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Path decomposition of defect 1-extendable bipartite graphs

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Abstract

A near perfect matching in a graph G is a matching saturating all but one vertex in G. If G is a connected graph and any n independent edges in G are contained in a near perfect matching of G where $n \leq (|V(G)| - 2)/2$, G is defect n-extendable. Let G be a bipartite graph with $|V(G)| \geq 5$ and let (U, W) be the bipartition of G such that $|W| = |U| + 1$. It is proved that G is defect 1-extendable if and only if G has a path decomposition, that is, $G = w + P_1 + P_2 + \ldots + P_{r-1} + P_r$ where $w \in W$ and P_i satisfies (1) or (2) as follows:

(1) P_i is an even path which begins with a vertex in $W \cap V(w + P_1 + P_2 + \ldots + P_{i-1})$ and has no other common vertex with $w + P_1 + P_2 + \ldots + P_{i-1}$;

(2) P_i is an odd path which has no common vertex with $w + P_1 + P_2 + \ldots + P_{i-1}$ except the two end vertices.

It is also shown that a defect 1-extendable bipartite graph G is minimal if and only if G contains no cycle.

1 Introduction and terminology

All graphs considered in this paper are undirected, finite and simple.

A **perfect matching** is a matching covering all vertices in a graph. A **near perfect matching** is a matching covering all but one vertex in a graph. Let G be a connected graph and $n \leq (|V(G)| - 2)/2$ be a positive integer. If any n independent edges in G are contained in a perfect matching, then G is ***n*-extendable**. Particularly, if G contains a perfect matching, then G is **0-extendable**. If any n independent edges in G are contained in a near perfect matching, then G is **defect *n*-extendable**. If for any edge e in a defect n-extendable graph G, $G - e$ is not defect n-extendable, then G is **minimal defect *n*-extendable**. A path that contains even edges is an **even path**, otherwise, it is an **odd path**.

We use $G = (X, Y)$ to denote a bipartite graph G with bipartition (X, Y). Let G_1 and G_2 be two graphs; then $G_1 + G_2$ denotes a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Let G be a graph and $S \subseteq V(G)$; then $\Gamma_G(S)$ denotes all the vertices in G that join to at least one vertex in S. Inserting vertices $x_0, x_1, \ldots, x_s$ to an edge xy means replacing the edge xy with the path $xx_0x_1 \ldots x_sy$.

For the other terminology and notation not defined in this paper, the reader is referred to [1].

Plummer [7] introduced the concept of an n-extendable graph in 1980. Since then, extensive research has been done on this topic. But n-extendable graphs are all of even order. To naturally extend the property of n-extendiblity to graphs of odd order, Lou and Wen [5] introduced the concept of defect n-extendable graphs. They showed that the connectivity of defect n-extendable bipartite graphs can be any integer. While Plummer [7] proved that the connectivity of a n-extendable graphs is not less than $n + 1$, which implies that the results on defect n-extendable graphs may not be trivially deduced from those of n-extendable graphs.

In fact, a few results on defect n-extendable graphs have been established until now. In [2], Grant, Holton and Little characterized defect 1-extendable graphs which were called 1-covered graphs in their paper. To combine the concept of n-extendable graphs and k-critical graphs, Liu and Yu [4] introduced (k, n, d)-graphs such that $(0, n, 1)$-graphs are the same as defect n-extendable graphs. They gave a Tutte style characterization and some properties of (k, n, d)-graphs.

In this paper, a characterization of defect 1-extendable bipartite graphs using path decomposition (defined in Section 3) is presented. Then by this characterization, we get a characterization and some properties of minimal defect 1-extendable bipartite graphs.

2 Preliminary results

In this section, two known results which will be used in the proof of our main theorems are given.

Lemma 1 (Lou and Wen [5]) *Let $n \geq 1$ and $G = (U, W)$ be a defect n-extendable bipartite graph with $|W| = |U| + 1$. Then for all $w \in W$, each component in $G - w$ is k-extendable where $k = \min(\kappa(G) - 1, n - 1)$.*

Lemma 2 (Hall [3]) *Let $G = (X, Y)$ be a bipartite graph. Then G has a matching of X into Y if and only if $|\Gamma_G(S)| \geq |S|$ for all $S \subseteq X$.*

3 Main results

First, we define path decomposition as follows.

