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Resolvable holey group divisible designs with block size three*

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Abstract

Holey group divisible designs (HGDDs) were first defined and investigated by R. Wei. In this paper, we study the existence of resolvable holey group divisible designs (RHGDDs). It is proved that the necessary conditions for the existence of RHGDDs with block size three and index unity are also sufficient with two exceptions.

1 Introduction

We assume that the reader is familiar with some basic concepts in design theory, otherwise the reader is referred to [4], [6] or [7]. A group divisible design K-GDD is a triple $(X, \mathcal{G}, \mathcal{A})$ where X is a finite set of points, $\mathcal{G}$ is a partition of X into subsets (called groups), and $\mathcal{A}$ is a family of subsets of X with sizes from K (called blocks) such that any pair of distinct points occurs in either one group or exactly one block, but not both. We use the usual "exponential" notation to describe the type of a GDD. A GDD of type $g_1^{u_1}g_2^{u_2}\ldots g_s^{u_s}$ means that it has u_i groups of size g_i, $1 \leq i \leq s$. We will use k-GDD to denote the GDD when $K=\{k\}$.

An α-parallel class of blocks in a GDD $(X, \mathcal{G}, \mathcal{A})$ is a subset $\mathcal{B} \subseteq \mathcal{A}$ such that each point $x \in X$ is contained in exactly α blocks in $\mathcal{B}$. When $\alpha = 1$, we will use the usual term parallel class. If the block set $\mathcal{A}$ can be partitioned into α-parallel classes, then the GDD is called α-resolvable (or just resolvable if $\alpha = 1$). A GDD $(X, \mathcal{G}, \mathcal{A})$ is called Γ-resolvable if its block set $\mathcal{A}$ admits a partition into subsets $P_1, P_2, \ldots, P_r$ where for each $i = 1, 2, \ldots, r$ there is an $\alpha_i \in \Gamma$ such that each point $x \in X$ is contained in exactly α_i blocks in P_i. It is not difficult to see that a Γ-resolvable

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GDD with $K = k$ must be uniform. We use K-RGDD to denote a resolvable K-GDD. The following result is taken from [3, 11, 14].

Theorem 1.1 *A 3-RGDD of type g^u exists if and only if $u \geq 3$, $gu \equiv 0 \pmod 3$, $g(u-1) \equiv 0 \pmod 2$ and $(g,u) \notin \{(2,3),\ (2,6),\ (6,3)\}$.*

A transversal design, TD(k,n), is a k-GDD of type n^k. A resolvable transversal design, denoted by RTD(k,n), is a TD(k,n) whose blocks can be partitioned into parallel classes. It is well known that an RTD(k,n) is equivalent to $k-1$ mutually orthogonal Latin squares (MOLS) of order n. From [6], we have the following result.

Theorem 1.2 (1) *An RTD$(3,n)$ exists for all $n \geq 3$ except for $n = 6$.*
(2) *An RTD$(4,n)$ exists for all $n \geq 4$ except for $n = 6$ and except possibly for $n = 10$.*
(3) *An RTD$(7,n)$ exists for all $n \geq 7$ except possibly for $n \in \{10, 12,\ 14,\ 15,\ 18,\ 20,\ 21,\ 22,\ 24,\ 26,\ 28,\ 30,\ 33,\ 34,\ 35,\ 36,\ 38,\ 39,\ 42,\ 44,\ 46,\ 48,\ 51,\ 52,\ 54,\ 55,\ 58,\ 60,\ 62,\ 66,\ 68,\ 74,\ 75\}$.*

An incomplete group divisible design (IGDD) with block size k is a quadruple $(X,\ \mathcal{G},\ \mathcal{H},\ \mathcal{A})$ where X is a finite set of points, $\mathcal{G}$ is a partition of the set X into subsets (called groups), H is a subset of X (called a hole) and $\mathcal{A}$ is a collection of subsets of X (called blocks) such that a group and a block contain at most one common point, and at the same time, each pair of points from distinct groups occurs either in exactly one block or in H, but not both. Such a design is denoted by k-IGDD.

If $\mathcal{G}=\{G_1, G_2, \ldots, G_n\}$, we define the type of the design by the multiset $\{(|G_i|, |G_i \cap H|) \mid 1 \leq i \leq n\}$. As with GDDs, we shall use an "exponential" notation to describe the type. When $H = \emptyset$, an IGDD of type $\{(|G_i|, 0) \mid 1 \leq i \leq n\}$ is just a GDD of type $\{|G_i| \mid 1 \leq i \leq n\}$.

A k-IGDD of type $(g,h)^k$ is called a k-ITD(g,h). A k-IGDD is said to be resolvable, denoted by k-IRGDD, if its blocks can be partitioned into parallel classes and partial parallel classes which partition $X \backslash H$. We use k-IRTD(g,h) to denote a k-IRGDD of type $(g,h)^k$. As for 4-ITD(g,h), we have the following result (see [6]).

Theorem 1.3 *If $g \geq 3h$ and $(g,h) \notin (6,1)$, then there exists a 4-ITD(g,h).*

By deleting all the points of one group in a 4-ITD(g,h), we obtain a 3-IRTD(g,h). So the following is obvious from the above theorem.

Lemma 1.4 *If $g \geq 3h$ and $(g,h) \notin (6,1)$, then there exists a 3-IRTD(g,h).*

If a GDD has several equal-sized holes which partition the point set of the GDD, then we call it a holey GDD, or an HGDD. We give a formal definition of an HGDD as follows (see [9]).

Let X be a set of hmn points which is partitioned into h-subsets X_{ij}, $1 \le i \le m$, $1 \le j \le n$. Let $\mathcal{A}$ be a collection of subsets of X (called blocks) with size of k, which satisfies the following conditions:
(1) every pair of point $x \in X_{i_1 j_1}$ and $y \in X_{i_2 j_2}$ is contained in exactly one block if $i_1 \neq i_2$ and $j_1 \neq j_2$;
(2) the pair of point x and y is not contained in any block if $i_1 = i_2$ or $j_1 = j_2$.

Then we call $(X, \mathcal{A})$ a holey group divisible design and denote it by k-HGDD of type (m, h^n). The subsets $\bigcup_{j=1}^{n} X_{ij}$, where $1 \le i \le m$, are called groups and the subsets $\bigcup_{i=1}^{m} X_{ij}$, where $1 \le j \le n$, are called holes.

HGDDs were first defined and investigated in [16]. An HGDD is a special case of double group divisible designs (DGDDs) which were introduced in [17]. Recently, in [5] Chang and Miao give some general constructions on DGDDs. If $h = 1$, an HGDD of type (m, h^n) is just a modified group divisible design (MGDD) which has been widely studied (see [1, 2, 8, 10]).

If the blocks of an HGDD can be partitioned into parallel classes, the HGDD is said to be resolvable and denoted by RHGDD. In this paper, we will focus on RHGDDs with block size three. The definition of an RHGDD implies the following fact that will be employed extensively in what follows.

Lemma 1.5 *A k-RHGDD of type (m, h^n) exists if and only if a k-RHGDD of type (n, h^m) exists.*

By simple calculation, the following necessary conditions for the existence of a k-RHGDD can be obtained.

Lemma 1.6 *The necessary conditions for the existence of a k-RHGDD of type (m, h^n) are $m \ge k$, $n \ge k$, $mnh \equiv 0 \pmod{k}$ and $h(m-1)(n-1) \equiv 0 \pmod{(k-1)}$.*

For $h = 1$, the following result has been established in [15].

Theorem 1.7 *There exists a 3-RHGDD of type $(m, 1^n)$ if and only if $m \ge 3$, $n \ge 3$, $mn \equiv 0 \pmod{3}$ and $(m-1)(n-1) \equiv 0 \pmod{2}$ except when $(m, n) = (3, 6)$ or $(6, 3)$.*

In this paper, we will employ both direct and recursive methods to show that the necessary conditions of Lemma 1.6 are also sufficient when $k = 3$ and $h > 1$. Combining the result in Theorem 1.7, we will obtain the following main theorem.

Theorem 1.8 *There exists a 3-RHGDD of type (m, h^n) if and only if $m \ge 3$, $n \ge 3$, $hmn \equiv 0 \pmod{3}$ and $h(m-1)(n-1) \equiv 0 \pmod{2}$ except when $(h, m, n) = (1, 3, 6)$ or $(1, 6, 3)$.*

2 Direct Constructions

This section serves to develop some direct constructions.

