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Path and cycle decomposition numbers

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Abstract

For a fixed graph H without isolated vertices, the H-decomposition number $d_H(G)$ of a graph G is the minimum number of vertices that must be added to G to produce a graph that can be decomposed into copies of H. In this paper, we find formulas for $d_H(G)$ in the cases where H is a path or a cycle and G is a path or a cycle. We also show a general lower bound which is useful in these cases and conjecture a formula for $d_{P_n}(K_{1,m})$.

1 Introduction

For a fixed graph H without isolated vertices, the H-decomposition number $d_H(G)$ of a graph G is the minimum number of vertices that must be added to G to produce a graph that can be decomposed into copies of H. (Any number of edges may be added incident with the new vertices.) Equivalently, it is $\min(|V(K)|-|V(G)|)$ where K is an H-decomposable graph with induced subgraph G. The H-decomposition number was previously studied in [1], where the authors show that the H-decomposition number is well-defined and give general upper bounds, as well as specific bounds and formulas when H is a path, cycle, or complete graph. In this paper, we present formulas for the H-decomposition number $d_H(G)$ when both H and G are restricted to the class of paths and cycles.

Our first result provides a general lower bound on the H-decomposition number of a graph G. We use $e(G)$ to represent the number of edges in a graph G and $\Delta(G)$ for the maximum degree of G.

Theorem 1. *The H-decomposition number of any graph G satisfies*

$$\left\lceil \frac{e(H)}{\Delta(H)} - \frac{e(G)}{\Delta(H)M} \right\rceil \leq d_H(G),$$

where $M = max\left(\lceil \frac{e(G)}{e(H)} \rceil, \lceil \frac{\Delta(G)}{\Delta(H)} \rceil\right)$.

* This paper is dedicated to the memory of Kevin McDougal.

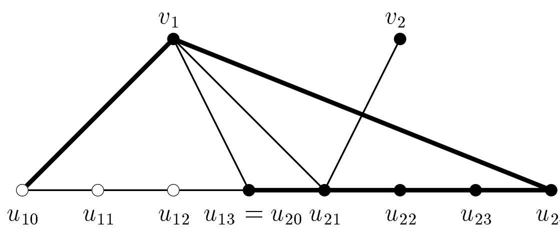

Figure 1: P_8 with the addition of two new vertices and several new edges is decomposed into copies of P_7, illustrating $d_{P_7}(P_8) = 2$.

Proof. Let $d = d_H(G)$.

Let K be a graph with induced subgraph G so that $|V(K)| = |V(G)| + d$ and K can be decomposed into k copies of H. Thus, K has exactly $k \cdot e(H)$ edges.

However, each new vertex in K can be incident with at most $\Delta(H)$ edges from each copy of H. Thus, K has at most $e(G) + \Delta(H)dk$ edges. We have $k \cdot e(H) \leq e(G) + \Delta(H)dk$, so $d \geq \frac{e(H)}{\Delta(H)} - \frac{e(G)}{\Delta(H)k}$. Since we must have enough copies of H to cover every edge of G, $k \geq \left\lceil \frac{e(G)}{e(H)} \right\rceil$. We must also have enough copies of H to cover every edge at a maximum-degree vertex of G, so $k \geq \left\lceil \frac{\Delta(G)}{\Delta(H)} \right\rceil$. Thus, $d \geq \frac{e(H)}{\Delta(H)} - \frac{e(G)}{\Delta(H)M}$. □

If $e(H)$ divides $e(G)$ and $\frac{e(G)}{e(H)} \geq \frac{\Delta(G)}{\Delta(H)}$, then this bound is trivial. In general, however, it is sharp. In the following sections, we show that $d_{P_m}(P_n)$ and $d_{P_m}(C_n)$ achieve this bound when $n > m$, and that $d_{C_m}(P_n)$ achieves the following slightly stronger version of the bound in Theorem 1 for $n \geq m$.

Corollary 1. *The C_m-decomposition number of any graph G that does not contain C_m as a subgraph is at least*

$$\left\lceil \frac{m}{2} - \frac{e(G)}{2\lceil \frac{e(G)}{m-2} \rceil} \right\rceil \leq d_{C_m}(G).$$

This is essentially Theorem 1 except for the denominator of $m - 2$. If the graph G does not contain any copies of C_m, then every copy of C_m in the decomposition will necessarily contain at least one vertex and two edges that are not in G. Thus, each copy uses at most $m - 2$ edges of G.

2 Decomposing Paths into Paths

First we consider $d_{P_m}(P_n)$. Figure 1 illustrates the method and the labelling that will be used in the next theorem with the example $d_{P_7}(P_8) = 2$. We will also need the following observation about the ceiling function.

Observation 1. *For any positive real number x, $\left\lceil \frac{1}{2} \lceil x \rceil \right\rceil = \left\lceil \frac{x}{2} \right\rceil$.*

Proof. Since $\lceil x \rceil \geq x$, we can see that $\left\lceil \frac{1}{2} \lceil x \rceil \right\rceil \geq \left\lceil \frac{x}{2} \right\rceil$. Since $2 \left\lceil \frac{x}{2} \right\rceil$ is an integer that is as least as large as x, we have $2 \left\lceil \frac{x}{2} \right\rceil \geq \lceil x \rceil$. If we divide both sides by 2, we see that $\left\lceil \frac{x}{2} \right\rceil$ is an integer at least as large as $\frac{1}{2} \lceil x \rceil$. The result follows. □

Theorem 2. *For any positive integers n and m with $m \geq 3$, we have*

$$d_{P_m}(P_n) = \begin{cases} \left\lceil \frac{m-1}{2} - \frac{n-1}{2\lceil \frac{n-1}{m-1} \rceil} \right\rceil & \text{if } n > m \\ m - n & \text{if } n \leq m. \end{cases}$$

Proof. When $n \leq m$, we must add enough vertices to complete a single copy of P_m. Thus, we will concentrate on the case $n > m$. The lower bound in this case comes from Theorem 1. We must show that this number can be achieved.

