A Note on Extending t-Designs

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Abstract. Khosrovshahi and Ajoodani-Namini give a new method for extending t-designs with $k = t + 1$. Based on their result, they obtain a recursive construction for t-designs and for large sets of disjoint $t-(v, k, \lambda)$ designs with $k = t + 1$. Independently, Teirlinck recursively constructs large sets with the same parameters using a different method. In this paper, we generalize their results to any $k \geq t + 1$ and construct a family of large sets of disjoint $3-(v, 5, \binom{v-3}{2}/3)$ designs. That is, the family of all 5-subsets of a v-set can be partitioned into 3 disjoint $3-(v, 5, \binom{v-3}{2}/3)$ designs with $v = 9m + 4$ $(m = 1, 2, 3, \ldots)$. To the author's knowledge, this family of large sets is new. We show that there is a large set of disjoint $4-(9m + 5, 6, \binom{9m+1}{2}/3)$ designs for any $m > 1$ if there is a large set of disjoint $4-(13, 5, 3)$ designs.

1 Introduction

We begin by giving some general definitions. A $t-(v, k, \lambda)$ design is a pair $(X, \mathcal{B})$ which satisfies the following properties:

(i) X is a set of v elements (called points);

(ii) $\mathcal{B}$ is a family of k-subsets of X (called blocks);

(iii) any t-subset of X is contained in exactly λ blocks.

A $t-(v, k, \lambda)$ design is called *simple* if it contains no repeated blocks. For a $t-(v, k, \lambda)$ design $(X, \mathcal{B})$ and any fixed subset Y of X with $|Y| = s \leq t$, let

$\mathcal{B}' = \{B\backslash Y : Y \subset B \in \mathcal{B}\}$ (Here B's are all blocks in $\mathcal{B}$ containing Y). Clearly $(X\backslash Y, \mathcal{B}')$ is a $(t-s)-(v-s, k-s, \lambda)$ design, and is called *a derived design* of $(X, \mathcal{B})$. It is well known that a $t-(v,k,\lambda)$ design is also an $s-(v,k,\lambda_s)$ design with $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Hence we have the following necessary condition for the existence of a $t-(v,k,\lambda)$ design:

$$\lambda\binom{v-s}{t-s} \equiv 0 \pmod{\binom{k-s}{t-s}} \quad (s = 0, 1, \ldots, t-1).$$

Given a v-set X, let $P_k(X)$ denote the set of all k-subsets of X. Suppose $(X, \mathcal{B}_1)$, $(X, \mathcal{B}_2), \cdots, (X, \mathcal{B}_n)$ are n simple $t-(v,k,\lambda)$ designs. If $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n$ forms a partition of $P_k(X)$ (namely, $\bigcup_{i=1}^{n} \mathcal{B}_i = P_k(X)$ and $\mathcal{B}_i \bigcap \mathcal{B}_j = \emptyset$ for all $1 \leq i < j \leq n$), then $(X, \mathcal{B}_1), (X, \mathcal{B}_2), \cdots, (X, \mathcal{B}_n)$ is called *a large set of disjoint $t-(v,k,\lambda)$ designs* (see [8]). Note that some use the term *uniform $t-(v,k,\lambda)$ partition.* In that terminology, only when λ is the smallest positive integer satisfying the necessary condition above, is the uniform $t-(v,k,\lambda)$ partition called a large set of disjoint $t-(v,k,\lambda)$ designs [2, 3, 6]. However, large sets with λ not necessarily the smallest integer are still very interesting and important.

