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On the local nature of some classical theorems on Hamilton cycles

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Abstract

We show that ten well-known global criteria for the hamiltonicity of a connected graph have equivalent formulations in terms of balls of constant radii not exceeding 4.

1 Introduction

We use [9] for terminology and notation not defined here and consider finite simple graphs only. Let $V(G)$ and $E(G)$ denote, respectively, the vertex set and edge set of a graph G, and let $d(u,v)$ denote the distance between vertices u and v in G. For each vertex u of G and a positive integer r, we denote by $N_r(u)$ and $M_r(u)$ the sets of all $v \in V(G)$ with $d(u,v) = r$ and $d(u,v) \leq r$, respectively. For a vertex u of a graph G the ball $G_r(u)$ of radius r centered at u is a subgraph of G induced by the set $M_r(u)$. We say that a vertex u is an interior vertex of a ball $G_r(x)$ if $M_1(u) \subseteq M_r(x)$. If $M_r(x) = V(G)$ then $G = G_r(x)$ and every vertex u of G is an interior vertex of $G_r(x)$.

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In [1–4] the present authors developed some local criteria for the existence of Hamilton cycles in a connected graph G, which generalize the classical global criteria due to Dirac [10], Ore [17], Nash-Williams [16], and Jung [15]. The idea was to show that if all balls of a small radius in G satisfy one of those global criteria K, or a variation of K, then G is hamiltonian. We called our results *localization theorems*.[1]

In this paper we show that ten well-known global criteria for the hamiltonicity of a connected graph have equivalent formulations in terms of balls of constant radii not exceeding 4. Indeed we show a little more. For each of those global criteria K we find a variation, K', and an integer $r(K') \leq 4$, such that

i) a connected graph G satisfies the criterion K if and only if every ball of radius $r(K')$ in G satisfies K',

ii) the class of graphs satisfying the criterion K is a proper subset of the class $\mathcal{B}(K')$ which consists of all graphs G where every ball of radius $r(K') - 1$ satisfies K'.

It follows from our results in [1–4] that if K is one of the criteria of Dirac [10], Ore [17], Nash-Williams [16], and Jung [15] then all graphs in the class $\mathcal{B}(K')$ are hamiltonian. For the other criteria K the following problem is open: are all graphs in the class $\mathcal{B}(K')$ hamiltonian? We formulate two conjectures concerning the criteria Fan [11] and Jackson [14].

2 Localizations with radius three

In this section we will show that the three classical criteria below (Theorems 2.1–2.3) have equivalent formulations in terms of balls of radius 3.

Theorem 2.1 (Ore's theorem [17]) *Let G be a graph on at least 3 vertices such that $d(u) + d(v) \geq |V(G)|$ for each pair of nonadjacent vertices u, v. Then G is hamiltonian.*

Theorem 2.2 (See, for example Jung [15]) *Let G be a 2-connected graph such that $d(u) + d(v) \geq |V(G)| - 1$ for each pair of nonadjacent vertices u, v. Then either G is hamiltonian or $G \in \mathcal{K}$ where $\mathcal{K} = \{G : K_{p,p+1} \subseteq G \subseteq K_p \vee \overline{K_{p+1}}$ for some $p \geq 2\}$ ($\vee$ denotes join).*

Theorem 2.3 (Nash-Williams [16]). *A 2-connected r-regular graph G is hamiltonian if $r \geq \frac{1}{2}(|V(G)| - 1)$.*

We need the following lemma:

Lemma 2.1. *Let G be a connected graph on at least 3 vertices where for every vertex $x \in V(G)$ the condition $d(u)+d(v) \geq |M_3(x)|-1$ holds for every pair of nonadjacent interior vertices u and v of the ball $G_3(x)$. Then the diameter of G does not exceed 2.*

[1] Similar localization results were obtained for other hamiltonian properties of a graph (see, for example, [5–7]).