We will divide the $n-1$ edges of P_n into $N = \left\lceil \frac{n-1}{m-1} \right\rceil$ subpaths so that the lengths of any two subpaths differs by at most 1. Let j_i be the number of edges in the ith subpath, $1 \leq i \leq N$, and notice that $j_i \leq m-1$ for each i. Label the vertices of the path, in order, $u_{10}, u_{11}, \ldots, u_{1j_1} = u_{20}, u_{21}, \ldots, u_{2j_2} = u_{0j_3}, \ldots, u_{(N-1)j_{N-1}} = u_{j0}$, $u_{j1}, \ldots, u_{Nj_N}$. Thus, the first subscript indicates which subpath the vertex belongs to and the second subscript indicates which vertex on that subpath; $N-1$ of the vertices belong to two different subpaths and so have two labels.

Let $d = \left\lceil \frac{m-1}{2} - \frac{n-1}{2\lceil \frac{n-1}{m-1} \rceil} \right\rceil$. First, we will add d new vertices $v_1, v_2, \ldots, v_d$. We wish to extend each of the subpaths into a path of length m. For each i, $1 \leq i \leq N-1$, join $u_{ij_i} = u_{(i+1)0}$ to v_1; join v_1 to $u_{(i+1)1}$; join $u_{(i+1)1}$ to v_2; and so forth, until we have a path $u_{i0}, u_{i1}, \ldots, u_{ij_i}, v_1, u_{(i+1)1}, v_2, u_{(i+1)2}, v_3, \ldots$ of length m for each i. Similarly, we join u_{Nj_N} to v_1, join v_1 to u_{11}, join u_{11} to v_2, and so forth, to extend the last subpath to a path of length m. We must check that both d and j_{i+1}, for each i, are large enough so that we can, in fact, extend each path to length m.

There are at least $\left\lfloor \frac{n-1}{\lceil \frac{n-1}{m-1} \rceil} \right\rfloor$ edges in each subpath, so at most $m-1-\left\lfloor \frac{n-1}{\lceil \frac{n-1}{m-1} \rceil} \right\rfloor = \left\lceil m-1-\frac{n-1}{\lceil \frac{n-1}{m-1} \rceil} \right\rceil$ additional edges are required for each path. For every two new edges needed on each path, we need one new vertex v_i. Since $d = \left\lceil \frac{m-1}{2} - \frac{n-1}{2\lceil \frac{n-1}{m-1} \rceil} \right\rceil$, it follows (see Observation 1) that there are enough new vertices to extend each subpath into a copy of P_m.

We will refer to the vertices $u_{i1}, u_{i2}, \ldots, u_{i(j_i-1)}$ as the *internal vertices* of the ith subpath. In order to extend the ith subpath as described above, we need at least $\left\lceil \frac{m-1}{2} - \frac{n-1}{2\lceil \frac{n-1}{m-1} \rceil} \right\rceil$ internal vertices on the $i+1$st subpath. The minimum number of vertices internal to any subpath is $\left\lfloor \frac{n-1}{\lceil \frac{n-1}{m-1} \rceil} - 1 \right\rfloor$. Thus, we need $\left\lfloor \frac{n-1}{\lceil \frac{n-1}{m-1} \rceil} - 1 \right\rfloor \geq \left\lceil \frac{m-1}{2} - \frac{n-1}{2\lceil \frac{n-1}{m-1} \rceil} \right\rceil$. With some algebra, we find that this inequality holds whenever

$\left\lceil \frac{n-1}{m-1} \right\rceil (m+3) \le 3(n-1)$. For $n > m \ge 5$ and $m \le n-2$, we have

$$\begin{aligned} \left\lceil \frac{n-1}{m-1} \right\rceil (m+3) &= \left\lceil \frac{n-1}{m-1} \right\rceil (m-1) + \left\lceil \frac{n-1}{m-1} \right\rceil (4) \\ &\le n-1+m-2+\left\lceil \frac{n-1}{m-1} \right\rceil (4) \\ &\le n-1+m+\left(\left\lceil \frac{n-1}{4} \right\rceil (4) - 2 \right) \\ &\le 3(n-1). \end{aligned}$$

For $m = n-1$, the inequality $\left\lfloor \frac{n-1}{\lceil \frac{n-1}{m-1} \rceil} - 1 \right\rfloor \ge \left\lceil \frac{m-1}{2} - \frac{n-1}{2\lceil \frac{n-1}{m-1} \rceil} \right\rceil$ becomes $\left\lfloor \frac{n-3}{2} \right\rfloor \ge \left\lceil \frac{n-3}{4} \right\rceil$. This inequality holds for $n = 5, 6, 7$ and, since $\left\lfloor \frac{n-3}{2} \right\rfloor \ge \frac{n-4}{2} \ge \frac{n}{4} \ge \left\lceil \frac{n-3}{4} \right\rceil$, for $n \ge 8$. We can check that $d_{P_3}(P_n)$ is 0 when n is odd and 1 when n is even, and $d_{P_4}(P_n)$ is 0 when 3 divides $n-1$ and 1 otherwise. $\square$

For a given n, the value of m that produces the largest decomposition number is $m = n-1$. In this case, our formula becomes $d_{P_{n-1}}(P_n) = \left\lceil \frac{n-3}{4} \right\rceil$. In particular, path into path decomposition numbers can be arbitrarily large. On the other hand, for n sufficiently large relative to m, we will show (in Theorem 3) that $d_{P_m}(P_n) \le 1$. First, we need the following number-theoretic lemma.