Khosrovshahi and Ajoodani-Namini (see [4]) give a new method of extending t-designs. Based on their result, they obtain a recursive construction for t-designs and for large sets of disjoint $t-(v,k,\lambda)$ designs with $k = t+1$. Independently, Teirlinck (see [8]) recursively constructs large sets with the same parameters using a different method. In this paper, we generalize their results to any $k \geq t+1$, and construct a family of large sets of disjoint $3-(v, 5, \binom{v-3}{2}/3)$ designs with $v = 9m+4$ $(m = 1, 2, 3, \ldots)$. This family of large sets is new, and the family of $3-(v, 5, \binom{v-3}{2}/3)$ designs, for $v = 9m+4$ $(m = 2, 3, \ldots)$, is not isomorphic to the known ones. We also show that there is a large set of disjoint $4-(9m+5, 6, \binom{9m+1}{2}/3)$ designs for any $m > 1$ if there is a large set of disjoint $4-(13, 5, 3)$ designs.

2 Main Results

Theorem 1 *Suppose*

(i) *that D_1 and D_2 are (simple) $t-(v_1,k,\lambda_1)$ and $t-(v_2,k,\lambda_2)$ designs, respectively, such that $\frac{\lambda_1}{\binom{v_1-t}{k-t}} = \frac{\lambda_2}{\binom{v_2-t}{k-t}} = s$; and*

(ii) *that there exist a large set of disjoint $(k-2)-(v_1-1,k-1,\frac{v_1-k+1}{n})$ designs and a large set of disjoint $(k-2)-(v_2-1,k-1,\frac{v_2-k+1}{n})$ designs, where n is an integer such that ns is an integer.*

Then there exists a (simple) $t-(v_1+v_2-k+1,k,\lambda)$ design D_3 with $\lambda = s\binom{v_1+v_2-k+1-t}{k-t}$, such that D_3 contains a copy of D_1 and a copy of D_2.

Note that for the special case $k = t+1$, Khosrovshahi and Ajoodani-Namini have already proved this theorem. We will give the proof in the next section. In the above theorem, if one of D_1 and D_2 is not a $(t+1)$-design, then D_3 as constructed in the proof is not a $(t+1)$-design, either. The following results for the special case $k = t+1$ can be found in [4], and Corollary 2 for $k = t+1$ can also be found in [8].

Theorem 2 *Suppose that there are large sets of disjoint t–$(v_1,k,\binom{v_1-t}{k-t}/n)$ and t–$(v_2,k,\binom{v_2-t}{k-t}/n)$ designs, respectively, and that there are large sets of disjoint $(k-2)$–$(v_1-1,k-1,\frac{v_1-k+1}{n})$ and $(k-2)$–$(v_2-1,k-1,\frac{v_2-k+1}{n})$ designs, respectively. Then there exists a large set of disjoint $t-(v_1+v_2-k+1,k,\binom{v_1+v_2-k+1-t}{k-t}/n)$ designs.*

We will give the proof in the next section. From the above two theorems we get:

Corollary 1 *Suppose that there exists a (simple) $t-(v,k,\lambda)$ design, and let $s = \frac{\lambda}{\binom{v-t}{k-t}}$. If there exists a large set of disjoint $(k-2)-(v-1,k-1,\frac{v-k+1}{n})$ designs, where n is an integer such that ns is an integer, then there is a (simple) $t-(v+m(v-k+1),k,s\binom{v-t+m(v-k+1)}{k-t})$ design for any $m > 0$.*

Corollary 2 *Suppose there exist a large set of disjoint* $t-(v,k,\binom{v-t}{k-t}/n)$ *designs and a large set of disjoint* $(k-2)-(v-1,k-1,\frac{v-k+1}{n})$ *designs. Then there is a large set of disjoint* $t-(v+m(v-k+1),k,\binom{v-t+m(v-k+1)}{k-t}/n)$ *designs for any* $m>0$.

Application. There is a large set of disjoint $3-(12,4,3)$ designs and a large set of disjoint $3-(13,5,15)$ designs which is not a large set of disjoint $4-(13,5,3)$ designs (see [2, 6]). By Corollary 2, there is a large set of disjoint $3-(9m+4,5,\binom{9m+1}{2}/3)$ designs for any $m>1$. Simple $3-(9m+4,5,\binom{9m+1}{2}/3)$ designs are already known to be existent which are also $4-(9m+4,5,3m)$ designs. But the large set of disjoint $3-(9m+4,5,\binom{9m+1}{2}/3)$ designs is new, and our $3-(9m+4,5,\binom{9m+1}{2}/3)$ (for $m>1$) designs are not isomorphic to the known ones.

