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On 3-choosability of plane graphs without 3-, 8- and 9-cycles

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Abstract

Steinberg (*Annals Discr. Math.* 55 (1993), 211–248) asked whether every planar graph without 4 and 5 cycles is 3-colorable. Borodin (*J. Graph Theory* 21(2) (1996), 183–186) showed that every planar graph without any cycles of length between 4 and 9 is 3-colorable. We improve this result by showing that every plane graph, which contains no triangles and contains no 8- and 9-cycles, is 3-choosable.

1 Introduction

In this paper, we consider only finite and simple graphs. Undefined terms may be found in [2]. Suppose k is a integer. Then k^+ and k^- denote integers $\geq k$ and $\leq k$, respectively. A vertex u is called a k-vertex if $d_G(u) = k$. A face f is called a k-face if $d_G(f) = k$. If no confusion can arise, $d(v)$ and $d(f)$ will be used instead of $d_G(v)$ and $d_G(f)$. A face of a plane graph is incident with all edges and vertices on its boundary. Two faces are adjacent if they have an edge in common. A k-cycle is a cycle on k vertices. The set of all k-cycles of G is denoted by C_k. A graph is called C_k*-free* if $C_k = \emptyset$. The girth of G is the length of a shortest cycle of G.

An h*-face* f is called a light h*-face* if all incidental vertices are 3^--vertices; otherwise a non-light h*-face* if it is incident with at least one 4^+-vertex. An h-face is called a minimal h-face if it lies on exactly one 4-vertex and 3^--vertex on other vertices; similarly, an h-face is called a non-minimal h-face if it lies on at least two 4^+-vertices.

A *list coloring* of G is an assignment of colors to $V(G)$ such that each vertex v receives a color from a prescribed list $L(v)$ of colors and adjacent vertices receive

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distinct colors (see [10]). $L(G) = \{L(v) \mid v \in V(G)\}$ is called a color list of G. The graph G is called *k-choosable* if G admits a list coloring for all color lists L with k colors in each list. Steiberg [7] asked whether every planar graph without 4 and 5 cycles is 3-colorable. Borodin [3] showed that every planar graph without any cycles of length between 4 and 9 is 3-colorable. Their results can easily be extended to choosability instead of colorability.

All 2-choosable graphs have been characterized by Erdős et al. [4]. In [8], Thomassen proved that every plane graph is 5-choosable. Voigt [11] showed that there are planar graphs which are not 4-choosable. It remains to decide whether a given plane graph is 4- or 3-choosable. Gutner [5] proved that these problems are NP-hard. So far, some sufficient conditions have been obtained and some constructions have been found. Alon and Tarsi [1] proved that every plane bipartite graph is 3-choosable. Thomassen [9] proved that every plane graph of girth at least 5 is 3-choosable. Lam[6] proved that every planar graph with girth at least 4 and without 5- and 6-cycles is 3-choosable. Zhang [12] proved that every plane graph with girth at least 4 and containing no 5-, 8- and 9-cycles is 3-choosable. In this paper, we will study a similar problem, the 3-choosability of graphs without 3-cycles. We show that if G contains no triangles and contains no 8-and 9- cycles, then G is 3-choosable.

2 Preliminaries

Before stating the main theorems, we shall first state the following necessary lemmas.

Lemma 1 [4] *Every cycle of even length is* 2*-choosable.*

Lemma 2 [12] *Let G be a non-*3*-choosable graph such that for any proper subset $V^* \subset V$, $G[V^*]$ is* 3*-choosable. Then any $2n$-cycle of G contains at least one 4^+-vertex.*

Lemma 3 [12] *Let G be a non-*3*-choosable graph such that for any proper subset $V^* \in V$, $G[V^*]$ is* 3*-choosable. If C_1 and C_2 are two* 4*-cycles with exactly one common vertex v_0, then at least one of C_1 and C_2 is a non-minimal cycle.*

