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Interpolation theorems in jump graphs

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Abstract

For positive integers m and n $(1 \leq m \leq \binom{n}{2})$, let $\mathbb{J}_m(n)$ be the class of all distinct subgraphs of K_n of size m. Let $G, H \in \mathbb{J}_m(n)$. Then G is said to be obtained from H by an edge jump if there exist four distinct vertices u, v, w, and x of K_n such that $e = uv \notin E(H)$, $f = wx \in E(H)$ and $\sigma(e,f)H := H + e - f = G$. The minimum number of edge jumps required to transform H to G is the jump distance from H to G. The graph $\mathbb{J}_m(n)$ is that graph having $\mathbb{J}_m(n)$ as its vertex set where two vertices of $\mathbb{J}_m(n)$ are adjacent if and only if the jump distance between the corresponding subgraphs is 1. Let $\mathbb{C}_m(n)$ be the subset of $\mathbb{J}_m(n)$ consisting of all connected graphs. We prove in this paper that the graph $\mathbb{J}_m(n)$ is connected and the subgraph of the graph $\mathbb{J}_m(n)$ induced by $\mathbb{C}_m(n)$ is also connected. Several graph parameters are proved to interpolate over the class $\mathbb{J}_m(n)$ and $\mathbb{C}_m(n)$. Algorithms for determining the extreme values for the chromatic number χ and the clique number ω are also provided.

1 Introduction

We limit our discussion to graphs that are simple and finite. For the most part, our notation and terminology follows that of Bondy and Murty [1]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{e_1, e_2, \cdots, e_m\}$. We use $|S|$ to denote the cardinality of a set S and define $n = \nu(G) = |V|$ to be the *order* of G and $m = \varepsilon(G) = |E|$ the *size* of G. We write $e = uv$ for an edge e that joins vertex u to vertex v. A *path* of order k in a graph G, denoted by P_k, is a sequence of distinct vertices $u_1u_2\cdots u_k$ of G such that for all $i = 1, 2, \cdots, k-1$, u_iu_{i+1} is an edge of G. The *degree* of a vertex v of

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a graph G is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree and the minimum degree of a graph G is usually denoted by $\Delta(G)$ and $\delta(G)$, respectively. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph of G induced by S. We also use the notation $\varepsilon(S)$ for $\varepsilon(G[S])$. For a graph G and $X \subseteq E(G)$, we denote by $G - X$ the graph obtained from G by removing all edges in X. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph G and $X \subseteq V(G)$, $G - X$ is the graph obtained from G by removing all vertices in X and all edges incident with vertices in X. For a graph G and $X \subseteq E(\overline{G})$, $G + X$ denotes the graph obtained from G by adding all edges in X. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs G and H are disjoint if $V(G) \cap V(H) = \emptyset$. For any two disjoint graphs G and H, $G \cup H$ is defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We can extend this definition to a finite union of pairwise disjoint graphs.

For positive integers m and n ($1 \le m \le \binom{n}{2}$), let $\mathbb{J}_m(n)$ be the class of all distinct subgraphs of K_n of size m. Let $G, H \in \mathbb{J}_m(n)$. Then G is said to be obtained from H by an *edge jump* if there exist four distinct vertices u, v, w, and x of K_n such that $e = uv \notin E(H)$, $f = wx \in E(H)$ and $\sigma(e, f)H := H + e - f = G$. The minimum number of edge jumps required to transform H to G is the jump distance from H to G. *The graph* $\mathbb{J}_m(n)$ is that graph having $\mathbb{J}_m(n)$ as its vertex set and where two vertices are adjacent if and only if the jump distance between the corresponding subgraphs is 1. Let $\mathbb{C}_m(n)$ be the subset of $\mathbb{J}_m(n)$ consisting of all connected graphs. It is clear that the graph $\mathbb{J}_m(n)$ has order $\binom{N}{m}$, where $N = \binom{n}{2}$.

Note that the operation $\sigma(e, f)$ is well defined on G if and only if $\sigma(f, e)$ is well defined on $\sigma(e, f)G$ and $\sigma(f, e)\sigma(e, f)G = G$. Thus the graph $\mathbb{J}_m(n)$ is simple.

