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Total domination and total domination subdivision numbers

O. Favaron

L.R.I., Bât. 490
Université Paris-Sud
91405 Orsay cedex
France
of@lri.fr

H. Karami

Department of Mathematics
Sharif University of Technology
P.O. Box 11365-9415 Tehran
I.R. Iran

S.M. Sheikholeslami*

Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz
I.R. Iran
s.m.sheikholeslami@azaruniv.edu

Abstract

A set S of vertices of a graph $G = (V, E)$ without isolated vertex is a *total dominating set* if every vertex of $V(G)$ is adjacent to some vertex in S.*The total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. *The total domination subdivision number* $sd_{\gamma_t}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the total domination number. We show that $sd_{\gamma_t}(G) \leq n - \gamma_t(G) + 1$ for any graph G of order $n \geq 3$ and that $sd_{\gamma_t}(G) \leq n - \gamma_t(G)$ except if $G \simeq P_3, C_3, K_4, P_6$ or C_6.

* Corresponding author

1 Introduction

Let $G = (V(G), E(G))$ be a graph of order n with no isolated vertex. The neighborhood of a vertex u is denoted by $N_G(u)$ and its degree $|N_G(u)|$ by $d_G(u)$ (briefly $N(u)$ and $d(u)$ when no ambiguity on the graph is possible). To work on the total domination, we must suppose the minimum degree δ of G is positive. We use [7] for terminology and notation which are not defined here.

A set S of vertices of G is a *total dominating set* if it is a dominating set of G with no isolated vertex, in other words if $N(S) = V$. The minimum cardinality of a total dominating set, denoted by $\gamma_t(G)$, is called the *total domination number* of G and a $\gamma_t(G)$-set is a total dominating set of G with cardinality $\gamma_t(G)$. When an edge uv of G is subdivided by inserting a vertex x between u and v, the total domination number cannot decrease. The *total domination subdivision number* $sd_{\gamma_t}(G)$ is the minimum number of edges of G that must be subdivided in order to increase the total domination number. Similar definitions exist for the domination number $\gamma(G)$ and the domination subdivision number $sd_\gamma(G)$ and, when G is connected, for the connected domination number $\gamma_c(G)$ and the connected domination subdivision number $sd_{\gamma_c}(G)$. The first of them was introduced for the usual domination number in Velammal's thesis [6].

It is rather difficult to construct graphs with large value of $sd_\gamma(G)$, $sd_{\gamma_t}(G)$ or $sd_{\gamma_c}(G)$ and the first conjecture on the subject was that $sd_\gamma(G) \leq 3$ for every G [6]. However it is now known that the three parameters can be arbitrary large and that there exist graphs of order n for which their order is $\log n$ (see [1] for sd_γ, [5] for sd_{γ_t}, [2] for sd_{γ_c}). It is also difficult to find general and good upper bounds on these parameters. Bhattacharya and Vijayakumar proved in [1] that if n is large enough, $sd_\gamma(G) \leq 4\sqrt{n}\ln n + 5$ and the authors of [4] asked if $sd_{\gamma_t}(G)$ is always at most n. Some bounds are given in terms of the corresponding domination parameters. For instance $sd_\gamma(G) \leq \gamma(G) + 1$ [1, 3] and $sd_{\gamma_c}(G) \leq n - \gamma_c(G) - 1$ with equality if and only if G is a path or a cycle [2].

Our purpose in this paper is to establish a bound of this type on $sd_{\gamma_t}(G)$. We prove that $sd_{\gamma_t}(G) \leq n - \gamma_t(G)$ (and thus $\leq n$) for every graph of order $n \geq 3$ without isolated vertex and different from P_3, C_3, K_4, P_6 and C_6. We will use the following results on $sd_{\gamma_t}(G)$.