3 Proof of the Main Theorem

Theorem 1 *Let G be a plane graph of girth not less than* 4*. If G contains no* 8*-and* 9*-cycles, then G is* 3*-choosable.*

Proof Suppose that G is a counterexample of minimum order. Then it is easy to see that δ (G)≥ 3. We assign a weight of $\sigma(x) = \frac{3d(x)}{10} - 1$ to each $x \in V(G)$, and $\sigma(x) = \frac{d(x)}{5} - 1$ to each $x \in F(G)$. Then by Euler's formula we have :

$$\sum_{v\in V(G)}(\frac{3d(v)}{10} - 1) + \sum_{f\in F(G)}(\frac{d(f)}{5} - 1) = -2. \quad (1)$$

if we obtain a new weight $\sigma^*(x)$ for all $x \in V \bigcup F$ by transferring weights from one element to another, then we also have $\sum_{x\in V\bigcup F} \sigma^*(x) = -2$. Moveover, if $\sigma^*(x) \geq 0$ for all $x \in V \bigcup F$, then the theorem is proved. A 4-face f is called a 4_i-face, for $i = 0, 1$ or 2, if f is adjacent to exactly i 4-faces. Weights will be transferred according to the following rules:

(R_1) From each 4- or 5^+-vertex to each incident 4-face, transfer $\frac{1}{10}$ or $\frac{1}{6}$ respectively.

(R_2) From each face to each incident 3-vertex, transfer $\frac{1}{30}$.

(R_3) From each 10^+-face to each adjacent 4-face which is called f', transfer:

- (R_{31}) $\frac{1}{10}$ if f' is a minimal 4-face;
- (R_{32}) $\frac{1}{15}$ if f' is a 4-face incident with one 5^+-vertex and three 3-vertices;
- (R_{33}) $\frac{1}{15}$ if f' is a 4-face incident with two 4-vertices and adjacent to three nonadjacent 5-faces;
- (R_{34}) $\frac{1}{30}$ otherwise.

(R_4) From each 10^+-face to each adjacent light 5-face transfer $\frac{7}{120}$.

(R_5) From each 10^+-face to each adjacent non-light 5-face transfer $\frac{1}{20}$.

(R_6) From each 7-face to each adjacent 5-face transfer $\frac{1}{20}$.

We shall make the following observations. Since G contains no 8- and 9-cycles, it follows that

(O_1) a 4-face is adjacent to at most two 4-faces;

(O_2) neither a 6-face nor a 7-face is adjacent to another 4-face;

(O_3) a 5-face is adjacent to at most two adjacent 4-faces;

(O_4) neither a 6-face nor a 5-face is adjacent to another 5-face.

Let v be a k- vertex of G.

If $k = 3$, then v is incident with three faces. Therefore according to R_2, $\sigma^*(v) = \sigma(v) + \frac{3}{30} = 0$.

If $k = 4$, then the total number of 4-faces incident with v is at most 2. According to R_1, $\sigma^*(v) \geq \sigma(v) - \frac{2}{10} = 0$. If $k \geq 5$, then at least two faces incident with v are not 4-faces. Therefore according to R_1, $\sigma^*(v) \geq \sigma(v) - \frac{k-2}{6} = \frac{2k-10}{15} \geq 0$.

Let f be an h-face of G. ($h = 4, 5, 6, 7, 10^+$).

First consider the case where $h = 4$.

By O_1 and O_2, f is adjacent only to 4-, 5- or 10^+-faces. And by Lemma 2, f is incident with at least one 4^+-vertex. Because transferring only happens from an adjacent 10^+-face to the 4-face, we must consider the condition when the number of adjacent 10^+-faces is as small as possible. This condition is considered the worst case.

Suppose f is a minimal 4-face.

If f is a 4_2-face, then f must be adjacent to two 10^+-faces; also by R_{31}, R_1, R_2, $\sigma^*(f) = \sigma(f) + \frac{1}{10} - \frac{3}{30} + \frac{2}{10} = 0$.