Lemma 1. *Let a be a positive integer. Let S be the set of nonnegative linear combinations of a, $a+1$, and $a+2$, that is,*

$$S = \{xa + y(a+1) + z(a+2) \mid x, y, z \text{ are non-negative integers}\}.$$

Then the largest integer not in S is

$$N(a) = \begin{cases} \frac{(a-2)(a+1)}{2} & \text{if } a \text{ is odd} \\ \frac{a^2-2}{2} & \text{if } a \text{ is even.} \end{cases}$$

Proof. Observe that if $n \in S$ for some positive integer n, then $n + ka \in S$ for every nonnegative integer k. This observation leads us to consider residue classes of integers modulo a. Let

$$R_a(n) = \{\ldots, n-2a, n-a, n, n+a, n+2a, n+3a, \ldots\}$$

be the *residue class modulo a* of the integer n. Notice that $n(a+1) \in S \cap R_a(n)$ for any integer n, so $S \cap R_a(n)$ is non-empty. Let $s(n)$ denote the least non-negative integer in the residue class $R_a(n)$ that is a member of S. Certainly the smallest member of $R_a(0)$ contained in S is 0. Note that

$$xa + y(a+1) + z(a+2) \equiv y + 2z \mod a.$$

Therefore, for $n = 1, 2, \ldots, (a-1)$, the least positive integer in $R_a(n)$ that is a member of S is $s(n) = y(a+1) + z(a+2) = (y+z)a + (y+2z)$ where y and z are chosen so that $y + 2z = n$ and $y + z$ is as small as possible. The values of y and z that satisfy these conditions are

$$\begin{array}{ll} y = 1, z = \frac{n-1}{2} & \text{if } n \text{ is odd} \\ y = 0, z = \frac{n}{2} & \text{if } n \text{ is even.} \end{array}$$

With these values for y and z, the value of $s(n) = (y + z)a + (y + 2z)$ is strictly increasing with n. Therefore the largest of these minimal values is $s(a - 1)$ which computes to

$$s(a-1) = \begin{cases} 1(a+1) + \frac{a-2}{2}(a+2) & \text{when } a \text{ is even} \\ 0(a+1) + \frac{a-1}{2}(a+2) & \text{when } a \text{ is odd} \end{cases}$$

or

$$s(a-1) = \begin{cases} \frac{a^2+2a-2}{2} & \text{when } a \text{ is even} \\ \frac{(a-1)(a+2)}{2} & \text{when } a \text{ is odd.} \end{cases}$$

The largest integer that is not a member of S will be $N(a) = s(a - 1) - a$, which simplifies to the formula in the assertion. □

Remark 1. *The conclusion of Lemma 1 is valid when the set S is restricted by requiring $y = 0$ or $y = 1$.*

Proof. This restriction on y leaves the set S unchanged since the number $n = xa + y(a + 1) + z(a + 2)$ may be written

$$n = \begin{cases} (x + \frac{y}{2})a + 0(a+1) + (z + \frac{y}{2})(a+2) & \text{when } y \text{ is even} \\ (x + \frac{y-1}{2})a + 1(a+1) + (z + \frac{y-1}{2})(a+2) & \text{when } y \text{ is odd.} \end{cases}$$

□

Now it can be shown that for n sufficiently large relative to m, the decomposition number $d_{P_m}(P_n)$ is at most 1.

Theorem 3. *Let n and m be positive integers with $n > m > 3$. Define*

$$B(m) = \begin{cases} \frac{(m-3)(m-4)}{2} & \textit{if } m \textit{ is even} \\ \frac{(m-3)^2}{2} & \textit{if } m \textit{ is odd.} \end{cases}$$

If $n > B(m)$, then $d_{P_m}(P_n) \le 1$. If $n = B(m)$, then $d_{P_m}(P_n) = 1$.

Proof. Suppose we have a decomposition of P_n into copies of P_m in which only one new vertex is added. Each copy of P_m has at most 2 edges not on the path P_n, and hence exactly $m - 1$, $m - 2$, or $m - 3$ edges on P_n. Conversely, if the $n - 1$ edges of P_n can be partitioned into subpaths of length $m - 1$, $m - 2$ and $m - 3$, then P_n can be decomposed into copies of P_m using only one additional vertex. We can add

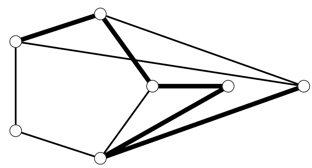

Figure 2: Example of construction for $n < m$, showing $d_{P_6}(C_5) = 2 = \lceil \frac{1}{2}(6-1) - \frac{1}{4}5 \rceil$.

the new edge to the right end of each subpath. For a subpath of length $m-3$, an additional path vertex is needed, but we can use an internal vertex of some other subpath. In the case $m=4$, there is at most one subpath of length $m-3=1$ and at least one of length $m-2=2$.

Thus, $d_{P_m}(P_n) \leq 1$ if and only if there exist non-negative integers x, y, and z such that $n-1 = x(m-1) + y(m-2) + z(m-3)$. From Lemma 1, the smallest integer $n-1$ that cannot be written in the form $n-1 = x(m-1)+y(m-2)+z(m-3)$ is $N(m-3)$. Thus, we know $d_{P_m}(P_n) \leq 1$ for $n-1 \geq N(m-3)+1$ which simplifies to the formula given. $\square$

3 Decomposing Cycles into Paths

Theorem 4. *For any positive integers n and m with $m \geq 3$, we have*

$$d_{P_m}(C_n) = \begin{cases} \left\lceil \frac{m-1}{2} - \frac{n}{2\lceil \frac{n}{m-1} \rceil} \right\rceil & \text{if } n \geq m \\ max(\lceil \frac{1}{2}(m-1) - \frac{1}{4}n \rceil, m-n) & \text{if } n < m. \end{cases}$$

Proof. In the case $n < m$, the cycle C_n should be divided into two subpaths of equal or nearly equal length. Each subpath can be extended by alternating new vertices with internal vertices of the other subpath. The shorter subpath has $\lfloor \frac{n}{2} \rfloor$ edges and needs $m - 1 - \lfloor \frac{n}{2} \rfloor = \lceil m - 1 - \frac{n}{2} \rceil$ more. Since we can add at most two new edges to this subpath for each new vertex, we need at least $\lceil \frac{1}{2} \lceil m - 1 - \frac{n}{2} \rceil \rceil = \lceil \frac{1}{2}(m-1) - \frac{1}{4}n \rceil$ new vertices (see Observation 1). However, each path will need m vertices total, including both new vertices and internal vertices of the other subpath, so we must also have at least $m-n$ new vertices.

