AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **38** (2007), Pages 155–162

The convex hull of every optimal pseudolinear drawing of K_n is a triangle

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* Supported by NSF Grants DMS-0302804, DMS-0603769 and DMS-0600303, and UIUC Campus Research Grant #06139.
† Supported by NSERC.
‡ Supported by FAI-UASLP and by CONACYT Grant 45903.

Abstract

A pseudolinear (respectively, rectilinear) drawing of a graph G is *optimal* if it has the smallest number of crossings among all pseudolinear (respectively, rectilinear) drawings of G. We show that the convex hull of every optimal pseudolinear drawing of the complete graph K_n is a triangle. This is closely related to the recently announced result that the convex hull of every optimal rectilinear drawing of K_n is a triangle.

1 Introduction

1.1 Our main result

A rectilinear (respectively, pseudolinear) drawing of a graph G is *optimal* if it has the smallest number of crossings among all rectilinear (respectively, pseudolinear) drawings of G. The following statement remained an important, open conjecture for a long time. Recently, a proof was announced by Aichholzer, Orden, and Ramos [3].

Theorem 1 ([3]) *The convex hull of every optimal rectilinear drawing of K_n is a triangle.*

Extending this conjecture to non-rectilinear drawings of K_n does not make much sense: there is no distinguished unbounded face if the rectilinear condition is altogether dropped, so a meaningful convex hull cannot even be defined. On the other hand, since the convex hull is well-defined for pseudolinear (which lie in between rectilinear and arbitrary) drawings, it makes sense to ask if a similar property holds for pseudolinear drawings. Our main result is that an analogous statement holds for pseudolinear drawings.

Theorem 2 (Main result) *The convex hull of every optimal pseudolinear drawing of K_n is a triangle.*

1.2 Pseudolinear drawings

Recall that a *pseudoline* in the projective plane $\mathbb{P}^2$ is a simple closed curve whose removal does not disconnect $\mathbb{P}^2$. A collection of pseudolines is a *pseudoline arrangement* if each two pseudolines intersect (necessarily cross) in exactly one point. A *generalized configuration Ω_P with point set P* consists of a finite set P of points, together with a pseudoline joining each pair, and it is *simple* if there is a single pseudoline for each pair.

Consider a good drawing $\mathcal{D}$ of K_n in the plane $\mathbb{R}^2$ (thus, every edge is represented by a simple curve), contained in a disk with radius $< R$ centered at the origin. Let D denote the disk with radius R, centered at the origin. By identifying antipodal points in the boundary of D (and discarding $\mathbb{R}^2 \setminus D$), we may regard $\mathcal{D}$ as (a new drawing $\mathcal{D}'$, as the host surface has changed) lying in the projective plane. Now if each edge e in $\mathcal{D}'$ can be extended to a pseudoline (an *extension of* e) so that the resulting structure is a simple generalized configuration Ω_P in which P is the set of n vertices, then the original drawing $\mathcal{D}$ is a *pseudolinear drawing of* K_n. The *pseudosegments* are the edges of a pseudolinear drawing; in pseudolinear drawings we use the term "edge" and "pseudosegment" interchangeably. If we start with a pseudolinear drawing of K_n (which, we emphasize, lies in $\mathbb{R}^2$), it is easy to see that we may equivalently stay (all along) in $\mathbb{R}^2$, and for each edge e construct an $\mathbb{R}^2$-*extension* ℓ_e, a set of points homeomorphic to a straight line, which contains e, whose removal disconnects $\mathbb{R}^2$ into two unbounded sets, and so that every pair of $\mathbb{R}^2$-extensions cross at exactly one point.

As we observed above, the convex hull in a pseudolinear drawing of K_n is a well-defined object that naturally generalizes the definition of the convex hull of a rectilinear drawing (the definition actually applies to quite more general objects, namely the CC-systems introduced by Knuth; see [5] and [7]). Consider a pseudolinear drawing $\mathcal{D}$ of K_n, and for each edge (pseudosegment) e construct an $\mathbb{R}^2$-extension ℓ_e as described above. An edge in $\mathcal{D}$ is a *convex hull edge* of $\mathcal{D}$ if the $n-2$ points (vertices of K_n) not incident with e lie on the same half-plane of ℓ_e, and the *convex hull* of $\mathcal{D}$ is the collection of all the convex hull edges and their incident vertices. It can be checked that convex hull edges are well-defined, that is, independent of the chosen $\mathbb{R}^2$-extensions.