Lemma 2.1 *There exists a* 3*-RHGDD of type* $(6, 4^6)$.

Proof. This design contains 50 parallel classes. We take $\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cup \mathbb{Z}_4 \times \mathbb{Z}_5 \times \{x\} \cup \mathbb{Z}_4 \times \{y\} \times \mathbb{Z}_5 \cup \mathbb{Z}_4 \times \{y\} \times \{x\}$ as the point set, $\{\mathbb{Z}_4 \times (\mathbb{Z}_5 \cup \{y\}) \times \{i\}\} (i \in \mathbb{Z}_5)$ together with $\mathbb{Z}_4 \times (\mathbb{Z}_5 \cup \{y\}) \times \{x\}$ as the groups, and $\{\mathbb{Z}_4 \times \{j\} \times (\mathbb{Z}_5 \cup \{x\})\} (j \in \mathbb{Z}_5)$ together with $\mathbb{Z}_4 \times \{y\} \times (\mathbb{Z}_5 \cup \{x\})$ as the holes. Cycling the following 24 initial base blocks by mod$(4, 5, 5)$ to obtain all the blocks of the desired design. In the following, we ensure that these blocks can be formed into fifty parallel classes. Notice that when adding 1 to the first component of each point of the following initial base blocks, we can get another 24 blocks. It is easy to check that the obtained 24 blocks together with the initial base blocks form a parallel class of the design. Then adding 2 to the first component of each point of the parallel class to obtain another one. Here the operation is taken modulo 4. To obtain all the parallel classes of the desired design, develop the blocks of the two parallel classes by mod$(-, 5, 5)$.

$\{(2,0,3),(3,2,1),(0,1,0)\}$,	$\{(2,1,0),(1,3,4),(3,2,3)\}$,	$\{(3,2,2),(3,0,0),(1,1,4)\}$,
$\{(0,4,3),(3,3,4),(2,1,2)\}$,	$\{(3,0,2),(1,2,3),(0,4,4)\}$,	$\{(3,2,4),(2,3,3),(3,4,0)\}$,
$\{(0,0,x),(0,y,0),(1,4,2)\}$,	$\{(0,1,x),(1,y,1),(3,0,4)\}$,	$\{(0,2,x),(2,y,0),(0,1,2)\}$,
$\{(0,3,x),(3,y,1),(1,0,2)\}$,	$\{(0,y,2),(2,1,1),(1,4,0)\}$,	$\{(0,y,3),(3,3,0),(3,4,1)\}$,
$\{(0,y,4),(0,0,1),(0,3,2)\}$,	$\{(2,y,2),(3,1,3),(1,4,1)\}$,	$\{(2,y,3),(1,2,4),(3,4,2)\}$,
$\{(2,y,4),(2,2,0),(2,4,3)\}$,	$\{(0,4,x),(2,3,2),(1,2,1)\}$,	$\{(2,0,x),(2,3,1),(3,1,4)\}$,
$\{(2,1,x),(0,3,3),(0,2,0)\}$,	$\{(2,2,x),(1,3,0),(0,0,3)\}$,	$\{(2,3,x),(2,4,4),(2,0,1)\}$,
$\{(2,4,x),(1,1,3),(1,2,2)\}$,	$\{(0,y,x),(0,1,1),(1,0,4)\}$,	$\{(2,y,x),(0,3,1),(1,0,0)\}$.

Lemma 2.2 *There exists a* 3*-RHGDD of type* $(3, 3^6)$.

Proof. We take $\mathbb{Z}_3 \times \mathbb{Z}_3 \times (\mathbb{Z}_5 \cup \{\infty\})$ as the point set, $\{\{i\} \times \mathbb{Z}_3 \times (\mathbb{Z}_5 \cup \{\infty\})\} (i \in \mathbb{Z}_3)$ as the groups, and $\{\mathbb{Z}_3 \times \mathbb{Z}_3 \times \{j\}\} (j \in \mathbb{Z}_5)$ together with $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \{\infty\}$ as the holes. Develop the initial base blocks listed below by mod$(-, 3, -)$ to obtain one base parallel class. Then develop the blocks of the base parallel class by mod$(3, -, 5)$ to get the desired design.

$\{(1,1,1),(2,0,0),(0,0,4)\}$,	$\{(1,2,2),(2,0,3),(0,2,0)\}$,	$\{(0,0,3),(1,0,4),(2,1,2)\}$,
$\{(0,0,\infty),(1,0,3),(2,0,1)\}$,	$\{(1,0,\infty),(0,1,2),(2,2,4)\}$,	$\{(2,0,\infty),(0,1,1),(1,2,0)\}$.

Lemma 2.3 *For any* $u \in \{4, 6, 8, 10, 14\}$*, there exists a* 3*-RHGDD of type* $(3, 2^u)$.

Proof. For each stated u, there exist $2(u-1)$ parallel classes. We take $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_{u-1} \cup \{\infty_{00}, \infty_{01}, \infty_{10}, \infty_{11}, \infty_{20}, \infty_{21}\}$ as the point set, $\{\{i\} \times \mathbb{Z}_2 \times \mathbb{Z}_{u-1} \cup \{\infty_{i0}, \infty_{i1}\}\} (i \in \mathbb{Z}_3)$ as the groups, and$\{\mathbb{Z}_3 \times \mathbb{Z}_2 \times \{j\}\} (j \in \mathbb{Z}_{u-1})$ together with $\{\infty_{00}, \infty_{01}, \infty_{10}, \infty_{11}, \infty_{20}, \infty_{21}\}$ as the holes. We list the base blocks as follows. Develop the blocks which form a parallel class by mod$(-, 2, u-1)$ to obtain all the parallel classes of the desired design.

u = 4

$\{(0,0,1),(1,1,2),(2,0,0)\}$, $\{(0,1,0),(1,0,2),(2,1,1)\}$, $\{\infty_{00},(1,1,1),(2,1,0)\}$,
$\{\infty_{01},(1,0,0),(2,0,1)\}$, $\{\infty_{10},(0,0,0),(2,1,2)\}$, $\{\infty_{11},(0,1,1),(2,0,2)\}$,
$\{\infty_{20},(0,0,2),(1,0,1)\}$, $\{\infty_{21},(0,1,2),(1,1,0)\}$.

u = 6

$\{(1,1,3),(2,0,2),(0,1,1)\}$, $\{(1,0,3),(2,0,0),(0,0,2)\}$, $\{(1,1,2),(0,0,3),(2,0,4)\}$,
$\{(0,1,3),(2,1,0),(1,1,1)\}$, $\{(2,0,3),(0,1,4),(1,0,2)\}$, $\{(1,0,1),(0,1,0),(2,1,4)\}$,
$\{\infty_{00},(2,0,1),(1,1,0))$, $\{\infty_{01},(2,1,2),(1,1,4))$, $\{\infty_{10},(0,0,0),(2,1,3)\}$,
$\{\infty_{11},(2,1,1),(0,0,4))$, $\{\infty_{20},(0,0,1),(1,0,0))$, $\{\infty_{21},(0,1,2),(1,0,4)\}$.

u = 8

$\{(1,0,5),(2,1,1),(0,1,0)\}$, $\{(1,0,4),(2,1,2),(0,0,0)\}$, $\{(1,1,2),(0,1,3),(2,0,6)\}$,
$\{(0,1,1),(2,1,6),(1,1,4)\}$, $\{(2,1,4),(0,1,2),(1,1,3)\}$, $\{(1,1,1),(0,0,4),(2,0,3)\}$,
$\{(2,0,2),(1,0,6),(0,1,5)\}$, $\{(0,1,6),(2,1,3),(1,0,2)\}$, $\{(0,0,5),(1,0,3),(2,0,1)\}$,
$\{(1,1,6),(0,1,4),(2,0,5)\}$, $\{\infty_{00},(1,0,1),(2,0,0)\}$, $\{\infty_{01},(1,0,0),(2,0,4)\}$,
$\{\infty_{10},(0,0,2),(2,1,0)\}$, $\{\infty_{11},(0,0,6),(2,1,5)\}$, $\{\infty_{20},(0,0,3),(1,1,5)\}$,
$\{\infty_{21},(1,1,0),(0,0,1)\}$.