We concentrate on the case $n \geq m$. The lower bound follows from Theorem 1. We must show that this bound can be achieved. The n edges of the cycle are divided into $\lceil \frac{n}{m-1} \rceil$ subpaths of nearly equal length less than m, and then these subpaths are extended using the new vertices and the internal vertices of the next subpath clockwise around the cycle alternately. The proof that there are enough new vertices and enough internal vertices in the next path clockwise to extend each subpath is very similar to the proof in Theorem 2 and therefore omitted. $\square$

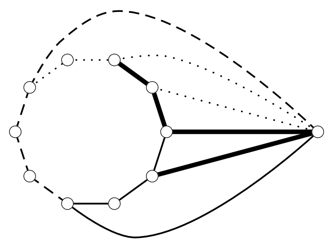

Figure 3: Example of construction for $n \geq m$, showing $d_{P_5}(C_{10}) = 1 = \left\lceil \frac{5-1}{2} - \frac{10}{2\lceil \frac{10}{5-1} \rceil} \right\rceil$.

4 Decomposing Paths into Cycles

Next, we will consider a formula for $d_{C_m}(P_n)$. First, we need a lemma about partitioning an integer into a sum of integers with desired properties. This lemma will be useful in dividing the path P_n into subpaths to form copies of C_m.

Lemma 2. *Let a and b be positive integers with $a > b$, and let q and r be positive integers so that $a = qb + r$, $0 \leq r \leq b - 1$. Then there is a partition of a into a sum of $q + 1$ positive integers, $a = j_1 + j_2 + \ldots + j_{q+1}$, so that each of the following holds:*

- *for $i < k$, $j_i \leq j_k$;*
- *for every i and k, $|j_k - j_i| \leq 2$;*
- *there is at most one i so that $j_i \not\equiv b$ modulo 2.*

Proof. Notice that $q \geq 1$ so $q + 1 \geq 2$. If the number $(b - r) - 2(q + 1) \left\lfloor \frac{b-r}{2(q+1)} \right\rfloor$ is even, it can be written as $2x$ for some integer x where $0 \leq x \leq q + 1$. In this case, we define

$$j_i = b - 2\left\lfloor \frac{b-r}{2(q+1)} \right\rfloor - 2 \qquad \text{for } 1 \leq i \leq x$$
$$j_i = b - 2\left\lfloor \frac{b-r}{2(q+1)} \right\rfloor \qquad \text{for } x + 1 \leq i \leq q + 1.$$

Notice that $j_1 + j_2 + \ldots + j_{q+1} = (q + 1)b - 2(q + 1) \left\lfloor \frac{b-r}{2(q+1)} \right\rfloor - 2x = qb + r = a$ and each $j_i \equiv b$ modulo 2.

Otherwise, $(b - r) - 2(q + 1) \left\lfloor \frac{b-r}{2(q+1)} \right\rfloor$ is odd and can be written as $2x + 1$ for some integer x where $0 \leq x \leq q$. In this case, we define

$$j_i = b - 2\left\lfloor \frac{b-r}{2(q+1)} \right\rfloor - 2 \qquad \text{for } 1 \le i \le x$$
$$j_i = b - 2\left\lfloor \frac{b-r}{2(q+1)} \right\rfloor - 1 \qquad \text{for } i = x+1$$
$$j_i = b - 2\left\lfloor \frac{b-r}{2(q+1)} \right\rfloor \qquad \text{for } x+2 \le i \le q+1,$$

and notice that $j_1 + j_2 + \ldots + j_{q+1} = (q+1)b - 2(q+1)\left\lfloor \frac{b-r}{2(q+1)} \right\rfloor - 2x - 1 = qb + r = a$. Each $j_i \equiv b$ modulo 2 except for j_{x+1}. □

For decomposing paths into cycles, we have the following formula. The construction in the proof also satisfies some additional conditions that will be useful in Section 5, when we decompose cycles into cycles.

Theorem 5. *For any positive integers m and n with $n \ge 2$ and $m \ge 3$, the decomposition number*

$$d_{C_m}(P_n) = \begin{cases} max\left(\left\lceil \frac{m}{2} - \frac{n-1}{2\lceil \frac{n-1}{m-2}\rceil} \right\rceil, 2\right) & \text{if } n \ge m \\ m - n & \text{if } n < m. \end{cases}$$

Furthermore, there is a C_m-decomposable graph G with order $n + d_{C_m}(P_n)$ and induced subgraph P_n such that the endpoints of P_n have degree 2 in G. If $d_{C_m}(P_n) > 2$ and $n \ge 4$, then G can be constructed so that the endpoints of P_n have no common neighbors.

Proof. For $n < m$, we must add enough vertices to form a single copy of C_m, so $d_{C_m}(P_n) = m - n$.

In the case $n \ge m$, we must decompose P_n into at least two different cycles C_m. Some vertex of P_n must lie on more than one cycle, and hence be adjacent to more than one new vertex. Thus, $d_{C_m}(P_n) \ge 2$ in this case. From Corollary 1, we have $d_{C_m}(P_n) \ge \left\lceil \frac{m}{2} - \frac{n-1}{2\lceil \frac{n-1}{m-2}\rceil} \right\rceil$. We proceed by showing that this lower bound can be achieved with specific conditions on n and m and then explore the remaining cases individually.