Proof. Suppose to the contrary that there is a pair of vertices u and v in G with $d(u,v)=3$. Let $uyxv$ be a (u,v)-path of length 3 in G. Then $M_1(u)\subset M_3(x)$ and $N(u)\cap N(v)=\emptyset$. Therefore

$$d(v)\le |M_3(x)-M_1(u)|-1=|M_3(x)|-|M_1(u)|-1=|M_3(x)|-d(u)-2.$$

Thus $d(u)+d(v)\le |M_3(x)|-2$ for the pair of nonadjacent interior vertices u and v of the ball $G_3(x)$, which contradicts the condition of the lemma. This implies that the diameter of G does not exceed 2. □

Now we will show that Ore's theorem is equivalent to the following proposition:

Proposition 2.1. *Let G be a connected graph on at least 3 vertices where for every vertex $x\in V(G)$ the condition $d(u)+d(v)\ge |M_3(x)|$ holds for every pair of nonadjacent interior vertices u and v of the ball $G_3(x)$. Then G is hamiltonian.*

If G satisfies the condition of Theorem 2.1 then it also satisfies the conditions of Proposition 2.1. Conversely, let G satisfy the conditions of Proposition 2.1. Then, by Lemma 2.1, the diameter of G does not exceed 2. This means that $M_3(x)=V(G)$ for each vertex x of G. Therefore Proposition 2.1 is equivalent to Theorem 2.1.

Using a similar argument we can prove that Theorem 2.2 is equivalent to the following proposition:

Proposition 2.2. *Let G be a 2-connected graph such that $d(u)+d(v)\ge |M_3(x)|-1$ for every pair of nonadjacent interior vertices u and v of the ball $G_3(x)$. Then either G is hamiltonian or $G\in\mathcal{K}$.*

Since the set $\mathcal{K}$ contains no regular graphs, Proposition 2.2 implies the following result:

Proposition 2.3. *Let G be a 2-connected r-regular graph where $r\ge \frac{1}{2}(|M_3(x)|-1)$, for each vertex x. Then G is a hamiltonian graph.*

Now we will show that Proposition 2.3 is equivalent to the theorem of Nash-Williams (Theorem 2.3).

If $r\ge (|V(G)|-1)/2$ then $r\ge (|M_3(x)|-1)/2$ for each vertex x.

Conversely, let $r\ge (|M_3(x)|-1)/2$ for each vertex x in an r-regular 2-connected graph G. Lemma 2.1 implies that the diameter of G does not exceed 2. Therefore $M_3(u)=V(G)$ for each vertex u. This means that $r\ge (|V(G)|-1)/2$.

We showed above that Propositions 2.1–2.3 are equivalent to Theorems 2.1–2.3, respectively. Now we will show that if we replace in Propositions 2.1–2.3 the set $M_3(x)$ by the set $M_2(x)$ and the graph $G_3(x)$ by the graph $G_2(x)$ then we obtain new propositions (Propositions 2.4–2.6) which describe larger classes of hamiltonian graphs than the classes given by the corresponding original theorems.

In [2] (see also [4]) we proved that a connected graph G with $|V(G)|\ge 3$ is hamiltonian if $d(u)+d(v)\ge |M_2(x)|$ for every path uxv with $uv\notin E(G)$. This implies the following result:

Proposition 2.4. *Let G be a connected graph with $|V(G)| \geq 3$ where for every vertex $x \in V(G)$ the condition $d(u) + d(v) \geq |M_2(x)|$ holds for every pair of nonadjacent interior vertices u and v of the ball $G_2(x)$. Then G is hamiltonian.*

Clearly, Proposition 2.4 generalizes Proposition 2.1. The following two results were obtained in [1]:

Proposition 2.5 [1] *Let G be a connected graph with $|V(G)| \geq 3$ where all balls of radius 2 are 2-connected and $d(u) + d(v) \geq |M_2(x)| - 1$ for every path uxv with $uv \notin E(G)$. Then either G is a hamiltonian graph or $G \in \mathcal{K}$.*

Proposition 2.6 [1] *Let G be a connected regular graph on at least 3 vertices where every ball of radius 2 is 2-connected and $d(u) \geq \frac{1}{2}(|M_2(u)| - 1)$, for each vertex u. Then G is a hamiltonian graph.*

Propositions 2.5 and 2.6 generalize, respectively, Theorem 2.2 and Theorem 2.3, because for each graph G satisfying one of those theorems we have $G_2(u) = G$ for each vertex u.