Let $G = (U, W)$ be a bipartite graph with $|W| = |U| + 1$. If $G = w + P_1 + P_2 + \ldots + P_r$ where $w \in W$ and path P_i satisfies one of the following conditions:

(1) P_i is an odd path joining two vertices in G_{i-1} and having no other vertices in common with G_{i-1} where $G_{i-1} = w + P_1 + P_2 + \ldots + P_{i-1}$.

(2) P_i is an even path beginning with a vertex in $W \cap V(G_{i-1})$ and having no other vertex in common with G_{i-1} where $G_{i-1} = w + P_1 + P_2 + \ldots + P_{i-1}$.

then $w + P_1 + P_2 + \ldots + P_{r-1} + P_r$ is a **path decomposition** of G.

Theorem 3 *Let $G = (U, W)$ be a bipartite graph with $|V(G)| \geq 5$ and $|W| = |U|+1$. Then G is defect 1-extendable if and only if G has a path decomposition.*

Proof: Let G be as defined in the Theorem.

Assume $G = w + P_1 + P_2 + \ldots + P_{r-1} + P_r$ is a path decomposition of G. We will show that G is defect 1-extendable.

Let $G_i = w + P_1 + P_2 + \ldots + P_{i-1} + P_i$, $1 \leq i \leq r$. Clearly, G_i is a connected bipartite graph. Since $|V(G)| \geq 5$, it suffices to prove that for any $1 \leq i \leq r$, each edge in G_i is contained in a near perfect matching of G_i. We prove this by induction on i.

(1) Clearly, G_1 is an even path as P_1 is an even path. So it is not difficult to see that each edge in G_1 is contained in a near perfect matching of G_1.

(2) Suppose each edge in G_k is contained in a near perfect matching of G_k.

(3) We shall prove that each edge in G_{k+1} is contained in a near perfect matching of G_{k+1}. Since G_k contains a near perfect matching, we may assume (U_k, W_k) is the bipartition in G_k such that $|W_k| = |U_k| + 1$. We discuss two cases.

Case 1: P_{k+1} is an odd path. Assume $P_{k+1} = v_0 v_1 \ldots v_{2s+1}$.

Select any near perfect matching M in G_k. Then $M \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq s\}$ is a near perfect matching in G_{k+1}. Further more, each edge in G_k is contained in a near perfect matching of G_k by induction hypothesis, so each edge in $E(G_k) \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq s\}$ is contained in a near perfect matching of G_{k+1}.

Clearly, $v_0 \in W_k$ or $v_{2s+1} \in W_k$. Without loss of generality, assume $v_0 \in W_k$.

We now prove that there is a perfect matching in $G_k - v_0$. Note that $|V(G_k)|$ is odd and $|V(G_k)| \geq 3$ by $k \geq 1$.

If $|V(G_k)| = 3$, $G_k = K_{2,1}$ as $|W_k| = |U_k| + 1$ and G_k is connected. So $G_k - v_0$ has a perfect matching.

If $|V(G_k)| \geq 5$, by induction hypothesis, G_k is defect 1-extendable. So Lemma 1 implies that $G_k - v_0$ has a perfect matching.

So there is a perfect matching M' in $G_k - v_0$ and a vertex $u \in V(G_k - v_0)$ such that $uv_{2s+1} \in M'$. Then $F = M' \backslash \{uv_{2s+1}\}$ is a near perfect matching in $G_k - v_0 - v_{2s+1}$ and $F \cup \{v_{2i}v_{2i+1} : 0 \leq i \leq s\}$ is a near perfect matching in G_{k+1}. Therefore, each edge in $\{v_{2i}v_{2i+1} : 0 \leq i \leq s\}$ is contained in a near perfect matching of G_{k+1}.

Case 2: P_{k+1} is an even path. Assume $P_{k+1} = x_0x_1 \ldots x_{2t}$ where $x_0 \in W_k$.