Claim 1. *The formula holds for $m \ge 5$, $n \ge m$, and $\frac{n-1}{\lceil \frac{n-1}{m-2}\rceil} \ge \lceil \frac{m+6}{3} \rceil$.*

Let $d = max\left(\left\lceil \frac{m}{2} - \frac{n-1}{2\lceil \frac{n-1}{m-2}\rceil} \right\rceil, 2\right)$. We will produce the graph G as follows. Begin with P_n and a set U of d additional vertices, $U = \{v_1, v_2, \ldots, v_d\}$. Partition the $n-1$ edges of the path P_n into $N = \lceil \frac{n-1}{m-2} \rceil$ subpaths with lengths $j_1, j_2, \ldots, j_N$, as described in Lemma 2, so that, for any $1 \le i < k \le N$, $j_i \le j_k \le j_i + 2$ and there is at most one i so that $j_i \not\equiv m$ modulo 2. Label the vertices of P_n, in order, $u_{10}, u_{11}, \ldots, u_{1j_1} = u_{20}, u_{21}, \ldots, u_{2j_2} = u_{30}, u_{31}, \ldots, u_{(N-1)j_{N-1}} = u_{N0}, u_{N1}, \ldots, u_{Nj_N}$.

For i odd, $1 \leq i \leq N$, add the edges $u_{i0}v_1$ and $u_{ij_i}v_1$, so the endpoints of the ith path are each joined to v_1. Let C_i be the cycle $u_{i0}, u_{i1}, \ldots, u_{ij_i}, v_1, u_{i0}$ on $j_i + 2 \leq m$ vertices. Similarly, for i even, add the edges $u_{i0}v_2$ and $u_{ij_i}v_2$. Let C_i be the cycle $u_{i0}, u_{i1}, \ldots, u_{ij_i}, v_2, u_{i0}$ on $j_i + 2 \leq m$ vertices. We now have a graph G that can be decomposed into $\left\lceil \frac{n-1}{m-2} \right\rceil$ cycles $C_1, C_2, \ldots, C_N$ of length at most m. We must adjust G, and the cycles $C_1, C_2, \ldots, C_N$, so that each cycle has length exactly m.

First we will describe the construction, then we will check that d and j_i, for $1 \leq i \leq N$, are large enough for this construction. Let $x = |\{C_i | 1 \leq i \leq N \text{ and order}(C_i) < m\}|$. If x is odd and $m - (j_1 + 2) = 2$, we will remove the edge $u_{10}v_1$ and replace it with the edges $u_{10}v_2$, v_2u_{21}, and $u_{21}v_1$. If x is odd and $m - (j_1 + 2) = 4$, we will replace the edge $u_{10}v_1$ with the edges $u_{10}v_2$, v_2, u_{21}, $u_{21}v_3$, v_3u_{22}, and $u_{22}v_1$. Similarly, for any odd value of x and any even value of $m-(j_1+2)$, we can adjust the cycle C_1 so that it has m vertices.

If $m - (j_1 + 2) = 1$, we replace the edge $u_{10}v_1$ with edges $u_{10}v_2$ and v_2v_1. If $m - (j_1 + 2)$ is odd and greater than one, we can adjust the cycle just as in the case when $m - (j_1 + 2)$ is even to produce a cycle of length $m + 1$, then replace the edges v_1u_{21} and $u_{21}v_2$ with the edge v_1v_2.

We may assume, then, that there are an even number of cycles C_i remaining with fewer than m vertices. Based on our construction, we may also assume that they appear in pairs C_i and C_{i+1} that share a vertex $u_{ij_i} = u_{(i+1)0}$. Without loss of generality, suppose currently the edge $u_{ij_i}v_1$ is used in cycle C_i and the edge $u_{(i+1)0}v_2$ is used in cycle C_{i+1}. We will replace the edge $u_{ij_i}v_1$ with the edges $u_{ij_i}v_2$, $v_2u_{(i+1)1}$, $u_{(i+1)1}v_3$,$v_3u_{(i+1)2}, \ldots$, joining the last vertex to v_1 to produce a new cycle for C_i with exactly m vertices. At the same time, we will replace the edge $u_{(i+1)0}v_2$ with the edges $u_{(i+1)0}v_1, v_1u_{(i+2)1}, u_{(i+2)1}v_3, v_3u_{(i+2)2}, \ldots$, joining the last vertex to v_2 to produce a new cycle for C_{i+1} that has exactly m vertices. (If necessary, take $i + 2$ modulo N.) Since there is only one cycle C_i that initially had parity j_i different from m, at most one of the new cycles will use an edge of the form v_kv_p. All other new edges added to each cycle are incident with different vertices of P_n, so we have not added any new edge twice.

The total number of edges missing from all of these cycles is $M = (m-2) \left\lceil \frac{n-1}{m-2} \right\rceil - (n - 1)$. Since we maintained the same parity as m in all but one of the cycles, we can think of these missing edges in pairs. Each cycle is missing at most $\left\lceil \frac{M/2}{\left\lceil \frac{n-1}{m-2} \right\rceil} \right\rceil$ *pairs* of edges. Since $d \geq \frac{m}{2} - \frac{n-1}{2\left\lceil \frac{n-1}{m-2} \right\rceil}$, it follows that no cycle is missing more than $d - 1$ pairs of edges.

The expansion of each cycle C_i also uses vertices $u_{i+11}, u_{i+12}, \ldots, u_{i+1j_{i+1}-1}$ from cycle C_{i+1}. We will refer to these vertices as the *internal path vertices* of C_{i+1}. We must check that each cycle C_{i+1} has enough internal path vertices for its neighbor C_i to use. Let p be the minimum number of internal path vertices on any one cycle, where the minimum is taken over the cycles C_i, $1 \leq i \leq N$. Taking into account the parity restrictions and the two vertices of each cycle that are joined to v_1 or v_2, we still have $p \geq \left\lfloor \frac{n-1}{\left\lceil \frac{n-1}{m-2} \right\rceil} \right\rfloor - 3$. The maximum number of internal-path vertices that must be added to any one cycle C_i is at most $\lfloor \frac{m-p-3}{2} \rfloor$, since the cycle C_i has $p+3$ vertices already and internal-path vertices will alternate with new vertices. Thus, we need $p \geq \lfloor \frac{m-p-3}{2} \rfloor$, or equivalently, $p \geq \frac{m-3}{3}$. It would suffice if $\frac{n-1}{\lceil \frac{n-1}{m-2} \rceil} - 3 \geq \lceil \frac{m-3}{3} \rceil$, or $\frac{n-1}{\lceil \frac{n-1}{m-2} \rceil} \geq \lceil \frac{m+6}{3} \rceil$.