If f is a 4_1-face, then at least two 10^+-faces are adjacent to f; also by R_{31}, R_1, R_2, $\sigma^*(f) \geq \sigma(f) + \frac{1}{10} - \frac{3}{30} + \frac{2}{10} = 0$.

If f is a 4_0-face, then at most two disjoint 5-faces are adjacent to f and the other two faces are 10^+-faces. Hence by R_{31}, R_1, R_2, $\sigma^*(f) \geq \sigma(f) + \frac{1}{10} - \frac{3}{30} + \frac{2}{10} = 0$.

Suppose f is incident with one 5^+-vertex and three 3-vertices; then by the above analysis and by R_{32}, R_1, R_2, $\sigma^*(f) \geq \sigma(f) + \frac{1}{6} - \frac{3}{30} + \frac{2}{15} = 0$.

Suppose f is incident with two 4^+-vertices and two 3-vertices. When f is a 4_1-face or a 4_2-face, f is adjacent to at least two 10^+-faces. Hence by R_{34}, R_1, R_2, $\sigma^*(f) \geq \sigma(f) + \frac{2}{10} - \frac{2}{30} + \frac{2}{30} = 0$. If f is a 4_0-face, then the worst condition is that f is adjacent to three 5-faces and one 10^+-face. Hence by R_{33}, R_1, R_2, $\sigma^*(f) \geq \sigma(f) + \frac{2}{10} - \frac{2}{30} + \frac{1}{15} = 0$.

Finally, we assume that f is incident with at least three 4^+-vertices. Even if no weight is transferred to f across the four faces, we also have

$$\sigma^*(f) \geq \sigma(f) + \frac{3}{10} - \frac{1}{30} > 0.$$

Consider $h = 5$.

By O_3 and O_4, f is adjacent only to 4-, 7- or 10^+-faces and f is adjacent to at most two 4-faces on two consecutive edges, otherwise G is not C_9-free.

Suppose f is a light 5-face. Because the weight, transferred to f from an adjacent 10^+-face, is more than the weight transferred from an adjacent 7-face, we also should consider 7-faces as much as possible. If f is adjacent to two 4-faces, then at most one 7-face is adjacent to f. Hence by R_6, R_4, $\sigma^*(f) \geq \sigma(f) + \frac{1}{20} - \frac{5}{30} + 2 \cdot \frac{7}{120} = 0$. If f is adjacent to at most one 4-face, then it is clear that $\sigma^*(f) \geq 0$.

Now we assume that f is a non-light 5-face. Since at most two 4-faces are adjacent to f and the other three then are 7^+-faces, hence by R_6, R_5, R_2 , $\sigma^*(f) \geq \sigma(f) - \frac{4}{30} + \frac{3}{20} > 0$.

Consider $h = 6$. Then at least one 4^+-vertex is incident with f by Lemma 2. So $\sigma^*(f) \geq \sigma(f) - \frac{5}{30} = \frac{1}{5} - \frac{1}{6} > 0$.

Consider $h = 7$.

If f is a light 7-face, the weights are transferred from f to at most three 5-faces, according to R_6. So $\sigma^*(f) \geq \sigma(f) - \frac{7}{30} - 3 \cdot \frac{1}{20} > 0$.

On the other hand, if f is a non-light 7-face, suppose there are r 5-faces adjacent to f $(4 \leq r \leq 7)$. Then at most $(14 - 2r)$ 3-vertices are incident with f. So $\sigma^*(f) \geq \sigma(f) - \frac{14-2r}{30} - \frac{r}{20} = \frac{5r-20}{300} \geq 0$. $(r \geq 4)$.

Now consider $h \geq 10$. In this case, to each vertex and edge on the boundary of f, we assign a quota of $\frac{1}{30}$ and $\frac{1}{15}$ respectively. By the discharging rules, f transfers the weights not only to 3-vertices on the boundary of f, but also to 4-faces and 5-faces adjacent to it. For each 4^+-vertex v on the boundary of f, the quota assigned to v can be donated to the edges incident with v on the boundary of f. If these quotas are enough to cover all transfers to incident vertices and adjacent 4- and 5-faces then

$$\sigma^*(f) \geq \sigma(f) - \frac{h}{30} - \frac{h}{15} \geq 0$$

and proof of the Theorem is completed.