Remark 2.1. The diameters of graphs satisfying the conditions of Theorem 2.1–2.3 do not exceed 2. In contrast with this, for every integer $n \geq 2$ there are graphs of diameter n which satisfy the conditions of Propositions 2.4–2.6. Consider, for example, the graph $G(p, 2n)$ which is defined as follows: its vertex set is $V_1 \cup \cdots \cup V_{2n}$, where $V_1, \ldots, V_{2n}$ are pairwise disjoint sets of cardinality $p \geq 2$ and two vertices of $G(p, 2n)$ are adjacent if and only if they both belong to $V_1 \cup V_{2n}$ or to $V_i \cup V_{i+1}$ for some $i \in \{1, 2, \ldots, 2n-1\}$. Clearly, $G(p, 2n)$ has diameter n and for each vertex x in $G(p, 2n)$ we have $|M_2(x)| = 5p$ and $d(x) = 3p - 1$. This implies that G satisfies the conditions of Propositions 2.4–2.6.

3 Localizations with radius four

In this section we give reformulation of seven well-known criteria for the hamiltonicity of a connected graph in terms of balls of radii four.

3.1 Reformulation of Dirac's theorem

Theorem 3.1 (Dirac's theorem [10]) *A graph G with $|V(G)| \geq 3$ is hamiltonian if $d(u) \geq \frac{1}{2}|V(G)|$ for each vertex u.*

In [3] (see also [4]) we obtained the following localization of Dirac's criterion:

Theorem 3.2 (Asratyan, Khachatryan [3,4]) *A connected graph G with $|V(G)| \geq 3$ is hamiltonian if $d(u) \geq \frac{1}{2}|M_3(u)|$ for each vertex u.*

The diameters of graphs satisfying the conditions of Dirac's theorem do not exceed 2. In contrast with this there is an infinite class of graphs of diameter 5 which satisfy the condition of Theorem 3.2. Consider, for example the graph G_n on $10n + 2$ vertices,

$n \geq 2$, which is defined as follows: its vertex set is $\cup_{i=0}^{5} V_i$, where $V_0, V_1, \ldots, V_5$ are pairwise disjoint sets of cardinality $|V_0| = |V_5| = n$, $|V_1| = |V_4| = 3n$, $|V_3| = |V_4| = n+1$ and two vertices of G_n are adjacent if and only if they both belong to $V_i \cup V_{i+1}$ for some $i \in \{0, 1, 2, 3, 4\}$. It is not difficult to see that G_n satisfies the condition of Theorem 3.2.

By considering the balls of radius 4 we obtain an evident corollary of Theorem 3.2:

Proposition 3.1. *A connected graph G with $|V(G)| \geq 3$ is hamiltonian if $d(u) \geq \frac{1}{2}|M_4(u)|$ for each vertex u.*

We will show that Proposition 3.1 is equivalent to Dirac's theorem.

If $d(u) \geq \frac{1}{2}|V(G)|$ then G is connected and $d(u) \geq \frac{1}{2}|M_4(u)|$ for each vertex u in G. Conversely, let $d(u) \geq \frac{1}{2}|M_4(u)|$ for each vertex u of a connected graph G, and let a vertex v have the minimum degree in G. We will show that $d(v) \geq \frac{1}{2}|V(G)|$. Suppose to the contrary that $d(v) < \frac{1}{2}|V(G)|$. Then $|M_4(v)| < |V(G)|$ and there is a vertex x in G with $d(x, v) = 3$. Clearly,

$$\begin{aligned} d(x) \leq |M_4(v) - M_1(v)| - 1 &= |M_4(v)| - |M_1(v)| - 1 \\ &= |M_4(v)| - d(v) - 2 \\ &\leq |M_4(v)| - 2 - \frac{1}{2}|M_4(v)| = \frac{1}{2}|M_4(v)| - 2. \end{aligned}$$

We have that $d(x) < d(v) = min_{u \in V(G)} d(u)$. This contradiction proves that $d(u) \geq \frac{1}{2}|V(G)|$ for each vertex u.