In this construction, the endpoints u_{10} and u_{Nj_N} of P_n each have degree 2 in G. Suppose $d > 2$ and u_{10} and u_{Nj_N} share a common neighbor v_i not on P_n. We would like to replace v_i in the cycle C_N with some other v_j, $j \neq i$. If $v_j \notin C_N$, we can simply replace v_i with v_j in C_N; otherwise, we may swap the roles of v_i and v_j in this cycle. The potential problem is that u_{N0} is adjacent to a vertex v_k in C_{N-1} and a vertex v_m in C_N, so the swap might create a double edge at u_{N0} if $\{k,m\} = \{i,j\}$. If u_{N0} is not adjacent to v_i, there is no difficulty. In the above construction, u_{N0} is adjacent to only two vertices of U and $|U| \geq 3$. Thus, if u_{N0} is adjacent to v_i, then we may choose v_j, $j \neq i$, so that u_{N0} is not adjacent to v_j. Thus, we have a construction in which u_{10} and u_{Nj_N} have no common neighbor.

Claim 2. *The formula holds for $n \geq m$ and $m \geq 22$.*

With a bit of algebra, we can show that if $n \geq \frac{(m+8)(m-3)}{2(m-7)} + 1$, then $\frac{n-1}{\lceil \frac{n-1}{m-2} \rceil} \geq \lceil \frac{m+6}{3} \rceil$. Since $m \geq 22$, we have $\frac{m+8}{2(m-7)} \leq 1$. Since $n \geq m$, it follows that $n \geq \frac{(m+8)(m-3)}{2(m-7)} + 1$.

Claim 3. *The formula holds for $n \geq m+9$ and $m \geq 11$.*

In this case, $\frac{m-3}{2(m-7)} \leq 1$, so again we have $n \geq \frac{(m+8)(m-3)}{2(m-7)} + 1$ and hence $\frac{n-1}{\lceil \frac{n-1}{m-2} \rceil} \geq \lceil \frac{m+6}{3} \rceil$.

If we wish to decompose P_n into copies of C_3, each edge of P_n must be used in a different copy of C_3, and each copy of C_3 will use exactly one new vertex. If P_n has vertices $u_1, u_2, u_3, \ldots, u_n$, we can add new vertices v_1 and v_2. Then for i odd, $0 \leq i \leq n-1$, u_i and u_{i+1} can be joined to v_1 to form a 3-cycle u_i, u_{i+1}, v_1, u_i. For i even, $1 \leq i \leq n-1$, u_i and u_{i+1} can be joined to v_2. Thus, $d_{C_3}(P_n) = 2$ for all $n \geq 3$.

To decompose $P_n : u_1, u_2, \ldots, u_n$ into copies of C_4, we add new vertices v_1 and v_2. For $i \equiv 1(\mod 4)$, $1 \leq i \leq n-2$, we join u_i and u_{i+2} to v_1 to form a cycle $u_i, u_{i+1}, u_{i+2}, v_1, u_i$. Similarly, for $i \equiv 3(\mod 4)$, $1 \leq i \leq n-2$, we join u_i and u_{i+2} to v_2. If n is odd, then we have P_n decomposed into copies of C_4. Otherwise, if $n-1 \equiv 1(\mod 4)$, add edges $u_{n-1}v_1$,v_1v_2, and v_2u_n to form the cycle $u_{n-1}, v_1, v_2, u_n, u_{n-1}$. If $n-1 \equiv 3(\mod 4)$, add edges $u_{n-1}v_2$, v_2v_1, and v_1u_n. We have $d_{C_4}(P_n) = 2$ for all $n \geq 4$.

If $n = k(m-2)+1$ for some integer k, then the number of edges in P_n is divisible by $m-2$. We may again add two new vertices v_1 and v_2; join u_1 and u_{m-1} to v_1, u_{m-1} and u_{2m-3} to v_2, and so forth. Therefore, $d_{C_m}(P_n) = 2$.

If $n = k(m-2)$ for some integer k, then we claim $d_{C_m}(P_n) = 2$. If we use the same construction as above, there are $m-3$ edges $u_{n-m+3}u_{n-m+4}$, $u_{n-m+4}u_{n-m+5}$, $\ldots$, $u_{n-1}u_n$ left over. We may add the edges $u_{n-m+3}v_1$, v_1v_2, and v_2u_n, or the edges $u_{n-m+3}v_2$, v_2v_1, and v_1u_n, depending on whether u_{n-m+3} is already adjacent to v_1 or v_2.

If $n = k(m-2)-1$, then our earlier construction with two new vertices v_1 and v_2 will leave $m-4$ edges $u_{n-m+4}u_{n-m+5}$, $u_{n-m+5}u_{n-m+6}, \ldots, u_{n-1}u_n$ left over. As long as $k \geq 2$ and $m \geq 5$, we may add edges $u_{n-m+4}v_1$, v_1u_{n-m+3}, $u_{n-m+3}v_2$, and v_2u_n, assuming (without loss of generality) that u_{n-m+4} is not already adjacent to v_1.