Theorem A [4] *If G is a graph of order $n \geq 3$ and $\gamma_t(G) = 2$ then $1 \leq sd_{\gamma_t}(G) \leq 3$.*

Theorem B [4] *If G is a graph of order $n \geq 3$ and $\gamma_t(G) = 3$ then $1 \leq sd_{\gamma_t}(G) \leq 3$.*

Theorem C [5] *For any connected graph G with adjacent vertices u and v, each of degree at least two,*

$$sd_{\gamma_t}(G) \leq d(u) + d(v) - |N(u) \cap N(v)| - 1 = |N(u) \cup N(v)| - 1.$$

2 An upper bound for total domination subdivision number

In this section we first prove for every connected graph G of order $n \geq 3$, $sd_{\gamma_t}(G) \leq n - \gamma_t(G) + 1$ and then we characterize the graphs achieving this bound. We start with the following lemma that will be used in the proof of Theorem 3.

Lemma 1 *Let G be obtained from any graph H on l vertices $w_1, \ldots, w_l$ by adding $2l+3$ new vertices u, v, v', y_i, z_i for $1 \leq i \leq l$ and the edges $uv, vv', uw_i, w_iy_i, y_iz_i$ for $1 \leq i \leq l$. Then $sd_{\gamma_t}(G) \leq \max\{3, l+1\}$.*

Proof. Clearly $\gamma_t(G) = 2l+2$. If H is not empty, let, say, $w_1w_2 \in E(H)$ and let G' be obtained from G by subdividing the edges w_1w_2 and y_iz_i for $1 \leq i \leq l$ and adding $l+1$ new vertices respectively called $a, b_1, \ldots, b_l$. Every total dominating set of G' contains at least two vertices in $\{u, v, v'\}$ and in each set $\{y_i, b_i, z_i\}$ with $1 \leq i \leq l$ and one vertex in $\{w_1, a, w_2\}$. Hence $\gamma_t(G') \geq 2l+3 > \gamma_t(G)$ and $sd_{\gamma_t}(G) \leq l+1$.

If H is empty, let G' be obtained from G by subdividing the edges w_1y_1, y_1z_1 and vv' and adding three new vertices respectively called a, b, c. Every total dominating set of G' contains at least two vertices in each set $\{v, c, v'\}$, $\{y_1, b, z_1\}$, $\{w_i, y_i, z_i\}$ for $2 \leq i \leq l$, and one vertex in $\{u, w_1, a\}$. Hence $\gamma_t(G') \geq 2l+3 > \gamma_t(G)$ and $sd_{\gamma_t}(G) \leq 3$.

Theorem 1 *For every connected graph G of order $n \geq 3$, $sd_{\gamma_t}(G) \leq n - \gamma_t(G) + 1$.*

Proof. If G is a star, then $\gamma_t(G) = 2$ and $sd_{\gamma_t}(G) = 2 \leq n - \gamma_t(G) + 1$. Otherwise, let u and v be two adjacent vertices of G of degree at least two, and let $X = V(G) \setminus (N(u) \cup N(v))$. Let $x_1, \ldots, x_k$ be the K_1-components, if any, of the subgraph $G[X]$ induced by X in G and for each vertex x_i, let $w_i \in N(x_i)$. Then $\{w_1, \ldots, w_k\} \cup (X \setminus \{x_1, \ldots, x_k\}) \cup \{u, v\}$ is a total dominating set of G with at most $|X|+2$ elements. Hence $\gamma_t(G) \leq |X| + 2$. From this inequality and by Theorem C we have

$$sd_{\gamma_t}(G) \leq |N(u) \cup N(v)| - 1 = |V(G) \setminus X| - 1 = n - |X| - 1 \leq n - \gamma_t(G) + 1. \quad (1)$$

Now we characterize the graphs achieving this bound. We start with the particular case where $\gamma_t(G) = 2$ or 3.