The concept of the edge jump is not new. Harary et al. [6, 7, 8, 9, 10, 11, 12] introduced this concept in late 1980 and called it an *edge exchange*. They used this concept to prove interpolation problems raised by Chartrand. Structure of specific classes of jump graphs were also studied by Chartrand et al. [2, 3].

Theorem 1.1 *Let $G, H \in \mathbb{J}_m(n)$. Then $G = H$ or there is a finite sequence of edge jumps $\sigma(e_1, f_1), \sigma(e_2, f_2), \cdots, \sigma(e_t, f_t)$ such that*
$H = \sigma(e_t, f_t)\sigma(e_{t-1}, f_{t-1}) \cdots \sigma(e_1, f_1)G$.

Proof. Let $G, H \in \mathbb{J}_m(n)$. If $G \ne H$, then $E(G) - E(H) \ne \emptyset$. Let $E(G) - E(H) = \{f_1, f_2, \cdots, f_t\}$ and $E(H) - E(G) = \{e_1, e_2, \cdots, e_k\}$. Since G and H are of the same size, it follows that $t = k$. Since $e_1 \notin E(G)$ and $f_1 \in E(G)$, it follows that $\sigma(e_1, f_1)$ is a well defined edge jump on G and $|E(G_1) \cap E(H)| = |E(G) \cap E(H)| + 1$, where $G_1 = \sigma(e_1, f_1)G$. Further, for all $i = 2, 3, \cdots, t$, $\sigma(e_i, f_i)$ is a well defined operation on G_{i-1} and $|E(G_i) \cap E(H)| = |E(G_{i-1}) \cap E(H)| + 1$, where $G_i = \sigma(e_i, f_i)G_{i-1}$. Thus $G_t = H$ and the proof is complete. □

Corollary 1.2 *The graph $\mathbb{J}_m(n)$ is connected.*

Theorem 1.3 *Let $G, H \in \mathbb{C}_m(n)$. Then $G = H$ or there is a finite sequence of edge jumps $\sigma(e_1, f_1)$, $\sigma(e_2, f_2)$, $\cdots$, $\sigma(e_t, f_t)$ such that for all $i = 1, 2, \cdots, t$,*

$\sigma(e_i, f_i)\sigma(e_{i-1}, f_{i-1}) \cdots \sigma(e_1, f_1)G \in \mathbb{C}_m(n)$ *and*

$$H = \sigma(e_t, f_t)\sigma(e_{t-1}, f_{t-1}) \cdots \sigma(e_1, f_1)G.$$

Proof. Note that $\mathbb{C}_m(n) \neq \emptyset$ if and only if $m \geq n-1$. If $m = n-1$, then $\mathbb{C}_m(n)$ contains all trees of order n. We first prove the theorem when $m = n-1$. Let $T_1, T_2 \in \mathbb{C}_{n-1}(n)$ and $T_1 \neq T_2$. Then $E(T_1) - E(T_2) \neq \emptyset$, $E(T_2) - E(T_1) \neq \emptyset$, and $|E(T_1) - E(T_2)| = |E(T_2) - E(T_1)|$. For any $e_1 \in E(T_2) - E(T_1)$, $T_1 + e_1$ contains a cycle. Thus there exists $f \in E(T_1) - E(T_2)$, say f_1, such that $T_1 + e_1 - f_1$ is a tree. In this case $|E(T_1 + e_1 - f_1) \cap E(T_2)| = |E(T_1) \cap E(T_2)| + 1$. Thus there exists a finite sequence of edge jumps $\sigma(e_1, f_1), \sigma(e_2, f_2), \cdots, \sigma(e_t, f_t)$ such that for all $i = 1, 2, \cdots, t$, $\sigma(e_i, f_i)\sigma(e_{i-1}, f_{i-1}) \cdots \sigma(e_1, f_1)G \in \mathbb{C}_{n-1}(n)$ and $T_2 = \sigma(e_i, f_i)\sigma(e_{i-1}, f_{i-1}) \cdots \sigma(e_1, f_1)T_1$. Let $G, H \in \mathbb{C}_m(n)$, $m \geq n$. Let T_1 and T_2 be spanning trees of G and H, respectively. Then, by the previous argument, we may assume that $T_1 = T_2$. Thus $E(G) \cap E(H)$ contains $E(T_1)$. Put $E(G) - E(H) = \{f_1, f_2, \cdots, f_t\}$ and $E(H) - E(G) = \{e_1, e_2, \cdots, e_t\}$. Thus there is a finite sequence of edge jumps $\sigma(e_1, f_1), \sigma(e_2, f_2), \cdots, \sigma(e_t, f_t)$ such that for all $i = 1, 2, \cdots, t$, $\sigma(e_i, f_i)\sigma(e_{i-1}, f_{i-1}) \cdots \sigma(e_1, f_1)G \in \mathbb{C}_m(n)$ and $H = \sigma(e_t, f_t)\sigma(e_{t-1}, f_{t-1}) \cdots \sigma(e_1, f_1)G$. □