3.2 Reformulation of Fan's theorem

Theorem 3.3 (Fan [11]) *Let G be a graph with $|V(G)| \geq 3$ such that*

$$\max(d(x), d(y)) \geq \frac{1}{2}|V(G)|$$

for every pair of vertices x, y with $d(x, y) = 2$. Then G is hamiltonian.

We prove that Fan's theorem is equivalent to the following proposition:

Proposition 3.2. *Let G be a connected graph with $|V(G)| \geq 3$ vertices such that the condition $\max(d(x), d(y)) \geq \frac{1}{2}\max(|M_4(x)|, |M_4(y)|)$ holds for each pair of vertices x, y with $d(x, y) = 2$. Then G is hamiltonian.*

If G satisfies the condition of Fan's theorem then clearly it also satisfies the conditions of Proposition 3.2. Conversely, let G satisfy the conditions of Proposition 3.2. Suppose to the contrary that there is a pair of vertices x and y with $d(x, y) = 2$ such that $\max(d(x), d(y)) < \frac{1}{2}|V(G)|$. Then this and the condition of Proposition 3.2 imply that the set

$$V_1 = \{u \in V(G) : \frac{1}{2}|M_4(u)| \leq d(u) < \frac{1}{2}|V(G)|\}$$

is not empty. Let v be a vertex in V_1 such that $d(v) = \min_{u \in V_1} d(u)$. Since $v \in V_1$, we have that $N_i(v) \neq \emptyset$ for $i = 1, \ldots, 5$. Consider a path $u_0 u_1 \ldots u_5$ where $u_0 = v$ and $u_i \in N_i(v)$ for $i = 1, 2, 3, 4, 5$. Clearly, $d(u_1, u_5) = 4$ and

$$\begin{aligned} d(u_3) \ \leq \ |N_2(v)| + |N_3(v)| + |N_4(v)| - 1 \ &= \ |M_4(v)| - |N_1(v)| - 2 \\ &\leq \ |M_4(v)| - \frac{1}{2}|M_4(v)| - 2 \\ &= \ \frac{1}{2}|M_4(v)| - 2 \ < \ d(v). \end{aligned}$$

Then $u_3 \notin V_1$ because v has the minimum degree among the vertices of V_1. Therefore $d(u_3) < \frac{1}{2}|M_4(u_3)|$. This and $d(u_3, u_1) = 2 = d(u_3, u_5)$ imply, by the condition of Proposition 3.2, that

$$\begin{aligned} \max(d(u_3), d(u_1)) \ &\geq \ \frac{1}{2}\max(|M_4(u_3)|, |M_4(u_1)|) \ \text{ and} \\ \max(d(u_3), d(u_5)) \ &\geq \ \frac{1}{2}\max(|M_4(u_3)|, |M_4(u_5)|). \end{aligned}$$

These two inequalities and the inequality $d(u_3) < \frac{1}{2}|M_4(u_3)|$ imply that $d(u_1) \geq \frac{1}{2}|M_4(u_3)|$ and $d(u_5) \geq \frac{1}{2}|M_4(u_3)|$. Then the distance between two interior vertices u_1 and u_5 of the ball $G_4(u_3)$ must be 2, a contradiction with $d(u_1, u_5) = 4$. This contradiction proves that the condition of Proposition 3.2 is equivalent to the condition of Fan's theorem.

The diameters of graphs satisfying the conditions of Fan's theorem do not exceed 6. If we replace in Proposition 3.2 the sets $M_4(x)$ and $M_4(y)$ by the sets $M_3(x)$ and $M_3(y)$ respectively, we obtain a new proposition such that the class of graphs given by it is larger than the class of graphs given by Fan's theorem. (Consider, for example, the graph H_n of diameter 7 which is defined as follows: its vertex set is $\cup_{i=0}^{7} V_i$, where $V_0, V_1, \ldots, V_7$ are pairwise disjoint sets of cardinality $|V_0| = |V_7| = 2n$ $(n \geq 1)$, $|V_1| = |V_6| = 2n + 2$, $|V_2| = |V_3| = |V_4| = |V_5| = 3n + 3$ and two vertices of G_n are adjacent if and only if they both belong to $V_i \cup V_{i+1}$ for some $i \in \{0, 1, 2, 3, 4, 5, 6\}$.)