Similar to the proof in Case 1, we obtain that each edge in $E(G_k) \cup \{x_{2i-1}x_{2i} : 1 \leq i \leq t\}$ is contained in a near perfect matching of G_{k+1} and there is a perfect matching M' in $G_k - x_0$. Clearly, $M' \cup \{x_{2i}x_{2i+1} : 0 \leq i \leq t-1\}$ is a near perfect matching in G_{k+1}, which implies that each edge in $\{x_{2i}x_{2i+1} : 0 \leq i \leq t-1\}$ is contained in a near perfect matching of G_{k+1}.

By the proof in Case 1 and Case 2, every edge in G_{k+1} is contained in a near perfect matching of G_{k+1} and this completes the proof of sufficiency.

We now prove the necessity. Assume G is defect 1-extendable. We shall prove that G has a path decomposition.

Select any vertex w in W. By Lemma 1, $G - w$ has a perfect matching M.

Select an edge e in G such that e is incident with w. Since G is defect 1-extendable, there is a near perfect matching F_e in G containing e. Let $P_1 = v_0v_1 \ldots v_s$ be the longest $F_e - M$ alternating path beginning with w.

Suppose $v_s \in U$. Then $v_{s-1}v_s \in E(G)\backslash M$ and there is a vertex u such that $uv_s \in M$. So $P_1 + uv_s$ is an $F_e - M$ alternating path beginning with w and $|E(P_1 + uv_s)| > |E(P_1)|$, contradicting the choice of P_1.

So $v_s \in W$. Since $v_0 = w \in W$, P_1 is an even path. If $G = w + P_1$, we are done.

If $G \neq w + P_1$, there is an edge f such that $f \in E(G)$, $f \notin E(w + P_1)$ and at least one end vertex x in f is on $w + P_1$. We now prove that there is a path containing f.

Since G is defect 1-extendable, there is a near perfect matching F_f in G containing f. We discusses two cases.

Case 1: $x \in U$.

Let P_2 be the $F_f - M$ alternating path starting at x and ending upon first return to $w + P_1$. Note that P_2 exists. Otherwise, assume $P = x_0x_1 \ldots x_s$ where $x_0 = x$ is the longest $F_f - M$ alternating path starting at x. Suppose $x_s \in U$. Then $x_{s-1}x_s \in M$ and there is a vertex $u \in W$ such that $x_su \in F_f$. Clearly $u \notin V(P)$. So $P + x_su$ is an $F_f - M$ alternating path starting at x and $|E(P + x_su)| > |E(P)|$, contradicting the choice of P. Suppose $x_s \in W$. Similar to the proof in the case of $x_s \in U$, we can also find a contradiction.

So P_2 exists. Observe that $f \in E(P_2)$ and P_2 has no vertex in common with

$w+P_1$ except its end vertices. Further, P_2 is an odd path as it begins and ends with an edge in F_f. So P_2 is the require path.

Case 2: $x \in W$.

Suppose there exists an $F_f - M$ alternating path P which begins with f and can return to a vertex of $w+P_1$. Let P_2 be the path when P first returns to $w+P_1$, then similar to the proof in Case 1, we obtain that P_2 is the required path.

We next suppose that there is no $F_f - M$ alternating path beginning with f and returning to a vertex of $w+P_1$. Let $Q = y_0 \ldots y_{t-1}y_t$ be the longest $F_f - M$ alternating path starting at x where $y_0 = x$. Then $(V(Q)\backslash\{x\}) \cap V(w+P_1) = \emptyset$.

Suppose $y_t \in U$. Then $y_{t-1}y_t \in F_f$ and there is a vertex u such that $y_t u \in M$. Clearly, $u \notin V(P)$. So $Q + y_t u$ is an $F_f - M$ alternating path starting at x such that $|E(Q+y_t u)| > |E(Q)|$, contradicting the choice of Q.

So $y_t \in W$. Since $x \in W$, Q is an even path. Further, observe that $f \in E(Q)$ and $V(Q) \cap V(w+P_1) = \{x\}$. So Q is the required path.

We may continue to find new paths until all edges of G lie in some path. □

Corollary 4 *Let G be a bipartite graph with $|V(G)| \geq 5$, M a near perfect matching of G and w the M-unsaturated vertex. Then G is defect 1-extendable if and only if G has a path decomposition $w + P_1 + P_2 + \ldots + P_{r-1} + P_r$ such that for all $1 \leq i \leq r$, P_i is an M-alternating path.*

Proof: The proof is similar to that of Theorem 3. □

Theorem 3 actually gives a simple method for constructing all defect 1-extendable bipartite graphs. We now present some properties of path decomposition.