There is a large set of disjoint $4-(14,6,15)$ designs (see [2, 3]). If we can construct a large set of disjoint $4-(13,5,3)$ designs, then there is a large set of disjoint $4-(9m+5,6,\binom{9m+1}{2}/3)$ designs for any $m>1$. The existence of simple $4-(9m+5,6,\binom{9m+1}{2}/3)$ design is believed to be unknown for $m>2$ (The $4-(23,6,57)$ design is in [5]).

3 Proofs of Main Results

Proof of Theorem 1. Let $X=\{1,2,\ldots,v_1+v_2-k+1\}$ and Denote all t-subsets of X by $T_1,T_2,\ldots,T_{\binom{v_1+v_2-k+1}{t}}$, respectively. Partition all k-subsets (called blocks) of X into the following $k+1$ disjoint classes:

$C_0=\{\{x_1,x_2,\ldots,x_k\}\in P_k(X):x_1<x_2<\cdots<x_k<v_1+1\}$,

$C_1=\{\{x_1,x_2,\ldots,x_k\}\in P_k(X):x_1<x_2<\cdots<x_{k-1}<v_1<x_k\}$,

$\ldots$,

$C_j=\{\{x_1,x_2,\ldots,x_k\}\in P_k(X):x_1<\cdots<x_{k-j}<v_1+1-j<x_{k-j+1}<\cdots<x_k\}$,

$\ldots$,

$C_{k-1}=\{\{x_1,x_2,\ldots,x_k\}\in P_k(X):x_1<v_1-k+2<x_2<x_3<\cdots<x_k\}$,

$C_k=\{\{x_1,x_2,\ldots,x_k\}\in P_k(X):v_1-k+1<x_1<x_2<\cdots<x_k\}$.

Let $n_{i,j}$ be the number of blocks B in C_j containing T_i. Since $C_0, C_1, C_2, \ldots, C_k$ is a partition of $P_k(X)$, $\sum_{j=0}^{k} n_{i,j}$ is the number of k-subsets of X containing T_i. So

$$\sum_{j=0}^{k} n_{i,j} = \binom{v_1 + v_2 - k + 1 - t}{k - t}.$$

Suppose we can construct a collection $\mathcal{B}_j$ of k-subsets of X from C_j such that any t-subset T_i of X is contained in $sn_{i,j}$ blocks in $\mathcal{B}_j$ $(j = 0, 1, \ldots, k)$. Then by the above equation, $(X, \bigcup_{j=0}^{k} \mathcal{B}_j)$ is the required $t-(v_1 + v_2 - k + 1, k, s\binom{v_1+v_2-k+1}{k-t})$ design. Now we try to construct such $\mathcal{B}_j$. Let

$$X_j = \{1, 2, \ldots, v_1 - j\}, \qquad j = 0, 1, \ldots, k-1;$$

$$Y_j = \{v_1 + 2 - j, v_1 + 3 - j, \ldots, v_1 + v_2 - k + 1\}, \qquad j = 1, 2, \ldots, k.$$

Note that $X_j \bigcup Y_j = X \setminus \{v_1 + 1 - j\}$ for $0 < j < k$.