Conjecture. Let G be a connected graph with $|V(G)| \geq 3$ vertices such that the condition $\max(d(x), d(y)) \geq \frac{1}{2}\max(|M_3(x)|, |M_3(y)|)$ holds for each pair of vertices x, y with $d(x, y) = 2$. Then G is hamiltonian.

3.3 Reformulation of results

In this subsection we give a reformulation of the results of Häggkvist-Nicoghossian [13], Bauer et al. [8] and Flandrin et al [12].

Let $\kappa(G)$ denote the connectivity of a graph G. The following results were obtained in [8,12,13]:

Theorem 3.4 (Bauer et al. [8]). *Let G be a 2-connected graph such that $d(x) + d(y) + d(z) \geq |V(G)| + \kappa(G)$ for each triple of independent vertices x, y, z. Then G is hamiltonian.*

Theorem 3.5 (Häggkvist and Nicoghossian [13]). *A 2-connected graph G is hamiltonian if $d(u) \geq \frac{1}{3}(|V(G)| + \kappa(G))$ for each vertex u.*

Theorem 3.6 (Flandrin et al. [12]). *Let G be a 2-connected graph such that $d(x) + d(y) + d(z) \geq |V(G)| + |N(x) \cap N(y) \cap N(z)|$ for any triple of independent vertices x, y, z. Then G is hamiltonian.*

We will show that the next three propositions are equivalent formulations of the above three theorems:

Proposition 3.3. *A 2-connected graph G is hamiltonian if for every vertex $w \in V(G)$ the condition $d(x) + d(y) + d(z) \geq |M_4(w)| + \kappa(G_4(w))$ holds for each triple of independent interior vertices x, y, z of the ball $G_4(w)$.*

Proposition 3.4. *A 2-connected graph G is hamiltonian if $d(u) \geq \frac{1}{3}(|M_4(w)| + \kappa(G_4(w))$ for each interior vertex u of the ball $G_4(w)$.*

Proposition 3.5. *Let G be a 2-connected graph such that $d(x) + d(y) + d(z) \geq |M_4(w)| + |N(x) \cap N(y) \cap N(z)|$ for each triple of independent interior vertices x, y, z of the ball $G_4(w)$. Then G is hamiltonian.*

We need the following lemma:

Lemma 3.1. *Let G be a connected graph such that for every vertex $w \in V(G)$ the condition $d(x) + d(y) + d(z) \geq |M_4(w)|$ holds for each triple of independent vertices x, y, z of the ball $G_4(w)$. Then there is a vertex $v \in V(G)$ such that $M_4(v) = V(G)$.*

Proof. Consider a vertex $v \in V(G)$ satisfying $|M_4(v)| = max_{u \in V(G)} |M_4(u)|$. Suppose that $M_4(v) \neq V(G)$. Then $N_5(v) \neq \emptyset$. Let x be a vertex in the set $N_3(v)$. If there is a vertex $y \in N_3(v)$ with $d(x, y) \geq 3$ then

$$d(v) \leq |M_4(v)| - |M_1(x)| - |M_1(y)| - 1 = |M_4(v)| - 3 - d(x) - d(y).$$

Thus $d(x)+d(y)+d(v) < |M_4(v)|$ for a triple of independent interior vertices x, y, v of the ball $G_4(v)$, which contradicts the condition of the lemma. Therefore $d(x, y) \leq 2$ for every $y \in N_3(v)$. Then $M_4(v) \cup N_5(v) \subseteq M_4(x)$. Since $M_4(v) \cap N_5(v) = \emptyset$ and $N_5(v) \neq \emptyset$, we have $|M_4(x)| > |M_4(v)|$ which contradicts our assumption. This contradiction proves that $M_4(v) = V(G)$. □

Now we prove that Theorem 3.4 is equivalent to Proposition 3.3.

Suppose that the conditions of Proposition 3.3 hold. Then, by Lemma 3.1, there is a vertex v such that $M_4(v) = V(G)$. We have that $G_4(v) = G$ and $\kappa(G) = \kappa(G_4(v))$. Therefore $d(x) + d(y) + d(z) \geq |M_4(v)| + \kappa(G_4(v)) = |V(G)| + \kappa(G)$ for each triple of independent vertices x, y, z of G. Thus the conditions of Theorem 3.4 hold.