Theorem 5 *Let $G = (U, W)$ be a defect 1-extendable bipartite graph with $|W| = |U| + 1$. Then the following statements hold:*

(1) *A path decomposition of G can be started with any vertex in W.*

(2) *Let $w + P_1 + P_2 + \ldots + P_r$ be a path decomposition of G and $G_i = w + P_1 + P_2 + \ldots + P_i$, $1 \leq i \leq r$. If $|V(G_i)| \geq 5$, then G_i is defect 1-extendable.*

(3) *There are $|E(G)| - |V(G)| + 1$ odd paths in any path decomposition of G.*

(4) *If H is a subgraph of G such that $G - V(H)$ contains a perfect matching, there is a decomposition $G = H + P_1 + P_2 + \ldots + P_r$ where P_i, $1 \leq i \leq r$, is as defined in path decomposition.*

(5) *Any graph obtained from G by inserting an even number of new vertices in an edge of G is also a defect 1-extendable bipartite graph.*

(6) *There exists a path decomposition $G = w + P_1 + P_2 + \ldots + P_r$ such that $|V(P_1)| \geq 5$.*

Proof:

(1) This follows immediately from the proof of Theorem 3.

(2) Assume $G_i = (U_i, W_i)$ where $U_i = V(G_i) \cap U$ and $W_i = V(G_i) \cap W$. By the definition of path decomposition, $|W_i| = |U_i| + 1$ and $w + P_1 + P_2 + \ldots + P_i$ is a path decomposition of G_i. So Theorem 3 implies that if $|V(G_i)| \geq 5$, G_i is defect 1-extendable.

(3) Assume there are r_1 odd paths P_1, P_2, ..., P_{r_1} and r_2 even paths Q_1, Q_2, ..., Q_{r_2} in a path decomposition of G. Assume $|V(P_i)| = v_i$ and $|V(Q_j)| = u_j$. Then $|V(G)| = 1+\sum_{i=1}^{r_1}(v_i-2)+\sum_{j=1}^{r_2}(u_j-1)$ and $|E(G)| = \sum_{i=1}^{r_1}(v_i-1)+\sum_{j=1}^{r_2}(u_j-1)$. So $|E(G)|-|V(G)| = (\sum_{i=1}^{r_1}(v_i-1)+\sum_{j=1}^{r_2}(u_j-1))-(1+\sum_{i=1}^{r_1}(v_i-2)+\sum_{j=1}^{r_2}(u_j-1)) = r_1 - 1$ and hence $r_1 = |E(G)| - |V(G)| + 1$.

(4) The proof of (4) is similar to that of the necessity in Theorem 3.

(5) Suppose G' is a graph obtained by inserting vertices $x_0, x_1, \ldots, x_{2k-1}$ into edge $xy \in E(G)$ in that order. Clearly G' is a bipartite graph. Since G is defect 1-extendable, by Theorem 3, G has a path decomposition $w + P_1 + P_2 + \ldots + P_r$. Assume xy is contained in path P_i where $1 \leq i \leq r$ and P_i' is the path obtained from P_i by replacing edge xy in P_i with path $xx_0x_1 \ldots x_{2k-1}y$. Clearly, $w + P_1 + \ldots + P_{i-1} + P_i' + P_{i+1} + \ldots + P_r$ is a path decomposition of G'. So by Theorem 3, G' is a defect 1-extendable bipartite graph.

(6) Since G is a defect 1-extendable bipartite graph, Theorem 3 implies that G has a path decomposition $w + P_1 + P_2 + \ldots + P_r$. Suppose $|V(P_1)| \geq 5$. Then we are done.

Suppose $|V(P_1)| < 5$. Note that P_1 is an even path by definition of path decomposition. So $|V(P_1)|$ is odd and hence $|V(P_1)| = 3$. Assume $P_1 = wv_1v_2$. By definition of path decomposition, $w \in W$, $v_2 \in W$ and $v_1 \in U$.