It is readily verified that no convex hull edge can cross another edge. Therefore Theorem 2 states that the obvious extension of Theorem 1 to pseudolinear drawings is true: the unbounded face in any drawing that attains the pseudolinear crossing number of K_n is incident with (exactly) 3 vertices and 3 edges.

1.3 Pseudolinear and rectilinear crossing numbers

If $\mathcal{D}$ is a drawing of K_n, then we let $\mathrm{cr}(\mathcal{D})$ denote the number of pairwise crossings of edges in $\mathcal{D}$. The *pseudolinear crossing number* $\widetilde{\mathrm{cr}}(K_n)$ is the minimum of $\mathrm{cr}(\mathcal{D})$ over all pseudolinear drawings $\mathcal{D}$ of K_n. The *rectilinear crossing number* $\overline{\mathrm{cr}}(K_n)$ of K_n is the minimum of $\mathrm{cr}(\mathcal{D})$ over all rectilinear drawings $\mathcal{D}$ of K_n. Since every rectilinear drawing of K_n is also a pseudolinear drawing, $\overline{\mathrm{cr}}(K_n) \geq \widetilde{\mathrm{cr}}(K_n)$.

If a pseudolinear drawing is combinatorially equivalent to a rectilinear drawing, then it is *stretchable*. Since almost all pseudolinear drawings are non-stretchable (see for instance [9]), it is conceivable that $\widetilde{\mathrm{cr}}(K_n) < \overline{\mathrm{cr}}(K_n)$ for some n. By computing good lower bounds for $\widetilde{\mathrm{cr}}(K_n)$, it is verified in [2] that $\widetilde{\mathrm{cr}}(K_n) = \overline{\mathrm{cr}}(K_n)$ for $n \leq 12$ and $n = 15$. On this basis, we put forward the following.

Conjecture 3 *For every* n, $\widetilde{\mathrm{cr}}(K_n) = \overline{\mathrm{cr}}(K_n)$.

Settling this conjecture in either direction would be quite interesting by itself: we would know whether or not there is anything to gain, with respect to crossing numbers, by considering non-stretchable pseudolinear drawings of K_n (over rectilinear ones).

2 Background: generalized configurations and allowable sequences

We recall that a *simple allowable sequence on n elements* Π is a doubly infinite sequence $(\ldots, \pi_{-1}, \pi_0, \pi_1, \ldots)$ of permutations of an n-element *ground* set (say $\{p_1, p_2, \ldots, p_n\}$), such that (i) any two consecutive permutations differ by exactly one transposition of two elements in adjacent positions; and (ii) after a move in which i and j switch, they do not switch again until every other pair has switched. If a transposition τ swaps elements p_i and p_j, so that p_i moves from position t to position $t+1$, and p_j moves from position $t+1$ to position t, then we write $\tau = [p_i|p_j]_t$. An allowable sequence $\Pi = (\ldots, \pi_{-1}, \pi_0, \pi_1, \ldots)$ on n elements is equivalently defined by its *transpositions sequence* $T(\Pi) = (\ldots, \tau_{-1}, \tau_0, \tau_1, \ldots)$, where τ_i is the transposition that transforms π_{i-1} into π_i.

It is straightforward to see that a simple allowable sequence on n elements has period $n(n-1)$. We shall be particularly interested in halfperiods of Π, that is, finite subsequences $(\pi_i, \pi_{i+1}, \ldots, \pi_{i+\binom{n}{2}})$. Note that the ending permutation of a halfperiod is the reverse permutation of the starting one.

Simple allowable sequences, introduced by Goodman and Pollack in an influential paper [6], are a fruitful tool to encode any generalized configuration of points: to each generalized configuration of points Ω_P with point set P, one can naturally associate a simple allowable sequence Π_{Ω_P} with ground set P, and, reciprocally, given a simple allowable sequence Π with ground set P one can obtain a generalized configuration of points Ω_P whose associated sequence is $\Pi_{\Omega_P} = \Pi$. The details of this relationship have been lucidly explained in [6] and in subsequent surveys (more recently in [1] or [8], precisely in the context of crossing numbers), so we shall omit them, and refer the interested reader to these sources.