Corollary 1.4 *The graph* $\mathbb{C}_m(n)$ *is connected.*

Let $\mathbb{G}$ be the class of all simple graphs. A function $\pi : \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $\pi(G) = \pi(H)$ for all isomorphic graphs G and H. A graph parameter π is called an *interpolation graph parameter over* $\mathbb{J} \subseteq \mathbb{G}$ if there exist integers x and y such that

$$\{\pi(G) : G \in \mathbb{J}\} := [x, y] := \{k \in \mathbb{Z} : x \leq k \leq y\}.$$

If π is an interpolation graph parameter over $\mathbb{J}$, then $\{\pi(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(\pi, \mathbb{J}) := \min\{\pi(G) : G \in \mathbb{J}\}$ and $\max(\pi, \mathbb{J}) := \max\{\pi(G) : G \in \mathbb{J}\}$. In the case where $\mathbb{J} = \mathbb{J}_m(n)$ we write $\min(\pi; m, n)$ and $\max(\pi; m, n)$ for $\min(\pi, \mathbb{J}_m(n))$ and $\max(\pi, \mathbb{J}_m(n))$, respectively, and in the case where $\mathbb{J} = \mathbb{C}_m(n)$ we write $\text{Min}(\pi; m, n)$ and $\text{Max}(\pi; m, n)$ for $\min(\pi, \mathbb{C}_m(n))$ and $\max(\pi, \mathbb{C}_m(n))$, respectively.

2 Interpolation Theorems

Studying interpolation theorems for graph parameters may be divided into two parts. The first part deals with the following question: Given a graph parameter π and a subset $\mathbb{J}$ of $\mathbb{G}$, does π interpolate over $\mathbb{J}$? If π interpolates over $\mathbb{J}$, then $\{\pi(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(\pi, \mathbb{J})$ and $\max(\pi, \mathbb{J})$. The second part of the interpolation theorems for graph parameters is to find the values of $\min(\pi, \mathbb{J})$ and $\max(\pi, \mathbb{J})$ for the corresponding interpolation graph parameters and this part is, in fact, the extremal problem in graph theory.

The interest in the interpolation properties of graph parameters was motivated by an open question posed by Chartrand during a conference held at Kalamazoo in 1980. He posed the following question: If a graph G contains spanning trees having m and n end-vertices, with $m < n$, does G contain a spanning tree with k end-vertices for every integer k with $m < k < n$? This question (which was answered affirmatively) led to a host of papers studying the interpolation properties of invariants of spanning trees of a given graph. The details can be found in [6, 7, 8, 9, 10, 11, 12]. In [16], several graph parameters were proved to interpolate over the class of all graphs with the same degree sequence and presented the two parts of the interpolation theorems.

For any integers m and n, it was shown in the previous section that the graph $\mathbb{J}_m(n)$ is connected and the subgraph of the graph $\mathbb{J}_m(n)$ induced by $\mathbb{C}_m(n)$ is also connected. Let π be a graph parameter. Thus we have the following theorems.

Theorem 2.1 *For a graph $G \in \mathbb{J}_m(n)$ and an edge jump $\sigma(e, f)$, if*

$$|\pi(G) - \pi(\sigma(e, f)G)| \leq 1,$$

then π is an interpolation graph parameter over $\mathbb{J}_m(n)$.

Theorem 2.2 *For a graph $G \in \mathbb{C}_m(n)$ and an edge jump $\sigma(e, f)$, if*

$$|\pi(G) - \pi(\sigma(e, f)G)| \leq 1,$$

then π is an interpolation graph parameter over $\mathbb{C}_m(n)$.