u = 10

$\{(0,1,4),(1,1,0),(2,0,3)\}$, $\{(1,0,3),(0,0,8),(2,0,7)\}$, $\{(2,1,6),(0,1,5),(1,1,8)\}$,
$\{(2,0,6),(0,1,1),(1,0,7)\}$, $\{(1,0,0),(0,0,3),(2,1,7)\}$, $\{(0,0,4),(1,0,2),(2,1,1)\}$,
$\{(2,1,5),(0,1,3),(1,1,4)\}$, $\{(0,1,2),(2,0,5),(1,1,1)\}$, $\{(1,0,4),(2,1,0),(0,0,2)\}$,
$\{(0,1,0),(2,0,2),(1,0,5)\}$, $\{(1,1,2),(0,0,1),(2,0,8)\}$, $\{(2,1,3),(0,1,6),(1,0,1)\}$,
$\{(1,1,3),(2,0,4),(0,0,0)\}$, $\{(0,0,7),(1,1,5),(2,1,8)\}$, $\{\infty_{00},(1,0,8),(2,0,1)\}$,
$\{\infty_{01},(1,1,6),(2,1,2)\}$, $\{\infty_{10},(0,1,8),(2,1,4)\}$, $\{\infty_{11},(0,0,6),(2,0,0)\}$,
$\{\infty_{20},(1,0,6),(0,1,7)\}$, $\{\infty_{21},(0,0,5),(1,1,7)\}$.

u = 14

$\{(2,0,1),(0,0,7),(1,1,11)\}$, $\{(0,0,6),(1,0,7),(2,1,3)\}$, $\{(1,1,12),(0,1,8),(2,1,7)\}$,
$\{(2,0,0),(1,0,3),(0,1,9)\}$, $\{(0,0,10),(1,1,7),(2,1,9)\}$, $\{(0,1,2),(1,0,10),(2,1,8)\}$,
$\{(2,0,10),(1,0,6),(0,0,0)\}$, $\{(1,0,5),(2,0,11),(0,0,2)\}$, $\{(2,0,8),(0,1,3),(1,1,0)\}$,
$\{(1,1,10),(0,0,1),(2,1,2)\}$, $\{(1,0,4),(0,1,12),(2,0,2)\}$, $\{(2,1,5),(1,0,12),(0,0,3)\}$,
$\{(2,0,6),(1,1,1),(0,0,11)\}$, $\{(2,1,6),(1,1,3),(0,1,4)\}$, $\{(2,1,12),(0,1,7),(1,1,5)\}$,
$\{(1,0,0),(0,0,5),(2,0,9)\}$, $\{(1,1,2),(0,1,0),(2,1,1)\}$, $\{(0,1,1),(1,0,2),(2,1,4)\}$,
$\{(1,0,11),(0,1,5),(2,0,12)\}$, $\{(1,1,8),(0,0,9),(2,0,7)\}$, $\{(1,1,4),(2,0,5),(0,1,10)\}$,
$\{(0,0,4),(2,1,0),(1,0,9)\}$, $\{\infty_{00},(1,0,1),(2,1,11)\}$, $\{\infty_{01},(1,1,9),(2,0,3)\}$,
$\{\infty_{10},(0,0,12),(2,1,10)\}$, $\{\infty_{11},(0,1,11),(2,0,4)\}$, $\{\infty_{20},(1,0,8),(0,1,6)\}$,
$\{\infty_{21},(0,0,8),(1,1,6)\}$.

Lemma 2.4 *For any* $u \in \{4,6,8,10,14\}$*, there exists a* 3*-RHGDD of type* $(6,2^u)$*.*

Proof. For each stated u, there exist $5(u-1)$ parallel classes. We take $\mathbb{Z}_2 \times \mathbb{Z}_{u-1} \times \mathbb{Z}_5 \cup \mathbb{Z}_2 \times \mathbb{Z}_{u-1} \times \{x\} \cup \mathbb{Z}_2 \times \{y\} \times \mathbb{Z}_5 \cup \mathbb{Z}_2 \times \{y\} \times \{x\}$ as the point set, $\mathbb{Z}_2 \times (\mathbb{Z}_{u-1} \cup \{y\}) \times \{i\} (i \in \mathbb{Z}_5)$ together with $\mathbb{Z}_2 \times (\mathbb{Z}_{u-1} \cup \{y\}) \times \{x\}$ as the groups, and $\{\mathbb{Z}_2 \times \{j\} \times (\mathbb{Z}_5 \cup \{x\})\} (j \in \mathbb{Z}_{u-1})$ together with $\mathbb{Z}_2 \times \{y\} \times (\mathbb{Z}_5 \cup \{x\})$ as the holes. We list the initial base blocks of $u = 4$ as follows and for other value of u, see the appendix. Cycle these blocks by $\mathrm{mod}(2,-,-)$ to obtain one parallel class. Then develop the parallel class by $\mathrm{mod}(-,u-1,5)$ to obtain all the parallel classes of the desired design.

u = 4

$\{(1,0,3),(0,1,1),(0,2,2)\}$, $\{(0,0,x),(0,y,0),(1,1,2)\}$, $\{(0,1,x),(1,y,1),(0,0,4)\}$,
$\{(0,2,x),(0,0,2),(1,1,4)\}$, $\{(0,y,2),(0,0,1),(0,2,3)\}$, $\{(0,y,3),(0,0,0),(0,2,1)\}$,
$\{(0,y,4),(1,1,3),(1,2,0)\}$, $\{(0,y,x),(0,1,0),(1,2,4)\}$.

Lemma 2.5 *For any $u \in \{5,7,11,13\}$, there exists a 3-RHGDD of type $(6,2^u)$.*

Proof. For each stated u, there exist $5(u-1)$ parallel classes. We take $\mathbb{Z}_2 \times \mathbb{Z}_{u-1} \times \mathbb{Z}_5 \cup \mathbb{Z}_2 \times \mathbb{Z}_{u-1} \times \{x\} \cup \mathbb{Z}_2 \times \{y\} \times \mathbb{Z}_5 \cup \mathbb{Z}_2 \times \{y\} \times \{x\}$ as the point set, $\mathbb{Z}_2 \times (\mathbb{Z}_{u-1} \cup \{y\}) \times \{i\} (i \in \mathbb{Z}_5)$ together with $\mathbb{Z}_2 \times (\mathbb{Z}_{u-1} \cup \{y\}) \times \{x\}$ as the groups, and $\{\mathbb{Z}_2 \times \{j\} \times (\mathbb{Z}_5 \cup \{x\})\} (j \in \mathbb{Z}_{u-1})$ together with $\mathbb{Z}_2 \times \{y\} \times (\mathbb{Z}_5 \cup \{x\})$ as the holes. We list the base blocks which form a parallel class for $u = 5$ as follows and for other value of u, see the appendix. Develop the parallel class by mod$(-, u-1, 5)$ to obtain all the parallel classes of the desired design.

u = 5

$\{(1,3,3),(0,2,2),(0,1,4)\}$, $\{(0,0,0),(1,3,1),(0,2,4)\}$, $\{(1,0,4),(1,2,3),(0,3,1)\}$,
$\{(1,3,0),(1,2,1),(1,1,3)\}$, $\{(0,0,x),(0,y,0),(1,3,2)\}$, $\{(0,1,x),(1,y,0),(0,0,2)\}$,
$\{(1,0,x),(0,y,1),(1,3,4)\}$, $\{(1,1,x),(1,y,1),(1,2,2)\}$, $\{(0,2,x),(1,0,1),(0,3,2)\}$,
$\{(0,3,x),(1,0,3),(0,1,0)\}$, $\{(1,2,x),(0,0,3),(0,3,4)\}$, $\{(1,3,x),(1,1,0),(0,2,1)\}$,
$\{(0,y,2),(0,1,3),(0,2,0)\}$, $\{(0,y,3),(1,0,2),(1,1,4)\}$, $\{(0,y,4),(0,3,3),(0,1,1)\}$,
$\{(1,y,2),(1,0,0),(1,1,1)\}$, $\{(1,y,3),(0,0,4),(1,2,0)\}$, $\{(1,y,4),(0,1,2),(0,2,3)\}$,
$\{(0,y,x),(0,0,1),(1,2,4)\}$, $\{(1,y,x),(0,3,0),(1,1,2)\}$.

From Lemma 2.3 to Lemma 2.5, we have the following lemma.