Suppose that $\mathcal{D}$ is a pseudolinear drawing of K_n, with underlying n-point set P. Thus (since $\mathcal{D}$ is pseudolinear) P is the point set of a simple generalized configuration Ω_P. We say that Ω_P is a generalized configuration *associated to* $\mathcal{D}$. Although Ω_P is not unique (as there are infinitely many ways to extend the pseudoedges to form pseudolines), the induced simple allowable sequence Π_{Ω_P} is unique, and thus it is consistent to call $\Pi_{\mathcal{D}} := \Pi_{\Omega_P}$ *the simple allowable sequence* associated to $\mathcal{D}$.

3 Allowable sequences and convex hulls: proof of Theorem 2

The encoding scheme from generalized configurations of points to simple allowable sequences [6] makes it particularly easy to identify the convex hull of a pseudolinear drawing of K_n, as follows.

Proposition 4 *Let $\mathcal{D}$ be a pseudolinear drawing of K_n, and let P denote the underlying n-point set. Let Π_0 be any halfperiod of the associated simple allowable sequence. Then a point p in P is in the convex hull of $\mathcal{D}$ iff it occupies either position 1 or position n in a permutation of Π_0.*

In view of this, in order to establish Theorem 2 it suffices to show that if $\mathcal{D}$ is *optimal* among pseudolinear drawings (that is, $\widetilde{\mathrm{cr}}(\mathcal{D}) = \widetilde{\mathrm{cr}}(K_n)$), then at most 3 elements in P ever occupy position 1 or position n in some permutation in Π_0 (any halfperiod of $\Pi_{\mathcal{D}}$). In order to prove such a result, we need a useful characterization of which simple allowable sequences are induced from optimal pseudolinear drawings of K_n.

Such a characterization can be obtained from results in [1] and [8] that give the crossing number in a pseudolinear drawing of K_n in terms of properties of its associated simple allowable sequence. In order to present this result, we need to define one local and one global function. Let $\tau = [p_i|p_j]_t$ be a transposition in the transpositions sequence of a simple allowable sequence Π. The *impact* $f(\tau)$ of τ is defined as follows:

$$f(\tau) = f([a|b]_t) = \left(\frac{n-2}{2} - (t-1)\right)^2. \tag{1}$$

Now if Π_0 is a halfperiod of a simple allowable sequence, then its *weight* $F(\Pi_0)$ is

$$F(\Pi_0) = \sum_{\tau} f(\tau), \tag{2}$$

where the summation is over all the τ_i's in the transpositions sequence of Π_0. That is, the weight of Π_0 is simply the sum of the impacts of all the transpositions in its transpositions sequence.

The relevance of the weight of a halfperiod of a simple allowable sequence induced by a pseudolinear drawing of K_n comes from the following result.

Theorem 5 ([1],[8]) *Let $\mathcal{D}$ be a pseudolinear drawing of K_n, and let Π be a halfperiod of its associated simple allowable sequence. Then*

$$\widetilde{\mathrm{cr}}(\mathcal{D}) = 3\binom{n}{4} - F(\Pi_0).$$

Our last required result, which is proved in Section 4, gives us a crucial piece of information on halfperiods that maximize F.

Proposition 6 *Let Π_0 be a halfperiod of a simple allowable sequence on n elements. Suppose that Π_0 maximizes F over all halfperiods of simple allowable sequences on n elements. Then there are (exactly) 3 elements that occupy either position 1 or position n in a permutation of Π_0.*

Proof of Theorem 2.

Since every simple allowable sequence can be induced from a pseudolinear drawing of K_n, it follows from Theorem 5 that a (pseudolinear) drawing attains the pseudolinear crossing number of K_n iff any halfperiod of its associated simple allowable sequence maximizes F over all possible halfperiods of simple allowable sequences. Propositions 6 and 4 complete the proof. ∎

4 Proof of Proposition 6

Throughout this proof, $\Pi_0 = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{\binom{n}{2}})$ is a halfperiod of a simple allowable sequence that minimizes F. Unless otherwise stated, all transpositions and permutations hereby mentioned occur are associated to Π_0.

Let us label the points so that the initial permutation is $a_1 a_2 \ldots a_n$.

Claim A *Let i satisfy $\lceil n/2 \rceil \le i < n$. Let τ_s be the transposition that moves a_n to position i. Suppose that a_ℓ is to the right of a_n in π_s. Then, after τ_s occurs, the first transposition that involves a_ℓ moves a_ℓ to the left, and the other element involved in the transposition is to the left of a_n in π_s.*

Proof. Seeking a contradiction, let i be smallest possible so that the statement is false. Label $b_1, b_2, \ldots, b_{n-i}$ the last $n-i$ elements in π_s, in the order in which they appear in π_s. Note that $\tau_s = [b_1|a_n]_i$.