Theorem 2 *Let G be a graph of order $n \geq 3$ with $\delta \geq 1$ and $\gamma_t(G) = 2$ or 3. Then $sd_{\gamma_t}(G) = n - \gamma_t(G) + 1$ if and only if $G \simeq P_3, C_3,$ or K_4.*

Proof. If $\gamma_t(G) = 2$ and $sd_{\gamma_t}(G) = n - \gamma_t(G) + 1 = n - 1$, then $n \leq 4$ by Theorem A. When $n = 3$, then G is necessarily isomorphic to P_3 or C_3 and for these two graphs, $sd_{\gamma_t}(G) = 2 = n - \gamma_t(G) + 1$. When $n = 4$, then G is connected and isomorphic to P_4, C_4, $K_{1,3}$, $K_{1,3} + e$, K_4 or $K_4 - e$. Since $sd_{\gamma_t}(P_4) = sd_{\gamma_t}(C_4) = 1 < n - \gamma_t(G) + 1$,

$sd_{\gamma_t}(K_{1,3}) = sd_{\gamma_t}(K_{1,3}+e) = sd_{\gamma_t}(K_4 - e) = 2 < n - \gamma_t(G) + 1$ and $sd_{\gamma_t}(K_4) = 3 = n - \gamma_t(G) + 1$, $G \simeq K_4$.

If $\gamma_t(G) = 3$ and $sd_{\gamma_t}(G) = n - \gamma_t(G) + 1 = n - 2$, then $n \leq 5$ by Theorem B. The only graphs with $\delta \geq 1$, $n \leq 5$ and $\gamma_t(G) = 3$ are P_5 and C_5 which both satisfy $sd_{\gamma_t}(G) = 1 < n - \gamma_t(G) + 1$. This completes the proof.

Theorem 3 *Let G be a connected graph of order $n \geq 3$ with $\gamma_t(G) \geq 4$. Then $sd_{\gamma_t}(G) = n - \gamma_t(G) + 1$ if and only if $G \simeq P_6$* or *C_6.*

Proof. If $G \simeq P_6$ or C_6 then $\gamma_t(G) = 4$ and since $\gamma_t(P_8) = \gamma_t(C_8) = 4$ and $\gamma_t(P_9) = \gamma_t(C_9) = 5$, $sd_{\gamma_t}(G) = 3 = n - \gamma_t(G) + 1$.

Suppose now that $sd_{\gamma_t}(G) = n - \gamma_t(G) + 1$. Since $\gamma_t(G) \geq 4$, G is not a star. Let u and v be any two adjacent vertices such that $\min\{d(u), d(v)\} \geq 2$. With the notation of Theorem 1, equality in (1) implies that $\gamma_t(G) = |X| + 2$ and $sd_{\gamma_t}(G) = n - |X| - 1 = |N(u) \cup N(v)| - 1$. In particular, $|N(u) \cup N(v)|$ does not depend on the choice of the pair $\{u, v\}$ of adjacent vertices of degree at least two.

Claim 1 Every connected component of $G[X]$ has order 1 or 2.

Proof of Claim 1. If G_1 is a component of $G[X]$ of order at least 3, then $\gamma_t(G_1) \leq |V(G_1)| - 1$. Let D_1 be a $\gamma_t(G_1)$-set. Let $x_1, \ldots, x_k$ be the K_1-components, if any, of $G[X]$ and for each vertex x_i, let $w_i \in N(x_i)$. Then $D_1 \cup (X \setminus (V(G_1) \cup \{x_1, \ldots, x_k\})) \cup \{w_1, \ldots, w_k\} \cup \{u, v\}$ is a total dominating set of G of order at most $|X| + 1 < \gamma_t(G)$, a contradiction. This proves the claim.

From now, we denote respectively by $x_1, \ldots, x_{l_1}$ and $y_1 z_1, \ldots, y_{l_2} z_{l_2}$ the K_1-components and the K_2-components of $G[X]$. Since $|X| = \gamma_t(G) - 2 \geq 2$, the integers l_1 and l_2 satisfy $l_1 \geq 0$, $l_2 \geq 0$ and $l_1 + 2l_2 \geq 2$.