Lemma 2.6 *For any $3 \le n \le 14$ and $n \notin \{9,12\}$, there exists a 3-RHGDD of type $(6,2^n)$.*

3 Recursive Constructions

To create more RHGDDs, we need some recursive constructions. First, the following concept is necessary for us.

An incomplete HGDD (IHGDD) is a quadruple $(X, \mathcal{G}, \mathcal{H}, \mathcal{A})$ where X is a finite set of points, $\mathcal{A}$ is a collection of subsets (called blocks) with size of k, $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$ is a partition of X and $\mathcal{H} = \{H_1, H_2, \ldots, H_{n+1}\}$ is another partition of X where $G_i \cap H_j = h$ for any $1 \le i \le m$, $1 \le j \le n$ and $G_i \cap H_{n+1} = g$ for any $1 \le i \le m$ such that each pair of points $x \in G_{i_1} \cap H_{j_1}$ and $y \in G_{i_2} \cap H_{j_2}$, where $1 \le i_1, i_2 \le m$ and $1 \le j_1, j_2 \le n+1$, is contained in exactly one block if $i_1 \ne i_2$ and $j_1 \ne j_2$, but the pair of points x and y is not contained in any block if $i_1 = i_2$ or $j_1 = j_2$.

We call such a design a k-IHGDD of type $(m, h^n g)$. The sets G_i, where $1 \le i \le m$, are called groups, the sets H_j, where $1 \le j \le n$, are called holes and the set H_{n+1} is called a distinguished hole. Notice that a k-IHGDD of type $(m, h^n h)$ is just a k-HGDD of type (m, h^{n+1}).

Furthermore, a k-IHGDD of type $(m, h^n g)$ with $g \ge h$ is said to be resolvable, denoted by k-IRHGDD if its blocks can be partitioned into parallel classes and partial parallel classes which partition $X\backslash H_{n+1}$. We remark that a k-IRHGDD of type $(m, h^n h)$ may be seen as a k-RHGDD of type (m, h^{n+1}) with the partial parallel class empty.

The next two lemmas are usually called "filling in holes" constructions

Lemma 3.1 *Let g and h be positive integers satisfying $h|g$. If a k-RHGDD of type (m, g^n) and a k-RHGDD of type $(m, h^{g/h})$ exist, then a k-RHGDD of type $(m, h^{ng/h})$ exists.*

Lemma 3.2 *Let g and h be positive integers satisfying $h|g$. If a k-IRHGDD of type $(m, h^n g)$ and a k-RHGDD of type $(m, h^{g/h})$ exist, then a k-RHGDD of type $(m, h^{n+g/h})$ exists.*

Weighting constructions are widely used in design theory. The following two are such constructions. For more information, the reader may refer to [6, 7].

Lemma 3.3 *Suppose that a k-RGDD of type g^m and a k-RHGDD of type (k, h^n) exist, then there exists a k-RHGDD of type $(m, (hg)^n)$. Furthermore, if a k-RHGDD of type (m, h^g) exists, then there exists a k-RHGDD of type (m, h^{ng}).*

Lemma 3.4 *If there exist a k-RHGDD of type (m, h^n) and an RTD(k, g), then there exists a k-RHGDD of type $(m, (hg)^n)$.*

In [11] Rees introduced two direct product type constructions for RGDDs and successfully solved the long standing open problem for the existence of 3-RGDDs with six groups. The two constructions are dramatically used in [12] and [13]. We make a slight modification for our use.

Lemma 3.5 (Rees [12], Construction 2.1) *Suppose that there is a K-GDD of type g^u in which there are s disjoint α-parallel classes of blocks of size $k \in K$, and that there is a TD(u, α). Then there is a K-GDD of type $(\alpha g)^u$ in which there are $s\alpha^2$ disjoint parallel classes of blocks of size k.*

In the above lemma, we give each point of the master design a GDD weight α, using a suitable TD$(|B|, \alpha)$ as an input design, to obtain the desired design. If we replace the master GDD with an HGDD, we can obtain a new HGDD (see Lemma 2.3 in [16]). As for how to ensure the resultant HGDD contains $s\alpha^2$ parallel classes, the proof is similar to Construction 2.1 of [12]. We state the result as follows.

Lemma 3.6 *Suppose that there is a K-HGDD of type (m, h^n) in which there are s disjoint α-parallel classes of blocks of size $k \in K$, and that there is a TD(m, α). Then there is a K-HGDD of type $(m, (\alpha h)^n)$ in which there are $s\alpha^2$ disjoint parallel classes of blocks of size k.*

As an immediately corollary, the following lemma is obvious.

Lemma 3.7 *If there is an α-resolvable k-HGDD of type (m, h^n) and a TD(m, α), then there is a k-RHGDD of type $(m, (\alpha h)^n)$.*

Lemma 3.8 (Rees [11], Construction 1) *Let $(X,\ \mathcal{G},\ \mathcal{A})$ be an Γ-resolvable k-GDD of type g^u in which for each $\alpha_i \in \Gamma$ there are r_i α_i-parallel classes of blocks. Suppose that there is a TD(u, h) admitting $(\mathcal{H}, *)$ as a group of automorphisms acting sharply transitively on the points of each group. Let H_j be a collection of subsets of $\mathcal{H}$, there being r_i such subsets of size α_i for each $\alpha_i \in \Gamma$, and suppose that the collection $\{b * \delta \mid b \in H_j,\ \delta \in \mathcal{H},\ j = 1, 2, \ldots, \sum_i r_i\}$ is resolvable on $\mathcal{H}$. Then there is a resolvable GDD of type $(hg)^u$.*

If we replace the master GDD in Lemma 3.8 with an HGDD, we can obtain a new HGDD (see Lemma 2.3 in [16]). As for how to ensure the resultant HGDD can be partitioned into parallel classes, the proof is similar to Lemma 3.8.

Lemma 3.9 *Let $(X,\ \mathcal{G},\ \mathcal{A})$ be an Γ-resolvable k-HGDD of type (u, g^n) in which for each $\alpha_i \in \Gamma$ there are r_i α_i-parallel classes of blocks. Suppose that there is a TD(u, h) admitting $(\mathcal{H}, *)$ as a group of automorphisms acting sharply transitively on the points of each group. Let H_j be a collection of subsets of $\mathcal{H}$, there being r_i such subsets of size α_i for each $\alpha_i \in \Gamma$, and suppose that the collection $\{b * \delta \mid b \in H_j,\ \delta \in \mathcal{H},\ j = 1, 2, \ldots, \sum_i r_i\}$ is resolvable on $\mathcal{H}$. Then there is an k-RHGDD of type $(u, (hg)^n)$.*

The following lemma is crucial to some cases in this paper and it is a generalization of Lemma 3.4 in [15].

Lemma 3.10 *Suppose that there exists an RTD$(m+1, t)$. If there exist a k-IRHGDD of type $(m, h^u e_1)$ and a k-IRGDD of type $(hu + e_i, e_i)^m$ for any i, $2 \le i \le t$, then there exists a k-IRHGDD of type $(m, h^{ut} e)$ with $e = \sum_{i=1}^{t} e_i$. Furthermore, if $h|e$ and a k-RHGDD of type $(m, h^{e/h})$ exists, then so does a k-RHGDD of type $(m, h^{ut+e/h})$.*

Proof. Let $I_n = \{1, 2, \ldots, n\}$. Let $(X, \mathcal{G}, \mathcal{A})$ be an RTD$(m+1, t)$ where m is a positive integer. $\mathcal{G} = \{G_1, G_2, \ldots, G_{m+1}\}$, $G_{m+1} = \{x_1, x_2, \ldots, x_t\}$. We shall construct a k-IRHGDD of type $(m, h^{ut} e)$ on the point set $X^* = ((X \backslash G_{m+1}) \times I_h \times I_u) \bigcup (I_m \times (\bigcup_{i=1}^{t} H_i))$, where $|H_i| = e_i$ $(1 \le i \le t)$ and $H_i \cap H_j = \emptyset$ for any $i \ne j$. Write $\mathcal{A} = \bigcup_{i=1}^{t} \mathcal{A}_i$ where $\mathcal{A}_i$ is a parallel class of the RTD$(m+1, t)$ for each i.