For $j = 0$, by the existence of D_1, we construct a collection $\mathcal{B}_0$ of k-subsets of X_0 such that $(X_0, \mathcal{B}_0)$ is a copy of D_1, i.e., a t–(v_1, k, λ_1) design. If $T_i \not\subset X_0$, then $n_{i,0} = 0$. If $T_i \subset X_0$, then $n_{i,0} = \binom{v_1 - t}{k-t}$ and $sn_{i,0} = \lambda_1$. T_i is thus contained in $sn_{i,0}$ blocks of $\mathcal{B}_0$ for every i. For $j = k$, we similarly construct a collection $\mathcal{B}_k$ of k-subsets of Y_k such that $(Y_k, \mathcal{B}_k)$ is a copy of D_2. So, T_i is contained in $sn_{i,k}$ blocks of $\mathcal{B}_k$ for every i.

We consider the general case $0 < j < k$. Let $(X_1, \mathcal{B}_{1,1})$, $(X_1, \mathcal{B}_{2,1})$, $\cdots$, $(X_1, \mathcal{B}_{n,1})$ be a large set of $(k-2)-(v_1 - 1, k-1, \frac{v_1 - k + 1}{n})$ designs. By deleting the points $v_1 + 1 - j$, $v_1 + 2 - j, \ldots$, $v_1 - 1$, we obtain the correspoding derived designs $(X_j, \mathcal{B}_{1,j})$, $(X_1, \mathcal{B}_{2,j})$, $\cdots$, $(X_1, \mathcal{B}_{n,j})$, which together form a large set of $(k-1-j)-(v_1 - j, k-j, \frac{v_1 - k + 1}{n})$ designs.

Similarly, let $(Y_{k-1}, \mathcal{B}'_{1,k-1})$, $(Y_{k-1}, \mathcal{B}'_{2,k-1})$, $\cdots$, $(Y_{k-1}, \mathcal{B}'_{n,k-1})$ be a large set of $(k-2)$–$(v_2 - 1, k-1, \frac{v_2 - k + 1}{n})$ designs. By deleting the points $v_1 + 3 - k$, $v_1 + 4 - k$, $\ldots$, $v_1 + 1 - j$, we have the corresponding derived designs $(Y_j, \mathcal{B}'_{1,j})$, $(Y_j, \mathcal{B}'_{2,j})$, $\cdots$, $(Y_j, \mathcal{B}'_{n,j})$ which together form a large set of $(j-1)-(v_2 - k + j, j, \frac{v_1 - k + 1}{n})$ designs.

Note that for any block $B \in C_j$, $|B \bigcap X_j| = k - j$ and $|B \bigcap Y_j| = j$. For every $B^{(1)} \in \mathcal{B}_{i_1,j}$ and every $B^{(2)} \in \mathcal{B}'_{i_2,j}$, $B^{(1)} \bigcup B^{(2)}$ is a block in C_j. Now given any permutation σ on $\{1,2,\ldots,n\}$, let

$$C_{(j,\sigma)} = \bigcup_{i=1}^{n} B_{i,j} \odot B'_{\sigma(i),j},$$

where $B_{i,j} \odot B'_{\sigma(i),j} = \{A \bigcup B : A \in B_{i,j}, B \in B'_{\sigma(i),j}\}$. Hence $C_{(j,\sigma)} \subset C_j$. We claim that T_i is contained in $\frac{n_{i,j}}{n}$ blocks in $C_{(j,\sigma)}$ for every i.

If $n_{i,j} = 0$, the claim is obvious. Assume $n_{i,j} \neq 0$. Then $v_1+1-j \notin T_i$. Let $T_i^{(1)} = \{t \in T_i : t < v_1+1-j\}$ ($\subset X_j$), $T_i^{(2)} = \{t \in T_i : t > v_1+1-j\}$ ($\subset Y_j$). $n_{i,j} \neq 0$ implies that $|T_i^{(1)}| \leq k-j$ and $|T_i^{(2)}| \leq j$. Let $l = j - |T_i^{(2)}|$. Then $|T_i^{(2)}| = j-l$, $|T_i^{(1)}| = |T_i| - |T_i^{(2)}| = t-j+l$ with $0 \leq l \leq k-t$. Choose any $(k-t-l)$-subset Z_1 of $X_j \backslash T_i^{(1)}$ and any l-subset Z_2 of $Y_j \backslash T_i^{(2)}$. Then $T_i \bigcup Z_1 \bigcup Z_2$ is a block in C_j. Clearly we have $\binom{v_1-t-l}{k-t-l}$ choices for Z_1 and $\binom{v_2-k+l}{l}$ choices for Z_2. Therefore $n_{i,j} = \binom{v_1-t-l}{k-t-l}\binom{v_2-k+l}{l}$.