Claim 2 There are no two vertices in $G[X]$ with a common neighbor in $N(u) \cup N(v)$.

Proof of Claim 2. Let, to the contrary, a_1 and a_2 be two vertices of X with a common neighbor a in $N(u) \cup N(v)$. If a_1 and a_2 are K_1-components of $G[X]$, we can assume $a_1 = x_1$ and $a_2 = x_2$. If $l_1 \geq 3$, let $w_i \in N(x_i)$ for $3 \leq i \leq l_1$. Then $(X \setminus \{x_1, \ldots, x_{l_1}\}) \cup \{w_3, \ldots, w_{l_1}\} \cup \{a, u, v\}$ is a total dominating set for G of order at most $|X| + 1$. If a_1 belongs to a K_2-component of $G[X]$, say $a_1 = y_1$, and a_2 is a K_1-component of $G[X]$, say $a_2 = x_1$, then $(X \setminus \{z_1, x_1, \ldots, x_{l_1}\}) \cup \{w_2, \ldots, w_{l_1}\} \cup \{a, u, v\}$, where $w_i \in N(x_i)$ for $2 \leq i \leq l_1$ if $l_1 \geq 2$, is a total dominating set for G of order at most $|X| + 1$. If a_1 and a_2 belong to the same K_2-component of $G[X]$, say $a_1 = y_1$ and $a_2 = z_1$, then $(X \setminus \{y_1, z_1, x_1, \ldots, x_{l_1}\}) \cup \{w_1, \ldots, w_{l_1}\} \cup \{a, u, v\}$, where $w_i \in N(x_i)$ for $1 \leq i \leq l_1$, is a total dominating set of $G[X]$ of order at most $|X| + 1$. Finally if a_1 and a_2 belong to different K_2-components of $G[X]$, say $a_1 = y_1$ and $a_2 = y_2$, then $(X \setminus \{z_1, z_2, x_1, \ldots, x_{l_1}\}) \cup \{w_1, \ldots, w_{l_1}\} \cup \{a, u, v\}$, where $w_i \in N(x_i)$ for $1 \leq i \leq l_1$, is a total dominating set of $G[X]$ of order at most $|X|+1$. The four cases contradict the fact that $\gamma_t(G) = |X| + 2$, which proves the claim.

In what follows, we choose the pair $\{u, v\}$ of adjacent vertices of degree at least two such that, if $\delta = 1$, then v has a neighbor v' of degree one. We consider two cases.

Case 1 There is a K_2-component of $G[X]$, say y_1z_1, such that

$$\min\{d(y_1), d(z_1)\} \geq 2.$$

Since $|N(u) \cup N(v)|$ doest not depend on the choice of the pair of adjacent vertices of degree at least two, $N(y_1) \cup N(z_1) = N(u) \cup N(v)$, which implies $\delta \geq 2$ and $G[X] \simeq K_2$ by Claim 2. Hence $\gamma_t(G) = |X| + 2 = 4$ and $sd_{\gamma_t}(G) = n - 3$. Moreover the symmetry between the pairs $\{u, v\}$ and $\{y_1, z_1\}$ shows that u and v have no common neighbor. Therefore $n = d(u) + d(v) + 2$ and $sd_{\gamma_t}(G) = d(u) + d(v) - 1$. Without loss of generality we can assume $N(u) \cap N(y_1) \neq \emptyset$. Let $y \in N(u) \cap N(y_1)$. Since $\{u, y, y_1\}$ cannot be a total dominating set of G, v has a neighbor v_1 which is not adjacent to any of u, y, y_1 and $v_1 \in N(v) \cap N(z_1)$.

