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# There exists no symmetric configuration with 33 points and line size 6

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**Abstract**

The nonexistence of a symmetric configuration with 33 points and line size 6 is established by means of a point-by-point backtrack search and clique searching. An equivalent result is that there exists no 6-regular bipartite graph with 66 vertices and girth 6. Combining with earlier results, it is now known that a symmetric configuration with line size 6 exists if and only if the number of points is 31 or at least 34. A further result obtained with essentially the same approach is that there exists no symmetric 2-configuration with 23 points and line size 7.

## 1 Introduction

A *configuration* with parameters $(v, r, b, k)$ consists of a set of $v$ *points*, a set of $b$ *lines*, and an *incidence relation* between points and lines such that (1) every point is incident with $r$ lines, (2) every line is incident with $k$ points, and (3) any two distinct points are incident with at most one line. Necessary existence conditions for a configuration are

$$vr = bk \qquad \text{and} \qquad v - 1 \geq r(k-1),$$

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obtained by counting in two ways the incidences of points with lines, and the incidences of pairs of points containing a given point with lines, respectively. The parameter $d = v - r(k-1) - 1 \geq 0$ is called the *deficiency* of a configuration. A configuration is *symmetric* if $v = b$ and hence $r = k$.

Configurations with zero deficiency—that is, Steiner 2-designs—have been extensively studied [4], while configurations with positive deficiency have received comparably less attention [8]. In the case of symmetric configurations, it is well known that the necessary existence condition $v \geq k^2 - k + 1$ is sufficient for $k = 3$ and $k = 4$, but for $k \geq 5$ gaps start to appear in the existence spectrum [5]. For $k = 5$, a symmetric configuration exists if and only if $v = 21$ or $v \geq 23$ [5]. For $k = 6$, a symmetric configuration exists if $v = 31$ (the projective plane of order 5), $v = 34$ [12], or $v \geq 35$ [5], and does not exist if $v = 32$ [9, 10] (see also [5]). Enumeration results for symmetric configurations with $k = 3, 4$ can be found in [1, 2, 6].

In this paper we settle the nonexistence of a symmetric configuration with $v = 33$ and $k = 6$ by means of an exhaustive computer search. An equivalent result is that there exists no 6-regular bipartite graph with 66 vertices and girth 6. This provides a negative answer to [3, Problem 397] and is apparently the first nonexistence result for a symmetric configuration with $d > 1$. Combining with results in [5, 9, 10, 12], we have the following theorem:

**Theorem 1** *A symmetric configuration with $k = 6$ exists if and only if $v = 31$ or $v \geq 34$.*

In a $\lambda$-configuration the constraint (3) is replaced with the constraint that any two distinct points have to be incident with at most $\lambda$ lines. We here also settle the nonexistence of a symmetric 2-configuration with $v = 23$ and $k = 7$, which is an open problem in [7, Remark 5.3].

## 2 The search

For a detailed introduction to classification algorithms for combinatorial designs, the reader is referred to [11].

A configuration can be represented by a 0-1 incidence matrix of size $v \times b$, with the rows corresponding to the points, the columns corresponding to the lines, and the 1s indicating incidence of a point with a line. The constraints (1), (2), and (3) are equivalent to requiring that (1′) every row sums to $r$, that (2′) every column sums to $k$, and that (3′) the inner product of any two distinct rows is at most one. Two matrices are *isomorphic* if one can be obtained from the other by permuting the rows and columns.

To establish that no symmetric configuration with $v = 33$ and $k = 6$ exists, it suffices to consider up to isomorphism all possibilities for constructing an associated incidence matrix one row at a time. Because each point $p$ is incident with $r = 6$ lines, and any other point is incident with at most one of these lines, up to isomorphism an incidence matrix has the decomposition in Fig. 1.

| | | |
|---|---|---|
| $p$ | 111111 | 000000000000000000000000000 |
| | 100000 | 111110000000000000000000000 |
| | 100000 | 000001111100000000000000000 |
| | 100000 | 000000000011111000000000000 |
| | 100000 | 000000000000000111110000000 |
| | 100000 | 000000000000000000001111100 |
| | 010000 | |
| | 010000 | |
| | 010000 | $A_2$ |
| | 010000 | |
| | 010000 | |
| | 001000 | |
| | 001000 | |
| | 001000 | $A_3$ |
| | 001000 | |
| | 001000 | |
| | $B$ | |

Figure 1: Incidence matrix of a symmetric configuration $v = 33$, $k = 6$

We carry out a row-by-row backtrack search dictated by the decomposition in Fig. 1 so that the rows in $A_2$ are completed first, followed by the rows in $A_3$. When completing a row, we require that (1′) and (3′) hold when restricted to the completed rows. The constraint (2′) cannot be violated when completing $A_2$ and $A_3$ in this manner.

Whenever a row is complete, the current matrix is transformed into canonical form using *nauty* [13] and compared against a record of matrices generated earlier. If the matrix appears on record, we disregard it from further consideration. To avoid an abundance of isomorphic matrices, when completing a row and there is a block of identical columns in the completed rows, the 1s are always inserted into the leftmost positions of such a block. Denoting by $w$ the number of complete rows in a matrix, for $w = 7, 8, \ldots, 16$ the numbers of nonisomorphic matrices we obtain are 3, 5, 5, 5, 4, 44, 1722, 30184, 79713, and 6509, respectively.

To complete the $33 - 16 = 17$ rows in $B$, we treat the row-by-row backtrack search as a clique search. For each of the 6509 nonisomorphic matrices with $A_2$ and $A_3$ completed, we first find all candidate rows that satisfy (1′) and (3′) and do not violate (2′) with respect to the completed rows. Next, we define a *compatibility graph* whose vertices are the candidate rows, and any two distinct rows are connected by an edge if and only if the inner product of the rows is at most one. We then search for cliques of size 17 in the compatibility graph—for algorithms, see [14, 15]. The search reveals that none of the 6509 compatibility graphs contains a clique of size 17, implying that no symmetric configuration with $v = 33$ and $k = 6$ exists. The number

$N(m)$ of compatibility graphs with maximum clique size $m$ is tabulated below for each $m$.

| $m$ | 3 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N(m)$ | 1 | 2 | 1 | 6 | 52 | 989 | 2600 | 1446 | 592 | 360 | 87 | 373 |

A similar approach shows the nonexistence of a symmetric 2-configuration with $v = 23$ and $k = 7$. We construct the incidence matrix up to 12 rows one row at a time so that the candidate rows are considered in decreasing lexicographic order, and the most significant column with sum less than $k$ is always set to 1. A matrix that is isomorphic to a matrix encountered earlier is rejected from consideration. In this way we obtain 1, 2, 5, 14, 8, 3, 3, 44, 5483, 153781, 349132, and 40324 nonisomorphic matrices up to 12 rows. The remaining 11 rows are completed by clique search. The maximum clique sizes we obtain are as follows.

| $m$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| $N(m)$ | 3 | 115 | 10427 | 28931 | 836 | 12 |

To gain confidence in the computed results, both authors independently implemented the algorithms and carried out the searches with identical results.

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