We claim that the first transposition τ_t after τ_s that involves an element in $\{b_1, b_2, \ldots, b_{n-i}\}$ must be the transposition swapping elements b_1 and b_2. Recall that Claim A holds if we substitute i by $i-1$. This implies, in particular, that the first element in $\{b_2, \ldots, b_{n-i}\}$ that gets involved in a transposition after τ_s must be b_2, and that the other element involved in the transposition is to the left of b_2 in π_s. Now the first transposition after τ_s that involves b_1 cannot involve an element to the left of b_1 in π_s, as otherwise (it is easy to check) Claim A would then also hold for i. Thus τ_t must involve b_1 and b_2, that is, $\tau_t = [b_1|b_2]_{i+1}$. Again using the assumption that Claim A holds for $i-1$, it follows that the last transposition τ_r before τ_s that involves an element in $b_1, b_2, \ldots, b_{n-i}$ is precisely the transposition that swaps b_2 and a_n, that is, $\tau_r = [b_2|a_n]_{i+1}$.

Thus, the following transpositions occur in the given order: $\tau_r = [b_2|a_n]_{i+1}$, $\tau_s = [b_1|a_n]_i$, and $\tau_t = [b_1|b_2]_{i+1}$. Moreover, the only transposition between τ_r and τ_t that involves an element in position $i+1$ or further right is precisely τ_s. This last observation implies that if we modify the transpositions sequence by delaying τ_r (if necessary) and letting it occur immediately before τ_s, and then accelerating τ_t (if necessary) and letting it occur immediately after τ_s, and leaving the transposition sequence otherwise unchanged, the resulting transpositions sequence will still correspond to a (valid) halfperiod $\widetilde{\Pi}_0$ of a simple allowable sequence. More precisely, if we let $\tau'_i = \tau_i$ for $1 \le i < r$, $\tau'_i = \tau_{i+1}$ for $r \le i \le s-2$, $\tau'_{s-1} = [b_1|b_2]_i$, $\tau'_s = [b_1|a_n]_{i+1}$, $\tau'_{s+1} = [b_2|a_n]_i$, $\tau'_i = \tau_{i-1}$ for $s+2 \le i \le t$, and $\tau'_i = \tau_i$ for $i > t$, then $\tau'_0, \tau'_1, \ldots, \tau'_{\binom{n}{2}}$ is the transpositions sequence of a simple allowable sequence $\overline{\Pi}_0$. Clearly, $\sum_{\tau_i \notin \{\tau_r, \tau_s, \tau_t\}} f(\tau_i) = \sum_{\tau'_i \notin \{\tau'_{s-1}, \tau'_s, \tau'_{s+1}\}} f(\tau'_i)$, Moreover, $f(\tau_r) = f(\tau'_s)$ and $f(\tau_s) = f(\tau'_{s-1})$, and so $\sum_{\tau_i \ne \tau_t} f(\tau_i) = \sum_{\tau'_i \ne \tau'_{s+1}} f(\tau'_i)$. However, $f(\tau_t) = (\frac{n-2}{2} - ((i+1)-1))^2 < (\frac{n-2}{2} - (i-1))^2 = f(\tau'_{s+1})$ (note that here we are using that $i \ge \lceil n/2 \rceil$). Therefore $F(\Pi_0) = \sum_{\tau_i} f(\tau_i) < \sum_{\tau'_i} f(\tau'_i) = F(\overline{\Pi}_0)$, contradicting the assumption that Π_0 maximizes F over all halfperiods of simple allowable sequences of size n. ∎

Claim B *Either a_1 moves a_n from position n or a_n moves a_1 from position 1.*

Proof of Claim B. We suppose that a_1 reaches position $\lceil n/2 \rceil$ before a_n reaches position $\lfloor n/2 \rfloor + 1$ (it is readily checked that these cannot occur simultaneously), and show that in this case a_1 moves a_n out of position n. The other possibility, that a_n reaches position $\lfloor n/2 \rfloor + 1$ before a_1 reaches position $\lceil n/2 \rceil$ (in which case the conclusion is that a_n moves a_1 from position 1), is dealt with in a totally analogous manner.