Claim 3 $|N(u) \cup N(v) \setminus \{u, v\}| = 2$

Proof of Claim 3. Let $Y = (N(u) \cup N(v)) \setminus \{u, v\}$ and suppose, to the contrary, $|Y| > 2$. Without loss of generality we can assume $d(v) \geq d(u)(\geq 2)$ and thus $d(v) \geq 3$. Let $N(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_m\}$ with $m \geq 2$. Let G' be obtain from G by subdividing the $d(v) + 1$ edges $uv, y_1z_1, vv_1, \ldots, vv_m$ and adding new vertices respectively called $a, b, t_1, \ldots, t_m$. Let $Y_1 = \{a, t_1, \ldots, t_m\}$, $Y_2 = \{y_1, b, z_1\}$, and let D be a $\gamma_t(G')$-set. We can remark that either $|D \cap Y_2| \geq 2$, or $|D \cap Y_2| = 1$ and $|D \cap Y| \geq 2$ (by Claim 2). Assume that $|D| = 4$. If $v \in D$ then $|D \cap (\{v\} \cup Y_1)| \geq 2$. Hence $|D \cap (\{v\} \cup Y_1)| = 2$ and $|D \cap (Y \cup Y_2)| = 2$. By the previous remark, $|D \cap Y_2| = 2$ and $|D \cap Y| = 0$. Since $D \cap Y = \emptyset$, D must contain a to dominate u. Thus $t_1 \notin D$ and since $D \cap (Y \cup Y_2) = \{b, y_1\}$ or $\{b, z_1\}$, either y or v_1 is not dominated by D, a contradiction. Therefore $v \notin D$ and D contains at least $m+2$ vertices in $Y_1 \cup \{u\} \cup Y$ (because the vertex dominating v cannot be isolated), and at least one in Y_2. This contradicts $m \geq 2$ and $|D| = 4$. Hence $\gamma_t(G') \geq 5 > \gamma_t(G)$ and $sd_{\gamma_t}(G) \leq d(v) + 1$. Since $sd_{\gamma_t}(G) = d(u) + d(v) - 1$, necessarily $d(u) = 2$. Exchanging the pairs u, v and u, y, we see that, since $|N(u) \cup N(v)| = |N(u) \cup N(y)|$ and v_1 and z_1 do not belong to $N(u) \cup N(y)$, y must be adjacent to every v_i with $2 \leq i \leq m$. If one of these vertices v_i, say v_2, is adjacent to z_1, then $\{y, v_2, z_1\}$ is a total dominating set of G. Otherwise, y_1 is adjacent to $v_2, \ldots, v_m$ and $\{y_1, v_2, v\}$ is a total dominating set of G. Both cases contradict $\gamma_t(G) = 4$, which proves that $|Y| = 2$ and completes the proof of Claim 3.

Claims 2 and 3 show that in Case 1, $G \simeq C_6$.

Case 2 Every K_2-component y_iz_i of $G[X]$ has a vertex of degree 1, say $d(z_i) = 1$, for $1 \leq i \leq l_2$.

Subcase 1 $l_1 \geq 2$. Let $w_i \in N(x_i)$ for $i = 1, 2$. Let G' be obtain from G by subdividing the three edges x_1w_1, x_2w_2, uv and adding the new vertices t_1, t_2, a, respectively. Let D be a $\gamma_t(G')$-set. Obviously $|D \cap N[y_i]| \geq 2$ for $1 \leq i \leq l_2$ (if $l_2 \geq 1$), $|D \cap N[x_j]| \geq 1$ for $3 \leq j \leq l_1$ (if $l_1 \geq 3$), $|D \cap (N_G[x_k] \cup \{t_k\})| \geq 2$ for $k = 1, 2$ and $|D \cap \{u, v, a\}| \geq 1$. Therefore $|D| \geq 2l_2 + l_1 + 2 + 1 = |X| + 3$ and thus, $\gamma_t(G') \geq \gamma_t(G) + 1$. Hence $sd_{\gamma_t}(G) = |N(u) \cup N(v)| - 1 \leq 3$ and we must have $|(N(u) \cup N(v)) \setminus \{u, v\}| = 2$, $l_2 = 0$, $l_1 = 2$ and $d(x_1) = d(x_2) = 1$. Let without loss of generality $uw_1 \in E(G)$. Since $\gamma_t(G) \geq 4$, $uw_2 \notin E(G)$, and so $vw_2 \in E(G)$, and $w_1v, w_1w_2 \notin E(G)$. This implies $G \simeq P_6$.