Theorem 2.3 *Let $\mathbb{J} \subseteq \mathbb{J}_m(n)$ and the subgraph of the graph $\mathbb{J}_m(n)$ induced by $\mathbb{J}$ be connected. For a graph $G \in \mathbb{J}$ and an edge jump $\sigma(e, f)$, if $|\pi(G) - \pi(\sigma(e, f)G)| \leq 1$, then π is an interpolation graph parameter over $\mathbb{J}$.*

3 Interpolation Graph Parameters

We will now present various graph parameters and prove interpolation results on the corresponding graph parameters with respect to $\mathbb{J}_m(n)$ and $\mathbb{C}_m(n)$. We first state the definition of χ. A *k-coloring* of a graph $G = (V, E)$ is a partition of its vertex set V as $V_1 \cup V_2 \cup \cdots \cup V_k$ such that no two vertices in V_i $(1 \leq i \leq k)$ are adjacent. The V_i's are called the *color classes.* A function $c : V \to \{1, 2, \cdots, k\}$ such that $c(v) = i$ for each $v \in V_i (1 \leq i \leq k)$ is called a *color function.* If G has a k-coloring, it is said to be *k-colorable* and the minimum integer k for which G is k-colorable is called the *chromatic number* of G and is denoted by $\chi(G)$. If $\chi(G) = k$, we say that G is *k-chromatic.*

Remark 3.1 In a proper coloring, each color class contains no edge, so G is k-colorable if and only if G is a k-partite graph. Thus a graph is 2-colorable if and only if it is bipartite. Thus a graph containing an odd cycle must be at least 3-colorable.

Theorem 3.2 *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on G. Then* $|\chi(G) - \chi(\sigma(e,f)G)| \leq 1$.

Proof. Let G be a graph. If $e \notin E(G)$, then $\chi(G) \leq \chi(G+e) \leq \chi(G)+1$ and if $f \in E(G)$, then $\chi(G+e)-1 \leq \chi(G+e-f) \leq \chi(G+e)$. Since either $\chi(G+e) = \chi(G)$ or $\chi(G+e) = \chi(G)+1$, it follows that $\chi(G) \leq \chi(G+e-f) \leq \chi(G)+1$. Hence $|\chi(G) - \chi(\sigma(e,f)G)| \leq 1$. □

A maximal complete subgraph of a graph G is called a *clique* of G. The maximum order of clique of G is called the *clique number* of G and is denoted by $\omega(G)$.

In general there is no formula for the chromatic number of a graph. Determining the chromatic number of even a relatively small graph is often a challenging problem. However, lower bounds for the chromatic number of a graph G can be given in term of the clique number of G. That is $\chi(G) \geq \omega(G)$, for any graph G.

Theorem 3.3 *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on G. Then* $|\omega(G) - \omega(\sigma(e,f)G)| \leq 1$.

Proof. Let G be a graph. If $e \notin E(G)$, then $\omega(G) \leq \omega(G+e) \leq \omega(G)+1$ and if $f \in E(G)$, then $\omega(G+e)-1 \leq \omega(G+e-f) \leq \omega(G+e)$. Since either $\omega(G+e) = \omega(G)$ or $\omega(G+e) = \omega(G)+1$, it follows that $\omega(G) \leq \omega(G+e-f) \leq \omega(G)+1$. Hence $|\omega(G) - \omega(\sigma(e,f)G)| \leq 1$. □

A graph containing no cycle as its subgraph is called an *acyclic graph*. An acyclic graph is called a *forest*. Thus each component of an acyclic graph is a tree.

Let G be a graph and $F \subseteq V(G)$. This subset F is called an *induced forest* of G if the subgraph of G induced by F contains no cycle. The maximum cardinality of an induced forest of a graph G is called the *forest number* of G and is denoted by $I(G)$. That is

$$I(G) := \max\{|F| : F \text{ is an induced forest in } G\}.$$

We have the following theorem.

Theorem 3.4 *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on G. Then* $|I(G) - I(\sigma(e,f)G)| \leq 1$.