Let $m+1$ be the position of a_1 immediately after it swaps with a_n. Thus, the transposition between a_1 and a_n is $[a_1|a_n]_m = \tau_q$ for some q. Since a_1 only moves right, and a_n only moves left, it follows that a_1 is in position $m \geq \lceil n/2 \rceil$ just before this permutation, that is, in π_{q-1}.

To prove the statement, for the rest of the proof we assume that $m < n-1$, and derive a contradiction.

Let b denote the element in position $m+2$ in π_{q-1} (and still there in π_q). Now b is to the right of a_n already in π_{q-1}. An application of Claim A with $i = m+1$ (that is, when a_n first moved into position $m+1$) yields that b could not have arrived to position $m+2$ (in π_{q-1}) by transposing with an element other than a_n. Thus b and a_n swap when b is in position $m+1$ (and a_n is in position $m+2$). Thus this transposition is $[b|a_n]_{m+1} = \tau_p$ for some $p < q$.

We note again that a_1 never moves left. Applying Claim A (again with $i = m+1$), we obtain that the transposition τ_r with $r > q$ smallest possible that involves an element in position $m+1$ or further right is the transposition that swaps a_1 and b. That is, $\tau_r = [a_1|b]_{m+1}$.

Thus, the following transpositions occur in the given order: $\tau_p = [b|a_n]_{m+1}$, $\tau_q = [a_1|a_n]_m$, and $\tau_r = [a_1|b]_{m+1}$. Moreover, τ_q is the only transposition between τ_p and τ_r that involves an element in position $m+1$ or further right (this follows again from Claim A). This observation implies that if we modify the transpositions sequence by delaying τ_p (if necessary) and letting it occur immediately before τ_q, and then accelerating τ_r (if necessary) and letting it occur immediately after τ_q, and leaving the transposition sequence otherwise unchanged, the resulting transpositions sequence will still induce a (valid) simple allowable sequence $\widetilde{\Pi}_0$. More precisely, if we let $\tau'_i = \tau_i$ for $1 \leq i < p$, $\tau'_i = \tau_{i+1}$ for $p \leq i \leq q-2$, $\tau'_{q-1} = [a_1|b]_m$, $\tau'_q = [a_1|a_n]_{m+1}$, $\tau'_{q+1} = [b|a_n]_m$, $\tau'_i = \tau_{i-1}$ for $q+2 \leq i \leq r$, and $\tau'_i = \tau_i$ for $i > r$, then $\tau'_0, \tau'_1, \ldots, \tau'_{\binom{n}{2}}$ is the transpositions sequence of a simple allowable sequence $\overline{\Pi}_0$. Clearly, $\sum_{\tau_i \notin \{\tau_p, \tau_q, \tau_r\}} f(\tau_i) = \sum_{\tau'_i \notin \{\tau'_{q-1}, \tau'_q, \tau'_{q+1}\}} f(\tau'_i)$, Moreover, $f(\tau_p) = f(\tau'_q)$ and $f(\tau_q) = f(\tau'_{q-1})$, and so $\sum_{\tau_i \neq \tau_r} f(\tau_i) = \sum_{\tau'_i \neq \tau'_{q+1}} f(\tau'_i)$. However, $f(\tau_r) = (\frac{n-2}{2} - ((m+1)-1))^2 < (\frac{n-2}{2} - (m-1))^2 = f(\tau'_{q+1})$. Therefore $F(\Pi_0) = \sum_{\tau_i} f(\tau_i) < \sum_{\tau'_i} f(\tau'_i) = F(\overline{\Pi}_0)$ (here we are using that $m \geq \lceil n/2 \rceil$), contradicting the assumption that Π_0 maximizes F over all halfperiods of simple allowable sequences of size n. ∎

Conclusion of proof of Proposition 6.

By Claim B, either a_1 moves a_n from position n or a_n moves a_1 from position 1. Suppose the former case holds. Let x be the element that moves a_1 from position 1. Immediately after a_1 and x transpose, x is in position 1, and a_n is in position

n. Thus another application of Claim B (with the suitable relabeling) implies that either x moves a_n out of position n or a_n moves x out of position 1. The former case is impossible, since $a_1 \neq x$ is the element that moves a_n out of position n. Thus a_n moves x out of position 1. Therefore, the only elements that ever occupy position 1 are a_1, x, and a_n, and the only elements that ever occupy position n are a_1 and a_n. ∎

A slightly different proof is given in [10].

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(Received 20 Apr 2006)