Similarly, if $n = k(m-2)-2$, $k \geq 2$, and $m \geq 5$, we may add two new vertices v_1 and v_2 and edges u_1v_1, $u_{m-1}v_1$, $u_{m-1}v_2$, $u_{2m-3}v_2$, $u_{2m-3}v_1$, $u_{3m-5}v_1$, etc., to form cycles of length m with $2m-7$ edges $u_{n-2m+7}u_{n-2m+8}$, $u_{n-2m+8}u_{n-2m+9}, \ldots$, $u_{n-1}u_n$ left over on the path. Assume without loss of generality that u_{n-2m+7} is not already adjacent to v_1. Then we may add the edges $u_{n-2m+7}v_1, v_1u_{n-1}, u_{n-1}v_2, v_2u_{n-m+3}$ to create one m-cycle and the edges $u_{n-m+3}v_1, v_1v_2, v_2u_n$ to create another m-cycle. Thus, $d_{C_m}(P_n) = 2$.

Similar constructions show $d_{C_m}(P_n) = 2$ for $n = k(m-2)-3$, $k \geq 2$, and $m \geq 6$. Taken together, these cases show that $d_{C_m}(P_n) = 2$ for $n \geq m$ and $m \leq 7$.

Furthermore, algebra shows that $\frac{n-1}{\lceil \frac{n-1}{m-2} \rceil} \geq \lceil \frac{m+6}{3} \rceil$ for $m = 8$ and $n \geq 26$, for $m = 9$ and $n \geq 16$, and for $m = 10$ and $n \geq 22$. There are only a finite number of remaining open cases, namely $m = 8$, $n = 8, 14, 20$; $m = 9$, $n = 9, 10$; $m = 10$, $n = 10, 11, 12, 18, 19, 20$; $m = 11$, $n = 11, 12, 13, 14$; $n = 12$, $m = 12, 13, 14, 15, 16$; $m = 13$, $13 \leq n \leq 18$; $m = 14$, $14 \leq n \leq 20$; $m = 15$, $15 \leq n \leq 22$; and $16 \leq m \leq 21$ with $m \leq n \leq m+8$. Each of these remaining cases is straightforward and has been checked individually. We leave out those details. □

Once again, when n is sufficiently large relative to m, the decomposition number is at most 2. We have the following result.

Theorem 6. *Let n and m be positive integers with $n > m > 4$. Define*

$$f(m) = \begin{cases} \frac{(m-5)(m-4)}{2} & \text{if } m \text{ is odd} \\ \frac{(m-4)^2}{2} & \text{if } m \text{ is even.} \end{cases}$$

Then $d_{C_m}(P_n) \leq 2$ whenever $n > f(m)$. Furthermore, if $m \leq 9$ and $n = f(m)$, then $d_{C_m}(P_n) = 2$.

Proof. The situation is similar to Theorem 3. We have $d_{C_m}(P_n) \leq 2$ if and only if the path P_n can be divided into subpaths of length $m-2$, $m-3$, and $m-4$, with at most one subpath of length $m-3$. Notice that to extend a subpath of length $m-3$ to a cycle of length m, we must use both new vertices and an edge between them. Equivalently, $d_{C_m}(P_n) \leq 2$ if and only if there exist non-negative integers x, y, and z such that $n-1 = x(m-2) + y(m-3) + z(m-4)$ where $y = 0$ or 1. From Lemma 1, the smallest integer that is not of the form $x(m-2) + y(m-3) + z(m-4)$ for some non-negative integers x, y, and z is $N(m-4)$. Therefore, the smallest n for which there does not exist a decomposition of P_n into copies of C_m with at most two additional vertices is $f(m) = N(m-4)+1$, which simplifies to the formula given. □

5 Decomposing Cycles into Cycles

Next we consider $d_{C_m}(C_n)$. Notice that for $n > m$, we necessarily have $d_{C_m}(C_n) \geq 2$. Any decomposition requires at least two cycles, and some vertex of C_n which appears in two cycles C_m must be incident with two new edges. In the next lemma, we explore the relationship between the decomposition numbers of paths and of cycles.

Lemma 3. *For any positive integers m and n, we have*

$$d_{C_m}(P_{n+1}) \leq d_{C_m}(C_n) \leq d_{C_m}(P_{n+1}) + 1$$

and furthermore, if $d_{C_m}(P_{n+1}) \geq 3$, then $d_{C_m}(C_n) = d_{C_m}(P_{n+1})$.

Proof. Let K be a graph with C_n as an induced subgraph, so that $|V(K)| = |V(C_n)| + d_{C_m}(C_n)$, and K can be decomposed into copies of C_m. Then some vertex u in the induced subgraph C_n has the property that its neighbors in C_n are not in the same copy of C_m. We can replace this vertex with two vertices u' and u'' and divide the edges incident with u among these two vertices to produce a graph K with induced subgraph P_{n+1} that can be decomposed into copies of C_m. Thus, $d_{C_m}(P_{n+1}) \leq d_{C_m}(C_n)$.

On the other hand, if we begin with a graph K with induced subgraph P_{n+1} that can be decomposed into copies of C_m, then we wish to associate the endpoints of P_{n+1} to produce a graph for C_n. The only difficulty is that the endpoints may share common neighbors. By Theorem 5, we may assume that the endpoints share at most one common neighbor. We may add one additional vertex to take the place of one of these common neighbors in one of the copies of C_m. If $d_{C_m}(P_{n+1}) > 2$, then we may assume that the endpoints have no common neighbor, so no additional vertex is needed. □

Thus, aside from the case $d_{C_m}(C_n) = 2$, the problem of determining $d_{C_m}(C_n)$ reduces to the problem of determining $d_{C_m}(P_{n+1})$.

Notice that for $n > 3$, $d_{C_3}(C_n)$ is 2 when n is even and 3 when n is odd. Each edge of C_n appears in a different copy of C_3. The remaining vertex of each C_3 must be different for adjacent edges, so the decomposition is analogous to an edge coloring, where the new vertices are the colors.