Step 1. For each block $B \in \mathcal{A}_1$, place a k-IRHGDD of type $(m, h^u e_1)$ on point set $((B \backslash G_{m+1}) \times I_h \times I_u) \bigcup (I_m \times H_1)$ so that the groups are $\{\{x\} \times I_h \times I_u \cup \{i\} \times H_1 : x \in G_i \cap B\}$ $(i \in I_m)$, the holes are $(B \backslash G_{m+1}) \times I_h \times \{i\}$ $(i \in I_u)$ and the distinguished hole is $I_m \times H_1$. Let the block set of this design be denoted by $\mathcal{B}_B$. Then it is easy to see that $\mathcal{B}_B$ can be partitioned into $\frac{hu(m-1)}{k-1}$ parallel classes and $\frac{(e_1-h)(m-1)}{k-1}$ partial parallel classes. Notice that when $e_1 = h$, a k-IRHGDD is just a k-RHGDD which has no partial parallel class.

Step 2. For each fixed i_0 ($2 \le i_0 \le t$) and any block $B \in \mathcal{A}_{i_0}$, place a k-IRGDD of type $(hu + e_{i_0}, e_{i_0})^m$ on the point set $((B \backslash G_{m+1}) \times I_h \times I_u) \bigcup (I_m \times H_{i_0})$ so that the groups are $\{\{x\} \times I_h \times I_u \cup \{j\} \times H_{i_0} : x \in G_j \cap B\}$ ($j \in I_m$) and the hole is $I_m \times H_{i_0}$. Let the block set of this design be denoted by $\mathcal{B}_B$. Then $\mathcal{B}_B$ can be partitioned into $\frac{hu(m-1)}{k-1}$ parallel classes and $\frac{e_{i_0}(m-1)}{k-1}$ partial parallel classes. Notice that when $e_{i_0} = 0$, a k-IRGDD is just a k-RGDD which has no partial parallel class.

It is routine work to check that the block set of the designs thus obtained can be partitioned into $\frac{hut(m-1)}{k-1}$ parallel classes and $\frac{(e-h)(m-1)}{k-1}$ partial parallel classes. Now let $\mathcal{G}^* = \{G_i \times I_h \times I_u \cup \{i\} \times \bigcup_{j=1}^{t} H_j \mid i \in I_m\}$, $\mathcal{H}^* = \{B \times I_h \times \{j\} \mid B \in \mathcal{A}_1,\ j \in I_u\} \cup I_m \times \bigcup_{i=1}^{t} H_i$ and $\mathcal{B}^* = \{\mathcal{B}_B \mid B \in \mathcal{A}\}$. Then $(X^*, \mathcal{G}^*, \mathcal{H}^*, \mathcal{B}^*)$ is a k-IRHGDD of type $(m, h^{ut}e)$ with $e = \sum_{i=1}^{t} e_i$. The second part of the lemma is obvious from Lemma 3.1. So the proof is complete. □

4 Main Results

In this section, we consider the existence problem of 3-RHGDDs. Among all the cases, 3-RHGDDs with three or six groups are relatively difficult to deal with because there is no 3-RGDD of type 2^3 or 2^6. First we consider 3-RHGDDs with three groups.

Lemma 4.1 *If $n \ge 3$ and $n \equiv 1 \pmod 2$, then there exists a 3-RHGDD of type $(3, 2^n)$.*

Proof. By Theorem 1.7, there exists a 3-RHGDD of type $(3, 1^n)$ which has $n-1$ parallel classes. Notice that $n-1$ is even. So the $n-1$ parallel classes can be paired to obtain a 2-resolvable 3-HGDD of type $(3, 1^n)$. Apply Lemma 3.7 with a TD(3,2) (see [6]) to obtain the desired design. □

Lemma 4.2 *For any $n \ge 6$ and $n \equiv 0 \pmod 6$, there exists a 3-RHGDD of type $(3, 2^n)$.*

Proof. For $n = 6$, there exists a 3-RHGDD of type $(3, 2^n)$ by Lemma 2.3. If $n \ge 12$, there exists a 3-RGDD of type 2^n by Theorem 1.1. Since a 3-RHGDD of type $(3, 1^3)$ exists from Theorem 1.7, we apply Lemma 3.3 and Lemma 1.5 to obtain the result. □

Lemma 4.3 *If $n \equiv 4, 8 \pmod{12}$, then there exists a 3-RHGDD of type $(3, 2^n)$.*

Proof. For $n = 4$ or $n = 8$, a 3-RHGDD of type $(3, 2^n)$ follows from Lemma 2.3. If $n \ge 16$, there exists a 3-RHGDD of type $(3, 1^{n/4})$ by Theorem 1.7. Apply Lemma 3.4 with an RTD(3,8) which exists by Theorem 1.2 to obtain a 3-RHGDD of type $(3, 8^{n/4})$. Using a 3-RHGDD of type $(3, 2^4)$ which has been constructed in Lemma 2.3, we obtain a 3-RHGDD of type $(3, 2^n)$ by Lemma 3.1. □

Lemma 4.4 *If $n \geq 10$ and $n \equiv 2, 10 \pmod{12}$, then there exists a 3-RHGDD of type $(3, 2^n)$.*

Proof. If $n = 10$ or $n = 14$, a 3-RHGDD of type $(3, 2^n)$ follows from Lemma 2.3.

If $n \geq 22$ and $n \equiv 10 \pmod{12}$, let $n = 12p+10$, $p \geq 1$. Then an RTD$(4, 6p+3)$ exists for all $p \geq 1$ by Theorem 1.2. A 3-IRTD$(6, 2)$ exists by Lemma 1.4 and a 3-RGDD of type 4^3, that is an RTD$(3, 4)$, exists by Theorem 1.2. In addition, there exists a 3-RHGDD of type $(3, 2^3)$ from Lemma 4.1. Apply Lemma 3.10 with $k = 3$, $m = 3$, $t = 6p+3$, $h = 2$, $u = 2$, $e_1 = 2$, $e_i = 2$ for $2 \leq i \leq 4$ and $e_i = 0$ for $5 \leq i \leq 6p+3$ to obtain a 3-IRHGDD of type $(3, 2^{12p+6}8)$. Furthermore, a 3-RHGDD of type $(3, 2^4)$ exists from Lemma 2.3. By Lemma 3.2, a 3-RHGDD of type $(3, 2^{12p+6+4})$ exists. The result is obtained.

If $n \geq 26$ and $n \equiv 2 \pmod{12}$, let $n = 12p+2$, $p \geq 2$. Here we write $n = 2(6p-1)+4$ and use an RTD$(3, 6p-1)$ which exists for all $p \geq 2$ by Theorem 1.2. The following work is completely similar to the above case. □

From Lemma 4.1 to Lemma 4.4, we have proved the following result.

Lemma 4.5 *For any $n \geq 3$, there exists a 3-RHGDD of type $(3, 2^n)$.*

Now we move to the case of six groups.

Lemma 4.6 *For any $n \geq 3$, there exists a 3-RHGDD of type $(6, 2^n)$.*

Proof. By Theorem 1.1, there exists a 3-RGDD of type 6^6. Apply Lemma 3.3 with a 3-RHGDD of type $(3, 1^3)$ to obtain a 3-RHGDD of type $(6, 6^3)$. A 3-RHGDD of type $(6, 2^9)$ follows from Lemma 3.1 by using a 3-RHGDD of type $(6, 2^3)$ which exists by Lemma 2.3 and Lemma 1.5. In a similar way, 3-RHGDDs of type $(6, 2^{12})$ and $(6, 2^{16})$ can be obtained. Combining Lemma 2.6 with these results, we know that a 3-RHGDD of type $(6, 2^n)$ exists for any n, $3 \leq n \leq 16$ and $n \neq 15$.