Proof. Let G be a graph and F be a maximum induced forest of G. For any $e \notin E(G)$ we see that the subgraph of $G+e$ induced by F contains at most one cycle. Thus $I(G+e) \geq I(G)-1$. It is clear that any induced forest F' of $G+e$ is an induced forest of G. Thus $I(G+e) \leq I(G)$. Therefore $I(G)-1 \leq I(G+e) \leq I(G)$. Let $f \in E(G)$. Then $I(G+e) \leq I(G+e-f) \leq I(G+e)+1$. Since either $I(G+e) = I(G)$ or $I(G+e) = I(G)-1$, it follows that $I(G)-1 \leq I(G+e-f) \leq I(G)$. Hence $|I(G) - I(\sigma(e,f)G)| \leq 1$. □

There is a counterpart graph parameter of I called the *decycling number*. Let G be a graph. The minimum number of vertices of G whose removal eliminates all cycles in the graph G is called the *decycling number* of G and is denoted by $\phi(G)$. Evidently, for a graph G of order n, $\phi(G) + I(G) = n$. Thus we obtain the following corollary.

Corollary 3.5 *Let* $G \in \mathbb{J}_m(n)$ *and* $\sigma(e, f)$ *be an edge jump on* G. *Then* $|\phi(G) - \phi(\sigma(e, f)G)| \leq 1$.

A subset U of the vertex set $V(G)$ of a graph G is said to be an *independent set* of G if the subgraph of G induced by U contains no edge. An independent set of G with maximum number of vertices is called a *maximum independent set* of G. The number of vertices of a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha_0(G)$.

It is clear that $\alpha_0(G) = \omega(\overline{G})$, for any graph G. Observe that a graph $G \in \mathbb{J}_m(n)$ if and only if $\overline{G} \in \mathbb{J}_{\binom{n}{2}-m}(n)$. Thus we have the following corollary as a direct consequence of Theorem 3.5.

Corollary 3.6 *Let* $G \in \mathbb{J}_m(n)$ *and* $\sigma(e, f)$ *be an edge jump on* G. *Then* $|\alpha_0(G) - \alpha_0(\sigma(e, f)G)| \leq 1$.

A vertex of a graph $G = (V, E)$ is said to cover the edges incident with it. A *vertex cover* of a graph G is a set of vertices covering all the edges of G. The minimum cardinality of a vertex cover of a graph G is called the *vertex covering number* of G and is denoted by $\beta_0(G)$.

A subset M of the edge set $E(G)$ of a graph G is an *independent edge set* or *matching* of G if no two distinct edges in M have a common vertex. A matching M of G is *maximum* if there is no matching M' of G with $|M'| > |M|$. The cardinality of a maximum matching of G is called the *matching number* of G and is denoted by $\alpha_1(G)$.

There is an analogous covering concept for edges. An edge of a graph G is said to cover two vertices incident with it. An *edge cover* of a graph G is a set of edges covering all the vertices of G. The minimum cardinality of an edge cover of G is called the *edge covering number* of G and is denoted by $\beta_1(G)$.

Theorem 3.7 *Let* $G \in \mathbb{J}_m(n)$ *and* $\sigma(e, f)$ *be an edge jump on* G. *Then* $|\alpha_1(G) - \alpha_1(\sigma(e, f)G)| \leq 1$.

Proof. Let G be a graph and M be a matching of G. For any $e \notin E(G)$ we see that M is also a matching of $G + e$. Thus $\alpha_1(G + e) \geq \alpha_1(G)$. On the other hand, for any matching M' of $G + e$ we see that M' is a matching of G if $e \notin M'$ and $M' - e$ is a matching of G if $e \in M'$. Thus $\alpha_1(G) - 1 \leq \alpha(G + e) \leq \alpha_1(G)$. Let $f \in E(G)$. Then $\alpha_1(G + e) - 1 \leq \alpha_1(G + e - f) \leq \alpha_1(G + e)$. Since either $\alpha_1(G + e) = \alpha_1(G)$ or $I(G + e) = I(G) - 1$, it follows that $\alpha_1(G) - 1 \leq \alpha_1(G + e - f) \leq \alpha_1(G)$. Hence $|\alpha_1(G) - \alpha_1(\sigma(e, f)G)| \leq 1$. □

Let n be a positive integer. The *star* of order $n+1$ is the complete bipartite graph $K_{n,1}$. The edges covered by one vertex in a vertex cover are the edges incident to it; they form a star. The vertex cover problem can be described as covering the edge set with the fewest number of stars. This is equivalent to our next graph parameter.