Case 1. $l = k-t$. Then $|T_i^{(1)}| = k-j$ and $T_i^{(1)}$ is a block in one and only one of the $(k-1-j)$–$(v_1-j, k-j, \frac{v_1-k+1}{n})$ designs $(X_j, B_{1,j})$, $(X_j, B_{2,j})$, $\cdots$, $(X_j, B_{n,j})$. $|T_i^{(2)}| = j-(k-t)$ and $T_i^{(2)}$ is contained in $\binom{v_2-t}{k-t}/n = n_{i,j}/n$ blocks of $(Y_j, B'_{u,j})$ $(u = 1,2,\ldots,n)$. Hence, T_i is contained in $n_{i,j}/n$ blocks in $C_{(j,\sigma)}$.

Case 2. $l = 0$. The discussion is similar to Case 1.

Case 3. $0 < l < k-t$. Then $T_i^{(1)}$ is contained in $\binom{v_1-t-l}{k-t-l}/n$ blocks of $(X_j, B_{u,j})$ $(u = 1,2,\ldots,n)$, and $T_i^{(2)}$ is contained in $\binom{v_2-k+l}{l}/n$ blocks of $(Y_j, B'_{u,j})$ $(u = 1,2,\ldots,n)$. So T_i is contained in $\sum_{u=1}^{n} \binom{v_1-t-l}{k-t-l}/n\binom{v_2-k+l}{l}/n = n_{i,j}/n$ blocks of $C_{(j,\sigma)}$.

Finally, let $m = sn$ and $\sigma_1, \sigma_2, \ldots, \sigma_m$ be m permutations on $\{1,2,\ldots,n\}$ and $B_j = \bigcup_{i=1}^{m} C_{(j,\sigma_i)}$. Then T_i is contained in $m(n_{i,j}/n) = sn_{i,j}$ blocks in B_j. Therefore $(X, \bigcup_{j=0}^{k} B_j)$ is the required t–$(v_1+v_2-k+1, k, s\binom{v_1+v_2-k+1}{k-t})$ design. If D_1 and D_2 are both simple, then $s = \lambda_1/\binom{v_1-t}{k-t} \leq 1$. Choose $\sigma_i = (1\ 2\ \cdots\ n)^i$, $1 \leq i \leq m$. Then the design $(X, \bigcup_{j=0}^{k} B_j)$ has no repeated blocks and thus is simple.

Proof of Theorem 2. We use the notations in the proof above. Let $(X_0, B_{1,0})$, $(X_0, B_{2,0})$, $\cdots$, $(X_0, B_{n,0})$ be a large set of disjoint t–$(v_1, k, \binom{v_1-t}{k-t}/n)$ designs, and $(Y_k, B'_{1,k})$, $(Y_k, B'_{2,k})$, $\cdots$, $(Y_k, B'_{n,k})$ a large set of disjoint $t-$

$(v_2, k, \binom{v_2-t}{k-t}/n)$ designs. Choose $\sigma_i = (1\ 2\ \cdots\ n)^i$, $i = 1, 2, \ldots, m$. Define

$$\mathcal{B}_i = \mathcal{B}_{i,0} \bigcup \mathcal{B}'_{i,k} \bigcup_{j=1}^{k-1} \mathcal{C}_{(j,\sigma_i)}.$$

Then we can verify that $(X, \mathcal{B}_1)$, $(X, \mathcal{B}_2), \cdots$, $(X, \mathcal{B}_n)$ is the required large set.
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