Notice that a 3-IRHGDD of type $(6, 2^2 2)$ is just a 3-RHGDD of type $(6, 2^3)$. A 3-IRGDD of type $(6, 2)^6$ can be found in [15] and a 3-RGDD of type 4^6 exists from Theorem 1.1. So if an RTD$(7, t)$ exists, applying Lemma 3.10 with $k = 3$, $m = 6$, $u = 2$, $h = 2$, $e_1 = 2$ and $e_i = 0$ or 2 for any $2 \leq i \leq t$, we obtain a 3-IRHGDD of type $(6, 2^{2t}e)$ where $e = \sum_{i=1}^{t} e_i$. Furthermore, if there exists a 3-RHGDD of type $(6, 2^{e/2})$, then there exists a 3-RHGDD of type $(6, 2^{2t+e/2})$. Let $a = |\{i | e_i = 2\}|$, then $e = 2a$. Write $n = 2t + a$. For $n = 15$ or n is from 17 to 28, we list n, t and a such that an RTD$(7, t)$ and a 3-RHGDD of type $(6, 2^a)$ both exist. Notice that a 3-RHGDD of type $(6, 2^1)$ always exists with the block set empty.

n	**15**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
t	7	7	7	7	8	8	8	8	9	9	9	11	11
a	1	3	4	5	4	5	6	7	6	7	8	5	6

We have proved above that a 3-RHGDD of type $(6, 2^n)$ exists for any $3 \le n \le 28$. We will use these small designs as ingredients in the following to obtain "bigger" designs. For any $n \ge 29$, we use the following recursion. Let t_1 be a positive integer such that an RTD$(7, t_1)$ exists. Similarly, we can apply Lemma 3.10 with k=3, $m = 6$, $u = 2$, $h = 2$, $e_1 = 2$ and $e_i = 0$ or 2 $(2 \le i \le t_1)$ to get a 3-RHGDD of type $(6, 2^p)$ for any $p \in [2t_1 + 3, 3t_1]$. If t_2 is the next integer such that an RTD$(7, t_2)$ exists, similarly we have a 3-RHGDD of type $(6, 2^p)$ for any $p \in [2t_2 + 3, 3t_2]$. We want $3t_1 \ge 2t_2 + 3$ so that the two intervals can lead to a longer one $[2t_1 + 3, 3t_2]$. By Theorem 1.2, $t_2 - t_1 \le 5$. If $t_1 \ge 13$, then $3t_1 - (2t_2 + 3) = t_1 - 2(t_2 - t_1) - 3 \ge 13 - 2 \times 5 - 3 = 0$. So we can use the recursion from $t_1 = 13$ to complete the proof. □

Lemma 4.7 *For any even $h \ge 2$, there exists a 3-RHGDD of type $(6, h^6)$.*

Proof. A 3-RHGDD of type $(6, 2^6)$ and a 3-RHGDD of type $(6, 4^6)$ come from Lemma 2.6 and Lemma 2.1, respectively. For any even $h \ge 6$ and $h \ne 12$, apply Lemma 3.4 with a 3-RHGDD of type $(6, 2^6)$ and an RTD$(3, h/2)$ which exists by Theorem 1.2 to obtain a 3-RHGDD of type $(6, h^6)$. For $h = 12$, there exists a 3-RHGDD of type $(6, h^6)$ by applying Lemma 3.4 with a 3-RHGDD of type $(6, 4^6)$ and an RTD$(3, 3)$ which exists by Theorem 1.2. □

Lemma 4.8 *If $h \equiv 1, 5 \pmod 6$ and $h \ge 5$, there exists a 3-RHGDD of type $(3, h^6)$.*

Proof. First we construct a $\{1, 2, 2\}$-resolvable 3-HGDD of type $(3, 1^6)$.
Points: $X = Z_{18}$
Groups: $\{\{0 + i, 3 + i, 6 + i, 9 + i, 12 + i, 15 + i\} : i = 0, 1, 2\}$
Holes: $\{\{0 + 3i, 1 + 3i, 2 + 3i\} : i = 0, 1, 2, 3, 4, 5\}$
Below are the required blocks.

$$\begin{array}{llll} \{1,6,11\}, & \{2,3,10\} & & \\ \{1,5,15\}, & \{0,10,17\}, & \{2,4,9\}, & \{2,6,13\} \\ \{4,6,17\}, & \{1,9,14\}, & \{0,4,8\}, & \{1,3,17\} \end{array}$$

Here all the base blocks are developed by $+6 \pmod{18}$. The two blocks in the first row generate a parallel class. The four blocks in each of the second and third rows generate a 2-resolvable parallel class, respectively.

Apply Lemma 3.9 with the $\{1, 2, 2\}$-resolvable 3-HGDD of type $(3, 1^6)$ and a TD$(3, h)$. Since $h \ge 5$ and $h \equiv 1, 5 \pmod 6$, an $(h, 3; 1)$-difference matrix exists (see [6] for details). Then it is easy to see that there exists a TD$(3, h)$ satisfying the conditions of Lemma 3.9, where $\mathcal{H} = Z_h$. Let $H_1 = \{0\}$, $H_2 = H_3 = \{0, 1\}$. It remains to be shown that the collection $\{H_j + \delta : \delta \in Z_h,\ j = 1, 2, 3\}$ is resolvable. We consider the following partition.

$$\{\{H_2 + 2i + k\} : i = 0, 1, 2, \cdots, \frac{h-3}{2}\} \cup \{h - 1 + k\}\ k = 0, 1;$$

and

$$\{\{H_3 + 2i + k + 2\} : i = 0, 1, 2, \cdots, \frac{h-3}{2}\} \cup \{1 + k\}\ k = 0, 1;$$

and

$$\{\{H_1 + i\} : i = 3, 4, \cdots, h - 2\} \cup \{h - 1, 0\} \cup \{1, 2\}.$$

So the proof is complete. □

Now we are in a position to establish the main result of this section.

Lemma 4.9 *If $h \geq 2$, $m \geq 3$, $n \geq 3$, $hmn \equiv 0 \pmod 3$ and $h(m-1)(n-1) \equiv 0 \pmod 2$, there exists a 3-RHGDD of type (m, h^n).*

Proof. Our construction splits into four cases depending on the values of $h \pmod 6$.

Case 1. $h \equiv 3 \pmod 6$.

In this case, we have $(m-1)(n-1) \equiv 0 \pmod 2$. Without loss of generality, we assume that m is odd and n is any positive integer not less than 3. By Theorem 1.1, there exists a 3-RGDD of type h^m. If $n \neq 6$, apply Lemma 3.3 with a 3-RHGDD of type $(3, 1^n)$ which follows from Theorem 1.7 to obtain the result. If $n = 6$ and $m \neq 3$, a 3-RHGDD of type $(m, 1^6)$ exists by Theorem 1.7. The result follows from Lemma 3.4 by using an RTD$(3, h)$ which exists for any stated value of h by Theorem 1.2. If $n = 6$ and $m = 3$, a 3-RHGDD of type $(3, 3^6)$ follows from Lemma 2.2. For $h \geq 9$, let $h = 3(2a+1)$, $a \geq 1$. Give each point of a 3-RHGDD of type $(3, 3^6)$ weight $2a+1$ by using Lemma 3.4 with an RTD$(3, 2a+1)$ which exists for any positive integer $a \geq 1$ to obtain the desired design.

Case 2. $h \equiv 0 \pmod 6$.

The proof of this case, where m and n are any positive integers not less than 3, is similar to case 1. If $n \neq 6$ and $(h, m) \neq (6, 3)$, we use Lemma 3.3 with a 3-RGDD of type h^m and a 3-RHGDD of type $(3, 1^n)$ which exist by Theorem 1.1 and Theorem 1.7, respectively, to get the result. If $n \neq 6$ and $(h, m) = (6, 3)$, apply Lemma 3.4 with a 3-RHGDD of type $(3, 2^n)$ in Lemma 4.5 and an RTD(3,3) to get the result. If $n = 6$ and $m \neq 6$, apply Lemma 3.3 with a 3-RGDD of type h^6 and a 3-RHGDD of type $(3, 1^m)$ to obtain a 3-RHGDD of type $(6, h^m)$, that is a 3-RHGDD of type (m, h^6) by Lemma 1.5. If $n = 6$ and $m = 6$, the result follows from Lemma 4.7.

Case 3. $h \equiv 1, 5 \pmod 6$.

In this case, $mn \equiv 0 \pmod 3$ and $(m-1)(n-1) \equiv 0 \pmod 2$. If $(m, n) \neq (3, 6)$ or $(6, 3)$, there exists a 3-RHGDD of type $(m, 1^n)$ from Theorem 1.7. An RTD$(3, h)$ exists for any stated value of h by Theorem 1.2. We use Lemma 3.4 to get the result. If $(m, n) = (3, 6)$ or $(6, 3)$, the conclusion comes from Lemma 4.8 and Lemma 1.5.

Case 4. $h \equiv 2, 4 \pmod 6$.

Here $mn \equiv 0 \pmod 3$. Without loss of generality, we assume that $m \equiv 0 \pmod 3$ and n is any positive integer not less than 3.