Subcase 2 $l_1 = 1$ or 0. Since $l_1 + 2l_2 \geq 2$, $G[X]$ has at least one K_2-component and by the choice of the pair u, v, the vertex v has a neighbor v' of degree one. For $1 \leq i \leq l_2$, let $t_i \in N(y_i) \setminus \{z_i\}$. When $l_1 = 1$, let x be the unique K_1-component of $G[X]$ and let $w \in N(x)$. Let G' be the graph obtained from G by subdividing the edges uv, vv', y_iz_i for $1 \leq i \leq l_2$, and xw when $l_1 = 1$ and adding $l_2 + l_1 + 2$ vertices respectively called $a, a', b_1, \ldots, b_{l_2}$ and c. Every total dominating set D of G' contains at least two vertices in each set $\{y_i, b_i, z_i\}$. Moreover D contains at least three vertices in $N_G[u] \cup N_G[v] \cup \{a, a'\}$ if $l_1 = 0$, or two vertices in $\{v, a', v'\}$ and two vertices in $N_G[x] \cup \{c\}$ if $l_1 = 1$. Therefore $\gamma_t(G') \geq 2l_2 + 3 + l_1 = |X| + 3 > \gamma_t(G)$ and $sd_{\gamma_t}(G) = n - |X| - 1 \leq l_2 + l_1 + 2$. Hence $n \leq |X| + (l_1 + l_2) + 3$. This implies by Claim 2 that $d(y_i) = 2$ for $1 \leq i \leq l_2$ and, if $l_1 = 1$, $d(x) = 1$. Then $n = 3l_2 + 2l_1 + 3$, $\gamma_t(G) = 2l_2 + l_1 + 2$ and $sd_{\gamma_t}(G) = l_2 + l_1 + 2$.

If $l_1 = 1$, then w is not adjacent to v, for otherwise $\{v, w, t_1, y_1, \ldots, t_{l_2}, y_{l_2}\}$ is a total dominating set of G of order $2l_2 + 2 < \gamma_t(G)$, and is thus adjacent to u. Let G' be obtained from G by subdividing the edges wx and y_iz_i for $1 \leq i \leq l_2$ and adding $l_2 + 1$ new vertices $c, z'_1, z'_2, \ldots, z'_{l_2}$. Every total dominating set D of G' contains at least two vertices in each set $\{x, c, w\}$, $\{z_i, z'_i, y_i\}$ for $1 \leq i \leq l_2$, and $N[v]$. Hence $\gamma_t(G') \geq 2l_2 + 4 > \gamma_t(G)$ and $sd_{\gamma_t}(G) \leq l_2 + 1$, a contradiction.

Thus $l_1 = 0$, $\gamma_t(G) = 2l_2 + 2$ and $sd_{\gamma_t}(G) = l_2 + 2$. If v is adjacent to one t_i, say to t_1, then $\{v, t_1, y_1, t_2, y_2, \ldots, t_{l_2}, y_{l_2}\}$ is a total dominating set of G of order $2l_2 + 1 < \gamma_t(G)$, a contradiction. Hence $vt_i \notin E(G)$ for $1 \leq i \leq l_2$ and all the vertices t_i are adjacent to u. The graph G is the graph described in Lemma 1 and satisfies $sd_{\gamma_t}(G) = l_2 + 2 \leq \max\{3, l_2 + 1\}$. Therefore $l_2 = 1$ and $G \simeq P_6$, which completes the proof.

The following corollary is an immediate consequence of Theorems 1, 2 and 3.

Corollary 1 *If G is a connected graph of order $n \geq 3$ different from P_3, C_3, K_4, P_6, C_6, then $sd_{\gamma_t}(G) \leq n - \gamma_t(G)$.*

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