Let xt, tu, uv, vw and wy be five consecutive edges of a 10^+-face f. Also let f_3, f_1, f', f_2 and f_4 be the faces adjacent to f at xt, tu, uv, vw and wy respectively. Let s be the weight transferred from f to f' across uv. By the discharging rules, f transfers the weights to 3-vertices on the boundary of f, also to 4-faces and 5-faces adjacent to it. So we assume that f' is a 4-face or a 5-face. Because we assigned $\frac{1}{15}$ to each edge on the boundary of f, if the situation which happens is $R_{32}, R_{33}, R_{34}, R_4, R_5$, then $s \leq \frac{1}{15}$. So we only need to consider the situation when f' is a minimal 4-face. Now there are the following situations.

Case 1: if u and v are 3- and 4- vertices respectively, the worst condition is that f_1 is a 4-face. Then by Lemma 3, f_2 is not a minimal face, and the weight transferred across vw is at most $\frac{1}{15}$ by R_{32}, R_{33}, R_{34}, R_4 and R_5. So $\frac{1}{30}$, which is the unused quota for the vertex v, may be donated to uv. Therefore, the quota of uv is adjusted to $\frac{1}{15} + \frac{1}{30} = \frac{1}{10} = s$ by R_{31}. The same conclusion holds if u and v are 4- and 3-vertices respectively.

Case 2: if both u and v are 3-vertices, then f' is adjacent to both f_1 and f_2.

Now assuming that f_1 is a 4-face, then f_2 must be a 10^+-face. So the weight transferred from f across vw is 0. And $\frac{1}{30}$, half of the unused quota of vw, may be donated to uv, adjusting its quota to $\frac{1}{15} + \frac{1}{30} = \frac{1}{10} = s$ by R_{31}. The same conclusion holds if f_2 is a 4-face .

It is clear that if either f_1 or f_2 is a 10^+-face, then by the same argument as above, $\frac{1}{30}$, which is half of the unused quota of vw or tu, may be donated to uv. Consequently, the quota of uv is adjusted to $\frac{1}{15} + \frac{1}{30} = \frac{1}{10} = s$ by R_{31}.

Assume that either f_1 or f_2 is a light 5-face. Without loss of generality, assuming that f_1 is a light 5-face, then f_2 is a 10^+-face or a 5-face. If f_2 is a 10^+-face, then the argument is the same as above. Otherwise, if f_2 is a 5-face, because f_1 is a light 5-face and t is a 3-vertex, then f_3 adjacent to f_1 must be a 10^+-face. So the weight transferred from f across xt is 0. And $\frac{1}{30}$, half of the unused quota of xt, may be donated to uv, adjusting its quota to $\frac{1}{15} + \frac{1}{30} = \frac{1}{10} = s$ by R_{31}.

If f_1 and f_2 are non-light 5-faces, then the weights transferred across tu and vw are both at most $\frac{1}{20}$ by R_5. And neither f_1 nor f_2 is adjacent to a 4-face at xt and wy. So, $\frac{1}{15} - \frac{1}{20}$, the remaining quota of the edge tu, need not be given to xt, and $\frac{1}{15} - \frac{1}{20}$, the remaining quota of vw, also need not be given to wy. Both of the remaining quotas should be donated to uv. Consequently, the quota of uv is adjusted to $\frac{1}{15} + 2 \cdot (\frac{1}{15} - \frac{1}{20}) = \frac{1}{10} = s$ by R_{31}.

In all of the above cases, the weight transferred across uv is less than or equal to the adjusted quota. Now, we get that $\sigma^*(x) \geq 0$ for each $x \in V \bigcup F$. It follows that

$$0 \leq \sum_{x \in V \bigcup F} \sigma^*(x) = -2.$$

This completes the proof.

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