If $h \geq 4$ and $n \neq 6$, apply Lemma 3.3 with a 3-RGDD of type h^m and a 3-RHGDD of type $(3, 1^n)$ to obtain the result. If $h \geq 4$ and $n = 6$, use Lemma 3.3 again with a 3-RGDD of type h^6 and a 3-RHGDD of type $(3, 1^m)$ where $m \neq 6$ to obtain a 3-RHGDD of type $(6, h^m)$, that is a 3-RHGDD of type (m, h^6) by Lemma 1.5. As for $m = 6$ and $n = 6$, the conclusion holds by Lemma 4.7.

If $h = 2$ and $m \geq 9$, the proof is similar to above. When $n \neq 6$, apply Lemma 3.3 with a 3-RGDD of type 2^m and a 3-RHGDD of type $(3, 1^n)$ to obtain the desired design. When $n = 6$, a 3-RHGDD of type $(m, 2^6)$ is just a 3-RHGDD of type $(6, 2^m)$ which exists by Lemma 4.6. If $h = 2$ and $m = 3$ or $m = 6$, the conclusion follows from Lemma 4.5 or Lemma 4.6.

So the proof is complete. □

Combining Lemma 4.9 and Lemma 1.6 with Theorem 1.7, we have proved the following.

Theorem 1.8 *There exists a 3-RHGDD of type (m, h^n) if and only if $m \geq 3$, $n \geq 3$, $hmn \equiv 0 \pmod 3$ and $h(m-1)(n-1) \equiv 0 \pmod 2$ except when $(h, m, n) = (1, 3, 6)$ or $(1, 6, 3)$.*

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Appendix

Following Lemma 2.4, we construct 3-RHGDDs of type $(6, 2^u)$ where $u \in \{6, 8, 10, 14\}$. For each stated u, we list the initial base blocks as follows. Develop those blocks by mod$(2, -, -)$ to obtain one parallel class. Then develop the parallel class by mod$(-, u-1, 5)$ to obtain the desired design. For more information, see Lemma 2.4.

$\mathbf{u = 6}$

$\{(0,4,2),(0,1,4),(0,0,1)\}$, $\{(0,4,4),(0,1,0),(0,2,1)\}$, $\{(1,0,0),(1,3,1),(0,1,2)\}$;
$\{(0,0,x),(0,y,0),(0,4,1)\}$, $\{(0,1,x),(1,y,1),(0,3,4)\}$, $\{(0,2,x),(1,4,0),(0,3,2)\}$;
$\{(0,3,x),(1,1,1),(1,2,3)\}$, $\{(0,4,x),(1,0,3),(0,2,4)\}$, $\{(0,y,2),(0,0,4),(1,1,3)\}$;
$\{(0,y,3),(0,0,2),(1,3,0)\}$, $\{(0,y,4),(0,2,2),(1,3,3)\}$, $\{(0,x,y),(0,2,0),(1,4,3)\}$;

u = 8

$\{(0,5,1),(0,1,3),(0,4,2)\}$, $\{(0,2,4),(0,0,0),(1,1,1)\}$, $\{(1,5,0),(1,2,2),(0,4,1)\}$;
$\{(0,0,3),(0,2,1),(0,6,0)\}$, $\{(1,2,0),(1,3,2),(0,6,4)\}$, $\{(0,0,x),(0,y,0),(0,5,3)\}$;
$\{(0,1,x),(1,y,1),(0,2,3)\}$, $\{(0,2,x),(1,1,0),(1,3,1)\}$, $\{(0,3,x),(1,6,3),(1,0,4)\}$;
$\{(0,4,x),(1,6,2),(0,3,3)\}$, $\{(0,5,x),(1,3,0),(0,1,4)\}$, $\{(0,6,x),(0,1,2),(0,3,4)\}$;
$\{(0,y,2),(1,0,1),(0,5,4)\}$, $\{(0,y,3),(0,0,2),(1,4,4)\}$, $\{(0,y,4),(0,4,0),(1,5,2)\}$;
$\{(0,y,x),(0,4,3),(1,6,1)\}$;

u = 10

$\{(0,0,0),(0,6,2),(1,2,4)\}$, $\{(0,6,0),(0,0,2),(1,8,1)\}$, $\{(0,6,4),(1,3,3),(1,1,1)\}$;
$\{(0,3,2),(0,0,1),(1,2,3)\}$, $\{(0,8,2),(0,7,1),(0,6,3)\}$, $\{(1,8,4),(1,0,3),(1,5,0)\}$;
$\{(1,3,4),(1,7,0),(0,4,2)\}$, $\{(0,0,x),(0,y,0),(0,1,2)\}$, $\{(0,1,x),(1,y,1),(0,3,0)\}$;
$\{(0,2,x),(1,5,3),(0,0,4)\}$, $\{(0,3,x),(1,2,1),(0,7,2)\}$, $\{(0,4,x),(0,7,3),(0,3,1)\}$;
$\{(0,5,x),(0,1,3),(0,2,0)\}$, $\{(0,6,x),(1,8,3),(1,1,4)\}$, $\{(0,7,x),(1,8,0),(1,4,1)\}$;
$\{(0,8,x),(1,4,3),(1,6,1)\}$, $\{(0,y,2),(0,1,0),(1,4,4)\}$, $\{(0,y,3),(0,2,2),(1,5,4)\}$;
$\{(0,y,4),(0,4,0),(1,5,2)\}$, $\{(0,y,x),(0,5,1),(1,7,4)\}$.

u = 14

$\{(1,1,2),(0,3,0),(1,7,3)\}$, $\{(0,5,4),(0,3,3),(0,0,1)\}$, $\{(0,2,2),(0,9,0),(0,7,1)\}$;
$\{(0,12,0),(1,3,2),(1,9,1)\}$, $\{(1,6,2),(0,11,3),(0,3,4)\}$, $\{(0,4,1),(0,8,2),(1,12,3)\}$;
$\{(0,3,1),(0,6,4),(0,2,0)\}$, $\{(0,11,4),(1,7,0),(0,12,2)\}$, $\{(1,8,1),(0,7,2),(1,5,0)\}$;
$\{(0,12,1),(0,6,3),(1,9,4)\}$, $\{(0,7,4),(0,4,0),(0,5,2)\}$, $\{(0,0,x),(0,y,0),(1,5,3)\}$;
$\{(0,1,x),(1,y,1),(1,0,2)\}$, $\{(0,2,x),(1,10,3),(0,8,4)\}$, $\{(0,3,x),(0,11,1),(1,4,2)\}$;
$\{(0,4,x),(1,1,3),(1,10,0)\}$, $\{(0,5,x),(1,1,1),(0,9,2)\}$, $\{(0,6,x),(0,5,1),(1,4,4)\}$;
$\{(0,7,x),(0,1,0),(0,9,3)\}$, $\{(0,8,x),(0,0,3),(1,2,4)\}$, $\{(0,9,x),(0,10,2),(0,6,0)\}$;
$\{(0,10,x),(1,12,4),(0,0,0)\}$, $\{(0,11,x),(1,2,3),(1,1,4)\}$, $\{(0,12,x),(0,8,3),(0,10,1)\}$;
$\{(0,y,2),(0,0,4),(1,6,1)\}$, $\{(0,y,3),(0,2,1),(1,8,0)\}$, $\{(0,y,4),(0,4,3),(1,11,0)\}$;
$\{(0,y,x),(0,10,4),(1,11,2)\}$.