A *dominating set* of a graph G is a subset D of $V(G)$ such that each vertex of $V(G)-D$ is adjacent to at least one vertex of D. The *domination number*, $\gamma(G)$, is the cardinality of a minimal dominating set with the least number of elements.

Theorem 3.8 *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on G. Then $|\gamma(G)-\gamma(\sigma(e,f)G)| \leq 1$.*

Proof. Let G be a graph and D be a dominating set of G. For any $e \notin E(G)$ we see that D is also a dominating set of $G+e$. Thus $\gamma(G+e) \leq \gamma(G)$. On the other hand, for any dominating set D' of $G+e$, where $e=uv$, we see that D' is a dominating set of G if either $u,v \in D'$ or $u,v \notin D'$, otherwise $D' \cup \{u\}$ or $D' \cup \{v\}$ is a dominating set of G. Thus $\gamma(G)-1 \leq \gamma(G+e) \leq \gamma(G)$. Let $f \in E(G)$. Then $\gamma(G+e) \leq \gamma(G+e-f) \leq \gamma(G+e)+1$. Since either $\gamma(G+e)=\gamma(G)$ or $\gamma(G+e)=\gamma(G)+1$, it follows that $\gamma(G) \leq \gamma(G+e-f) \leq \gamma(G)+1$. Hence $|\gamma(G)-\gamma(\sigma(e,f)G)| \leq 1$. □

Gallai [5] and Norman-Rabin [14] proved the following results concerning relationship between α_0 and β_0, and between α_1, and β_1, respectively.

Theorem 3.9 *For any graph G of order n, $\alpha_0+\beta_0=n$.*

Theorem 3.10 *For any graph G of order n and $\delta \geq 1, \alpha_1+\beta_1=n$.*

As a direct consequence of the previous two theorems, we have the following corollary.

Corollary 3.11 *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on G. Then $|\beta_0(G)-\beta_0(\sigma(e,f)G)| \leq 1$. and $|\beta_1(G)-\beta_1(\sigma(e,f)G)| \leq 1$.*

The following theorems are obtained as consequences of our results.

Theorem 3.12 *χ, ω, I, ϕ, α_0, α_1, β_0, β_1 and γ are interpolation graph parameters over $\mathbb{J}_m(n)$.*

Theorem 3.13 *χ, ω, I, ϕ, α_0, α_1, β_0, β_1 and γ are interpolation graph parameters over $\mathbb{C}_m(n)$.*

Theorem 3.14 *Let $\pi \in \{\chi,\omega,I,\phi,\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma\}$. Then there exist integers $a := \min(\pi;m,n)$ and $b := \max(\pi;m,n)$ such that there exists $G \in \mathbb{J}_m(n)$ with $\pi(G)=c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

Theorem 3.15 *Let $\pi \in \{\chi,\omega,I,\phi,\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma\}$. Then there exist integers $a := \mathrm{Min}(\pi;m,n)$ and $b := \mathrm{Max}(\pi;m,n)$ such that there exists $G \in \mathbb{C}_m(n)$ with $\pi(G)=c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

4 Extremal Problems for Graph Parameters

We have already proved interpolation property of various graph parameters in Section 3. Thus for an interpolation graph parameter π over $\mathbb{J} \subseteq \mathbb{G}$, the values of $\{\pi(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(\pi, \mathbb{J})$ and $\max(\pi, \mathbb{J})$. The problem of finding $\min(\pi, \mathbb{J})$ and $\max(\pi, \mathbb{J})$ is so called the extremal problem in graph theory.

An *extremal problem* asks for minimum and maximum values of a function over a class of objects.

Remark 4.1 Proving that $A = \min\{\pi(G) : G \in \mathbb{J}\}$ requires showing two things:

1. $\pi(G) \geq A$ for all $G \in \mathbb{J}$.
2. $\pi(G) = A$ for some $G \in \mathbb{J}$.

The proof of the bound must apply to every $G \in \mathbb{J}$. For equality it suffices to obtain an example in $\mathbb{J}$ with the desired value of π.

Changing " $\geq$ " to " $\leq$ " yields the criteria for a maximum.