Assume P_2 is an even path. Then either w or v_2 is an end vertex of P_2. Let $P_1' = P_1 + P_2$. Then P_1' is an even path and both end vertices of P_1' are in W. Assume w' is an end vertex in P_1'. Then $w' + P_1' + P_3 + P_4 + \ldots + P_r$ is a path decomposition of G. Further, $|V(P_1')| = |E(P_1)| + |E(P_2)| + 1 \geq 2 + 2 + 1 = 5$. So $w' + P_1' + P_3 + P_4 + \ldots + P_r$ is the required path decomposition of G.

Assume P_2 is an odd path. Then either w and v_1 or v_1 and v_2 are the two end vertices of P_2. Without loss of generality, assume v_1 and v_2 are the end vertices of P_2. Let $P_1'' = wv_1 + P_2$ and $P_2'' = v_1v_2$. Then P_1'' is an even path and both end vertices of P_1' are in W. So $w+P_1''+P_2''+P_3+P_4+\ldots+P_r$ is a path decomposition of G. Since v_1 and v_2 are the two end vertices of P_2, P_2 contains at least three edges. So P_1'' contains at least four edges and hence $|V(P_1)''| \geq 5$. Thus $w + P_1'' + P_2'' + P_3 + P_4 + \ldots + P_r$ is the required path decomposition of G. □

Since Theorem 3 implies that a bipartite graph obtained by adding an edge to a defect 1-extendable bipartite graph remains defect 1-extendable, it's natural to study minimal defect 1-extendable bipartite graphs. A defect 1-extendable bipartite graph

G is minimal if $G{-}e$ is not defect 1-extendable for every edge e of G.

Theorem 6 *Let G be a defect 1-extendable bipartite graph. G is minimal if and only if G contains no cycle.*

Proof: Since G is defect 1-extendable, Theorem 3 implies that there is a path decomposition $w{+}P_1{+}P_2{+}\dots{+}P_r$ of G.

Suppose G contains no cycle. Then G is a tree and hence for any edge $e{\in}E(G)$, $G{-}e$ is disconnected, that is, $G{-}e$ is not defect 1-extendable. So G is minimal.

Conversely, assume G is minimal. Suppose to the contrary G contains a cycle. Then it is easy to see that there exists a path P_k, $1 \leq k \leq r$, such that P_k is an odd path. Assume $P_k = v_0 v_1 \dots v_s$ and $G = (U, W)$ such that $|W| = |U| + 1$. Then $v_0 \in U$ or $v_s \in U$. Without loss of generality, assume $v_s \in U$. Let edge $e = v_{s-1}v_s$ and $P'_k = P_k - e$. Then P'_k is an even path beginning with $v_0 \in W$. Further, since P_k contains at least three edges as G is minimal, P'_k contains at least two edges. So $w + P_1 + \dots + P_{k-1} + P'_k + P_{k+1} \dots + P_r$ is a path decomposition of $G - e$ and hence $G - e$ is defect 1-extendable, contradicting the assumption that G is minimal. So G contains no cycle. □

By Theorems 3 and 6, a minimal defect 1-extendable bipartite graph can be constructed by repeatedly adding even paths.

Since determining whether a graph G is a defect 1-extendable bipartite graph needs $O(|E(G)|)$ time when the maximum matching of G is known [6]. Further, identifying whether G contains a cycle also needs $O(|E(G)|)$ time. So it takes $O(|E(G)|)$ time to determine whether G is a minimal defect 1-extendable bipartite graph when the maximum matching of G is known.

Corollary 7 *Let G be a minimal defect 1-extendable bipartite graph, $w + P_1 + P_2 + \dots + P_r$ be a path decomposition of G and $G_i = w + P_1 + P_2 + \dots + P_i$, $1 \leq i \leq r$. Then for any $1 \leq i \leq r$, if $|V(G_i)| \geq 5$, G_i is minimal defect 1-extendable.*

Proof: If $|V(G_i)| \geq 5$, Theorm5(2) implies that G_i is defect 1-extendable. Since G is a minimal defect 1-extendable bipartite graph, Theorem 6 implies that G contains no cycle and hence G_i contains no cycle. Thus by Theorem 6 again, G_i is minimal defect 1-extendable. □

Corollary 8 *If G is a minimal defect 1-extendable bipartite graph, then $\delta(G) = 1$.*

Proof: This follows immediately from Theorem 6. □

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