Following Lemma 2.5, we construct 3-RHGDDs of type $(6,2^u)$ where $u \in \{7,11,13\}$. For each stated u, we list the base blocks as follows which form a parallel class. Develop the parallel class by $\mathrm{mod}(-,u-1,5)$ to obtain the desired design. For more information, see Lemma 2.5.

u = 7

$\{(0,0,0),(1,4,1),(1,3,4)\}$, $\{(1,2,0),(0,1,1),(0,3,2)\}$, $\{(0,1,4),(0,5,2),(1,0,3)\}$;
$\{(0,4,4),(0,0,3),(0,1,0)\}$, $\{(1,3,2),(1,1,4),(0,0,1)\}$, $\{(1,4,2),(0,5,1),(1,1,0)\}$;
$\{(1,4,0),(1,2,4),(1,1,1)\}$, $\{(1,2,1),(1,4,3),(0,0,4)\}$, $\{(0,0,x),(0,y,0),(1,3,1)\}$;
$\{(0,1,x),(1,y,0),(1,2,3)\}$, $\{(1,0,x),(0,y,1),(0,1,2)\}$, $\{(1,1,x),(1,y,1),(1,5,3)\}$;
$\{(0,2,x),(0,1,3),(0,3,1)\}$, $\{(0,3,x),(0,0,2),(0,5,4)\}$, $\{(0,4,x),(1,0,1),(0,2,3)\}$;
$\{(0,5,x),(1,3,3),(1,4,4)\}$, $\{(1,2,x),(1,1,3),(0,4,1)\}$, $\{(1,3,x),(1,5,1),(1,0,0)\}$;
$\{(1,4,x),(0,3,4),(1,5,2)\}$, $\{(1,5,x),(0,2,4),(0,3,0)\}$, $\{(0,y,2),(1,5,4),(1,3,0)\}$;
$\{(0,y,3),(0,4,0),(1,2,2)\}$, $\{(0,y,4),(0,3,3),(0,4,2)\}$, $\{(1,y,2),(0,2,1),(0,5,3)\}$;
$\{(1,y,3),(0,2,0),(1,1,2)\}$, $\{(1,y,4),(0,2,2),(1,5,0)\}$, $\{(0,y,x),(0,4,3),(1,0,4)\}$;
$\{(1,y,x),(0,5,0),(1,0,2)\}$.

u = 11

$\{(0,6,1),(1,4,3),(1,1,2)\}$,	$\{(1,8,4),(0,0,3),(0,4,1)\}$,	$\{(1,4,2),(0,5,1),(1,8,0)\}$;
$\{(0,5,4),(0,6,2),(1,3,3)\}$,	$\{(0,1,0),(1,7,1),(1,6,3)\}$,	$\{(1,2,3),(0,4,0),(1,6,4)\}$;
$\{(0,1,4),(0,6,0),(0,8,3)\}$,	$\{(1,2,2),(0,5,0),(0,6,4)\}$,	$\{(1,6,1),(1,9,0),(1,7,3)\}$;
$\{(0,8,1),(0,1,3),(0,7,4)\}$,	$\{(1,7,2),(0,6,3),(1,2,0)\}$,	$\{(0,0,1),(0,7,3),(0,8,4)\}$;
$\{(0,0,0),(0,5,2),(0,7,1)\}$,	$\{(0,1,1),(1,4,4),(0,7,0)\}$,	$\{(1,9,1),(1,0,0),(1,2,4)\}$;
$\{(0,2,1),(1,9,4),(1,6,2)\}$,	$\{(0,0,x),(0,y,0),(0,7,2)\}$,	$\{(0,1,x),(1,y,0),(0,3,1)\}$;
$\{(1,0,x),(0,y,1),(0,9,4)\}$,	$\{(1,1,x),(1,y,1),(0,5,3)\}$,	$\{(0,2,x),(0,1,2),(0,3,3)\}$;
$\{(0,3,x),(1,2,1),(0,9,0)\}$,	$\{(0,4,x),(0,8,0),(1,9,3)\}$,	$\{(0,5,x),(0,8,2),(1,3,4)\}$
$\{(0,6,x),(0,4,2),(1,0,1)\}$,	$\{(0,7,x),(0,2,4),(1,3,1)\}$,	$\{(0,8,x),(1,1,4),(1,0,3)\}$
$\{(0,9,x),(1,0,4),(1,6,0)\}$,	$\{(1,2,x),(1,8,3),(1,4,1)\}$,	$\{(1,3,x),(0,4,4),(0,0,2)\}$;
$\{(1,4,x),(1,1,0),(0,2,3)\}$,	$\{(1,5,x),(1,9,2),(0,0,4)\}$,	$\{(1,6,x),(0,2,2),(1,5,4)\}$;
$\{(1,7,x),(0,9,2),(1,8,1)\}$,	$\{(1,8,x),(1,1,1),(1,3,2)\}$,	$\{(1,9,x),(0,2,0),(1,7,4)\}$;
$\{(0,y,2),(1,1,3),(0,9,1)\}$,	$\{(0,y,3),(0,3,4),(1,4,0)\}$,	$\{(0,y,4),(1,0,2),(1,5,3)\}$;
$\{(1,y,2),(0,3,0),(1,5,1)\}$,	$\{(1,y,3),(0,3,2),(1,5,0)\}$,	$\{(1,y,4),(1,5,2),(1,7,0)\}$;
$\{(0,y,x),(0,4,3),(1,8,2)\}$,	$\{(1,y,x),(0,9,3),(1,3,0)\}$.	

u = 13

$\{(0,5,3),(0,1,4),(1,0,0)\}$,	$\{(1,1,4),(0,3,3),(1,6,0)\}$,	$\{(1,3,4),(0,2,3),(0,10,2)\}$;
$\{(1,4,3),(1,7,4),(1,10,2)\}$,	$\{(1,10,1),(0,7,0),(1,9,3)\}$,	$\{(0,7,4),(0,8,3),(1,2,0)\}$;
$\{(0,10,1),(1,9,4),(1,7,2)\}$,	$\{(1,4,1),(1,11,3),(1,1,2)\}$,	$\{(0,7,2),(0,2,0),(0,10,3)\}$;
$\{(1,10,3),(1,11,0),(1,6,1)\}$,	$\{(1,2,4),(0,11,1),(1,10,0)\}$,	$\{(1,5,2),(1,6,3),(1,9,0)\}$;
$\{(0,7,3),(1,9,2),(1,5,1)\}$,	$\{(1,11,4),(0,5,1),(0,3,0)\}$,	$\{(1,0,4),(0,4,3),(1,2,2)\}$;
$\{(0,10,0),(0,1,2),(0,9,4)\}$,	$\{(0,0,3),(0,6,4),(0,1,1)\}$,	$\{(0,11,3),(0,6,2),(1,4,4)\}$;
$\{(0,2,2),(1,8,3),(0,1,0)\}$,	$\{(1,9,1),(0,8,4),(0,1,3)\}$,	$\{(0,0,x),(0,y,0),(1,7,1)\}$;
$\{(0,1,x),(1,y,0),(0,0,4)\}$,	$\{(1,0,x),(0,y,1),(1,5,3)\}$,	$\{(1,1,x),(1,y,1),(1,3,0)\}$;
$\{(0,2,x),(0,3,1),(0,5,0)\}$,	$\{(0,3,x),(0,8,0),(0,5,2)\}$,	$\{(0,4,x),(0,8,2),(1,2,1)\}$;
$\{(0,5,x),(1,8,4),(1,7,0)\}$,	$\{(0,6,x),(1,7,3),(0,3,4)\}$,	$\{(0,7,x),(0,2,4),(1,3,2)\}$;
$\{(0,8,x),(1,2,3),(0,6,1)\}$,	$\{(0,9,x),(1,6,2),(1,1,0)\}$,	$\{(0,10,x),(0,4,2),(0,6,0)\}$;
$\{(0,11,x),(1,10,4),(1,4,2)\}$,	$\{(1,2,x),(1,11,2),(1,1,3)\}$,	$\{(1,3,x),(0,11,4),(0,9,2)\}$;
$\{(1,4,x),(0,9,0),(1,11,1)\}$,	$\{(1,5,x),(0,3,2),(1,8,0)\}$,	$\{(1,6,x),(1,0,3),(0,5,4)\}$;
$\{(1,7,x),(0,2,1),(0,11,2)\}$,	$\{(1,8,x),(1,0,1),(0,10,4)\}$,	$\{(1,9,x),(1,5,0),(0,0,1)\}$;
$\{(1,10,x),(0,11,0),(1,8,2)\}$,	$\{(1,11,x),(1,0,2),(0,8,1)\}$,	$\{(0,y,2),(0,0,0),(0,6,3)\}$;
$\{(0,y,3),(0,0,2),(1,3,1)\}$,	$\{(0,y,4),(0,4,1),(1,3,3)\}$,	$\{(1,y,2),(0,4,0),(1,5,4)\}$;
$\{(1,y,3),(0,4,4),(1,8,1)\}$,	$\{(1,y,4),(0,7,1),(1,4,0)\}$,	$\{(0,y,x),(0,9,3),(1,1,1)\}$;
$\{(1,y,x),(0,9,1),(1,6,4)\}$.		

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