A study of extremal problems for graph parameters was motivated by Dirac's conjecture as state in the following.

In the graph-theoretic colloquium at Smolenice in 1963, Dirac conjectured that the chromatic number of a proper regular subgraph of a complete n-gon is at most $\frac{3n}{5}$. Erdős and Gallai answered this conjecture immediately and presented their result during the conference. Their article was entitled "Solution to a problem of Dirac" [4], and it appeared in the proceedings of the symposium, Smolenice, in 1964. In fact, the result was more than Dirac asked for. In addition, they found all regular graphs reaching the upper bound as stated in the following theorem.

Theorem 4.2 *An r-regular graph G of order $n > r + 1$ has chromatic number $k \leq \frac{3n}{5}$, with equality holds if and only if the complementary graph $\overline{G}$ of G is a union of 5-cycles.*

In [15] we generalized the result of the above theorem by introducing the notion of $F(j)$-graphs and using these graphs to obtain generalized results.

Let j be a positive integer. An $F(j)$-*graph* is a $(j-1)$-regular graph G of minimum order $f(j)$ satisfying $\chi(\overline{G}) > f(j)/2$.

It is easy to see that an $F(3)$-graph is C_5 and $f(3) = 5$. We will see that $F(j)$-graphs, $j \geq 5$, are not unique.

We have found $F(j)$-graphs for all odd integers j as we state in the following theorem.

Theorem 4.3 *For odd integers j with $j \geq 3$, we have $f(j) = \frac{5}{2}(j-1)$ if $j \equiv 3 \pmod 4$ and $f(j) = 1 + \frac{5}{2}(j-1)$ if $j \equiv 1 \pmod 4$.*

The following result can be found in [15] which is a generalization of Theorem 4.2.

Theorem 4.4 *Any r-regular graph of order n with $n - r = j$ odd and $j \geq 3$ has chromatic number at most* $\frac{f(j)+1}{2f(j)} \cdot n$*, and this bound is achieved precisely for those graphs with complement equal to a disjoint union of $F(j)$-graphs.*

As we have already proved the interpolation property of χ with respect to $\mathbb{J}_m(n)$ and $\mathbb{C}_m(n)$ in Section 3, it is natural to ask the same for $\min(\chi; m, n)$, $\max(\chi; m, n)$, $\text{Min}(\chi; m, n)$ and $\text{Max}(\chi; m, n)$. We will consider this question in this section.

First, we state some elementary facts concerning the chromatic number of graphs of order n and size m.

1. Every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges. Equality holds for the graph $K_k \cup (n-k)K_1$.
2. If the connectivity is concerned, one can ask for the minimum size among connected k-chromatic graphs with n vertices. This question was answered in [18]: the minimum size of such a graph is at least $\binom{k}{2} + n - k$. Equality holds for a graph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{v_1 v_i : 2 \leq i \leq n\} \cup \{v_i v_j : 2 \leq i < j \leq k\}$.
3. Mantel proved in [13] that the maximum size of a triangle-free graph of order n is $\lfloor \frac{n^2}{4} \rfloor$. Equality holds for the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Mantel's theorem provides the maximum number of edges that a 2-chromatic graph of order n can have. On the other hand the minimum number of edges in a 2-chromatic graph of order $n \geq 2$ is 1 and the minimum number of edges in a 2-chromatic connected graph of order $n \geq 2$ is $n - 1$. Turán [17] extended the result of Mantel by introducing the *Turán graph.* The famous result of Turán is viewed as the origin of extremal graph theory.

The *Turán graph* $T_{n,r}$ is the complete r-partite graph of order n whose partite sets differ in size by at most 1.

Theorem 4.5 *Among the graphs of order n containing no complete subgraph of order $r + 1$, $T_{n,r}$ has the maximum number of edges.*

In order to apply Turán's theorem in our context, we would like to state the following facts.

1. If $n = rq + t$, $0 \leq t < r$, then $T_{n,r}$ consists of t partite sets of cardinality $\lceil \frac{n}{r} \rceil$ and $r - t$ partite sets of cardinality $\lfloor \frac{n}{r} \rfloor$.
2. Let $G \in \mathbb{J}_m(n)$. If $\omega(G) \leq r$, then $m \leq \varepsilon(T_{n,r})$.
3. $\varepsilon(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

4. Let $t(n,r) = \varepsilon(T_{n,r})$. Then for a fixed n, by using elementary arithmetic, we have $t(n,r-1) < t(n,r)$ for all r, $2 \le r \le n$. In fact $t(n,r) - t(n,r-1) \ge \binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

We have the following theorems.

