$

$$b_2 + 3b_3 + 6b_4 = N(N - 1)/2.$$

It follows that $3b_1 + 4b_2 + 3b_3 = N(16 - N)$. So the covering with the maximal number of geometric points has N = 16. This leads to the fact that the the geometric points form the 16-point affine geometry comprising 20 quadruples. Each of the 5 resolution class must be extended by adding pairs ab to the first class, cd to the second, ef to the third, gh to the fourth, and ij to the last class. Again, one has to get all missing pairs in four blocks and this is easily achieved by considering the covering set of triples on the five elements ab, cd, ef, gh, ij; thus we may take the last four blocks as abcdef, abghij, cdghij, efghij.

5. The Covering Number C(2,6,21).

Discussion of the case when v = 21 can be carried out in a similar manner. If the result were 14, all points would be geometric and so the blocks would all have weight 13 - 6(3), and this is not possible. If the result were 15, we find that blocks of 6 or 5 geometric points would have negative weight; so there is a total of at most 6(15) = 90 geometric pairs in the covering array. But the number of geometric points is at least 15 and so there must be at least 105 geometric pairs. Again we have a contradiction.

If we try C(2,6,21) = 16, we can also rule out blocks containing 6 or 5 geometric points. If there were a block with 4 geometric points, then its weight would be $15 - (18 + \alpha + \beta) + (r_{ab} - 1)$; this is not possible since it would require the frequency of the pair ab to exceed the frequency $4 + \alpha$ of the element a.

Now we suppose that there are N geometric points; clearly, $N \geq 9$. But the equations (again, b_i denotes the number of blocks containing i geometric points)

$$b_1 + 2b_2 + 3b_3 + 4b_4 = 4N,$$

$$b_2 + 3b_3 + 6b_4 = N(N - 1)/2,$$

show that $N \leq 9$. Hence N = 9, and the situation is promising. We get 12 blocks by adjoining elements abc, def, ghi, jkm, to the four resolution classes of the (9,12,4,3,1) design. The question then becomes one of whether we can get all the missing letter pairs from abc, def, ghi, jkm, into four sextuples.

Each of the letters occurs exactly twice in these four sextuples. Consider the 3 elements abc. If any sextuple contains abc, then there are at most 3 + 5 places for other letters to appear with a; but there are 9 other letters. Hence the letters appear in the blocks in the form ab, ac, b, c. Again this leads to an impossibility (there are only 8 places for the other letters to occur with a).

We thus miss out on the possibility that C(2,6,21) = 16. On the other hand, a covering in 17 blocks is readily obtained as follows. Take a block 1234ab, where the first four elements are geometric; let the other 15 elements be represented by A,B,C,D,E,F, g,h,i. Here the 6 capital letters represent pairs of elements and so represent 12 elements in all. The letters g,h,i, represent single elements. Write down the following array:

1234ab	1ABg, 1CDh, 1EFi	
	2ACi, 2BEh, 2DFg	
	3ADi,3BFh, 3CEg	
	4AEh, 4BDi, 4CFg	
	abAF, abDE, abCB	abghi

This array covers all pairs (the last block can be filled in by any of the 12 elements from the capital pairs).

6. An Example with k = 7.

We close with an illustration of the use of geometric points in discussing C(2,7,19). The bound for C(2,7,19) is equal to 9; if this bound is achieved, there must be at least 13 geometric points. It is easy to calculate weights and find that no block may contain 7, 6, or 5 geometric points; but nine 4-sets of geometric points can produce only 36 pairs and there are 13 geometric points. Hence the bound can not be achieved.

If C(2,7,19) = 10, there are at least 6 geometric points. A similar calculation shows that there can not be blocks with 7, 6, or 5 geometric points. If there were a block with four geometric points 1,2,3,4, then all other blocks would contain one of these points and this would not allow a further geometric point. Hence there can be at most 3 geometric points in a block. If we write down the equations

$$b_1 + 2b_2 + 3b_3 = 3N,$$

$$b_2 + 3b_3 = N(N - 1)/2,$$

we find that $N \leq 7$. For N = 7, the geometric points form a Fano Geometry and this requires too many abnormal points to complete the 7 blocks containing the geometric points. For N = 6, the geometric points form a PBD 12, 34, 56, 135, 146, 236, 245, and this can not be embedded in a covering. However, $N \geq 6$, and so there would have to be 15 blocks if all geometric pairs occurred separately. Hence C(2,7,19) ≥ 10.

If C(2,7,19) = 11, there need not be any geometric points. Let us try to find a covering with as many geometric points as possible. We easily rule out the possibilities that there can be 7, 6, 5, or 4 geometric points in a block. If there are 3 geometric points in a block (123abcd), there are 6 more blocks that contain them (two each for 1, 2, and 3). Since abcd are non-geometric, they must occur at least 3 more times each. Hence the last blocks can be taken as abcdefg, abcdhij, abcdkmn, abcdpqr. We can now write the other blocks as 1efghij, 1kmnpqr, 2efgkmn, 2hijpqr, 3efgpqr, 3hijkmn. This shows that C(2,7,19) = 11.

7. Conclusion.

We have tried to illustrate the use of geometric points in discussing a number of the coverings which possess them; further illustrations can be found in papers [1] and [2].

REFERENCES

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