Theorem 7. *For positive integers n and m, with $n > m \geq 4$, we have $d_{C_m}(C_n) = 2$ if and only if n can be written in the form $n = 2(m-2)k + (m-3)p + (m-4)l$ where k, p, and l are nonnegative integers with $k + l \geq 1$ and p is equal to either 0 or 1.*

Proof. Suppose K is a graph with induced subgraph C_n so that $V(K) = V(C_n) \cup \{a, b\}$ and K can be decomposed into copies of C_m. The edges of C_n are partitioned by the cycles C_m. A cycle C_m may consist of $m-2$ edges of C_n, one of the vertices a or b, and two edges incident with that vertex; $m-3$ of the edges of C_n, both of the vertices a and b, an edge incident with each of them, and the edge in between them; or $m-4$ of the edges of C_n along with both a and b and two edges incident with each of them. Since there is only one edge ab, there can be at most one cycle using exactly $m-3$ of the edges of C_n; hence, p is either 0 or 1.

Each of the cycles that use either $m-3$ or $m-4$ edges of C_n will use either one or two strings of edges in C_n with one endpoint of each string adjacent to a and the other adjacent to b. Each cycle that uses $m-2$ edges of C_n will use a string of edges with both endpoints adjacent to a or both endpoints adjacent to b.

Suppose we fix a particular direction to travel around the cycle C_n. Then we may call a cycle C_m that uses $m-2$ edges of C_n an *a-cycle* if it includes vertex a and a *b cycle* if it includes vertex b. A cycle C_m that uses $m-3$ or $m-4$ edges of C_n will be an *ab-cycle* if the first vertex of this cycle that we reach, when traveling around C_n, is adjacent to a and the last vertex of this cycle on C_n is adjacent to b. Similarly, a cycle C_m that uses $m-3$ or $m-4$ edges of C_n might be an *ba-cycle*. We cannot have multi-edges incident with vertex a or b. Thus, every ab-cycle must be immediately followed by either an a-cycle or another ab-cycle, and every ba-cycle must be followed by either a b-cycle or another ba-cycle.

It follows that we must have an even number of cycles C_m that use $m-2$ edges of C_n. Thus, if $d_{C_m}(C_n)=2$, we must have $n=2(m-2)k+(m-3)p+(m-4)l$ as required.

Now, suppose n can be written in the desired form. Then the cycles C_m described above may be arranged around the cycle C_n to create the desired decomposition. We must ensure that the additional vertex needed for each cycle that uses $m-4$ edges of C_n is available. If $m\geq 6$, then each cycle using $m-4$ edges of C_n has at least two consecutive edges on C_n. Each such cycle may use an internal vertex of one other such cycle. If $m=5$, then we may assume without loss of generality that $l\leq 5$. If $k\geq 2$ or if $k=1$ and $p=1$ or if $k=1$ and $l\leq 4$, then there are enough internal vertices available on C_n.

The remaining cases are $m=5$, $k=1$, $p=0$, $l=5$; $m=5$, $k=0$, $p=0,1$, $l=0,1,2,3,4,5$; $m=4$. When $m=4$, we have $n=4k+p$ where $p=0$ or 1. If $n=4k$, then C_n may be divided into an even number of subpaths of length 2. These subpaths may be joined with either vertex a or vertex b alternately to create cycles of length 4. If $n=4k+1$, then C_n can be similarly divided into subpaths, with one subpath of length 1. This subpath can be joined to both a and b and the edge ab to create a cycle of length 4. If $m=5$ and $k=0$, then since $m<n$ we must have either $n=7$, $p=1$, $l=5$ or $n=6$, $p=1$, $l=5$. The graph C_6 may be divided into two subpaths of length 4; we can join one to vertex a and the other to vertex b to create the 5-cycles. The graph C_7 may be divided into two subpaths of length 4 and one of length 1. One subpath of length 4 will be joined to a, the other to b. The subpath of length 1 will be joined to both a and b, which in turn will be joined to an internal vertex of a different subpath. For $m=5$, $k=1$, $p=0$ and $l=5$, we have $n=11$. One possible decomposition for this graph is shown in Figure 2.

□

6 Remark

Since any path P_n requires n vertices, it follows that $d_{P_n}(K_{1,m})\geq n-(m+1)$. In the attainment of $d_{P_n}(K_{1,m})$, no P_n can consist of only existing edges between vertices

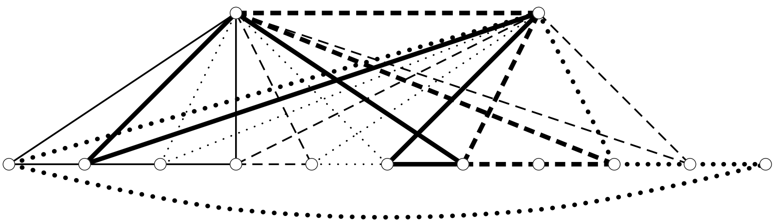

Figure 4: The cycle C_{11} with two additional vertices, decomposed into copies of C_5.

of $K_{1,m}$, so one sees that

$$d_{P_n}(K_{1,m}) \geq \begin{cases} \left\lceil \frac{n-3}{2} \right\rceil = n - (1 + \left\lceil \frac{n}{2} \right\rceil) & \text{when } m \text{ is even} \\ \left\lceil \frac{n-3}{2} \right\rceil = n - (1 + \left\lfloor \frac{n}{2} \right\rfloor) & \text{when } m \text{ is odd.} \end{cases}$$

Combining this information, we conjecture that

$$d_{P_n}(K_{1,m}) = \begin{cases} n - 1 - \min\{\left\lceil \frac{n}{2} \right\rceil, m\} & \text{when } m \text{ is even} \\ n - 1 - \min\{\left\lfloor \frac{n}{2} \right\rfloor, m\} & \text{when } m \text{ is odd.} \end{cases}$$

This equality holds when m is even and n is odd and in numerous other cases using the results and methods in [1] but several open cases remain unproven.

Thanks

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References

[1] Brian A. Keller, Robert Vandell and Steven J. Winters, *H*-decomposition numbers of graphs, *Vishwa International Journal of Graph Theory* **2** (1993), 131–149.

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