Theorem 4.6 *Let n, m and k be positive integers satisfying $n \ge k \ge 3$ and $\binom{k}{2} \le m < \binom{k+1}{2}$. Then* $\max(\chi; n, m) = k$.

Proof. By above observation, we see that every k-chromatic graph has at least $\binom{k}{2}$ edges. Thus for n, m and k satisfying conditions of the theorem, $\max(\chi; n, m) \le k$. Observe that if $n = k$, then $m = \binom{k}{2}$. Thus $\max(\chi; k, \binom{k}{2}) = k$. Suppose $n > k$. Since $\binom{k}{2} \le m < \binom{k+1}{2}$, it follows that $m - \binom{k}{2} < \binom{k+1}{2} - \binom{k}{2} = k$. Let G be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{v_iv_j : 1 \le i < j \le k\} \cup \{v_nv_i : 1 \le i \le m - \binom{k}{2}\}$. It is easy to see that $G \in \mathbb{J}_m(n)$ and $\chi(G) = k$. Thus $\max(\chi; n, m) = k$. □

Theorem 4.7 *Let n, m and $k \ge 2$ be positive integers satisfying $t(n, k-1) < m \le t(n,k)$. Then* $\min(\chi; n, m) = k$.

Proof. By definition of $T_{n,r}$ and above observation, we find that if $t(n, k-1) < m \le t(n,k)$, them $\min(\chi; m, n) \ge k$. Let G be a graph of order n obtained from removal $t(n,k) - m$ edges from $T_{n,k}$. Thus $G \in \mathbb{J}_m(n)$ and G is a subgraph of $T_{n,k}$. Therefore $\chi(G) \le \chi(T_{n,k}) = k$. This proves that $\min(\chi; m, n) = k$. □

As we have mentioned earlier, the graph parameters χ and ω are closely related in the sense that $\chi(G) \ge \omega(G)$. However, it is known that for any positive integer t, there exists a graph G such that $\chi(G) - \omega(G) \ge t$. But this situation does not occur from the results of the previous two theorems. The results are still true if we replace χ by ω as state in the following corollaries.

Corollary 4.8 *Let n, m and k be positive integers satisfying $n \ge k$ and $\binom{k}{2} \le m < \binom{k+1}{2}$. Then* $\max(\omega; n, m) = k$.

Corollary 4.9 *Let n, m and k be positive integers satisfying $t(n, k-1) < m \le t(n,k)$. Then* $\min(\omega; n, m) = k$.

Since $\mathbb{C}_m(n) \subseteq \mathbb{J}_m(n)$, one sees that $\min(\chi; m, n) \le \mathrm{Min}(\chi; m, n)$ and $\mathrm{Max}(\chi; m, n) \le \max(\chi; m, n)$. A graph G that we constructed in the proof of Theorem 4.7 contains enough edges to be connected. Thus $\min(\chi; m, n) = \mathrm{Min}(\chi; m, n)$.

Corollary 4.10 *Let n, m and $k \ge 2$ be positive integers satisfying $t(n, k-1) < m \le t(n,k)$. Then* $\mathrm{Min}(\chi; n, m) = k$.

Corollary 4.11 *Let n, m and $k \geq 2$ be positive integers satisfying $t(n, k-1) < m \leq t(n, k)$. Then* $\mathrm{Min}(\omega; n, m) = k$.

Results can be obtained similarly as stated in the following theorems.

Theorem 4.12 *Let n, m and k be positive integers satisfying $n \geq k \geq 3$ and $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - k - 1$. Then* $\mathrm{Max}(\chi; n, m) = k$.

Theorem 4.13 *Let n, m and k be positive integers satisfying $n \geq k \geq 3$ and $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - k - 1$. Then* $\mathrm{Max}(\omega; n, m) = k$.

Thus all extreme values of χ and ω in $\mathbb{J}_m(n)$ and $\mathbb{C}_m(n)$ are obtained in all situations.

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