Conversely, suppose that the condition of Theorem 3.4 holds for a 2-connected graph G. First we will show that the diameter of G does not exceed 5. Suppose to the contrary that there is a pair of vertices u, v in G such that $d(u, v) \geq 6$. Let $v_0 v_1 \dots v_r$

be a (u,v)-path where $v_0 = u$, $v_r = v$ and $r = d(u,v)$. Then $d(v_0, v_3) = d(v_3, v_6) = 3$ and

$$d(v_0) + d(v_3) + d(v_6) \le |V(G) \setminus \{v_0, v_3, v_6\}| = |V(G)| - 3.$$

Thus the condition of Theorem 3.4 does not hold for independent vertices v_0, v_3, v_6, which contradicts our assumption. Therefore $d(x,y) \le 5$ for each pair of vertices x, y in G, that is, the diameter of G does not exceed 5.

Now we will show that $|V(G)| + \kappa(G) \ge |M_4(w)| + \kappa(G_4(w))$ for each vertex w. Suppose to the contrary that $|V(G)| + \kappa(G) < |M_4(w)| + \kappa(G_4(w))$ for some vertex w. Then $|M_4(w)| < |V(G)|$ and $\kappa(G) < \kappa(G_4(w))$. This implies that $N_5(w) \ne \emptyset$ and $V(G) = M_4(w) \cup N_5(w)$ because the diameter of G does not exceed 5. Let $S = \{u_1, \ldots, u_\kappa\}$ be a vertex cut of G, $\kappa = \kappa(G)$, and let $H_1, \ldots, H_r$ be the components of the graph $H = G - S$, $r \ge 2$. Since $\kappa(G) < \kappa(G_4(w))$, the graph $G_4(w) - S$ is connected. Therefore $M_4(w) \subseteq S \cup V(H_i)$, for some $i, 1 \le i \le r$. Without loss of generality we assume that $M_4(w) \subseteq S \cup V(H_1)$. Then $V(H_2) \subseteq N_5(w)$ because $V(G) = M_4(w) \cup N_5(w)$. Consider three independent vertices w, y, z where $y \in N_3(w)$ and $z \in V(G_2) \subseteq N_5(w)$. Then $d(z) \le |N_5(w)| - 1 + \kappa(G)$ and

$$\begin{aligned} & d(w) + d(y) + d(z) \\ \le\ & |N_1(w)| + (|N_2(w)| + |N_3(w)| - 1 + |N_4(w)|) + |N_5(w)| - 1 + \kappa(G) \\ =\ & |V(G)| - 3 + \kappa(G). \end{aligned}$$

Thus the condition of Theorem 3.4 does not hold for the independent vertices w, y, z, which contradicts our assumption. Therefore $|V(G)| + \kappa(G) \ge |M_4(w)| + \kappa(G_4(w))$ for each vertex w. This implies that the conditions of Proposition 3.3 hold.

Therefore Proposition 3.3 is equivalent to Theorem 3.4.

Using similar arguments we can show that Theorem 3.5 is equivalent to Proposition 3.4 and Theorem 3.6 is equivalent to Proposition 3.5.

3.4 Reformulation of Jackson's theorem and its generalization

The following results were obtained in [14,18]:

Theorem 3.7 (Jackson [14]). *A 2-connected r-regular graph G is hamiltonian if $r \ge \frac{1}{3}|V(G)|$.*

Theorem 3.8 (Zhu, Liu, Yu [18]). *Let G be a 2-connected r-regular graph with $r \ge \frac{1}{3}(|V(G)| - 1)$. Then either G is hamiltonian or G is the Petersen graph.*

We will prove that Jackson's theorem is equivalent to the following proposition:

Proposition 3.6. *A 2-connected r-regular graph G is hamiltonian if $r \ge \frac{1}{3}|M_4(w)|$ for each vertex w.*

If $r \ge \frac{1}{3}|V(G)|$ then clearly $r \ge \frac{1}{3}|M_4(w)|$ for each vertex w in G.

Conversely, let $r \geq \frac{1}{3}|M_4(w)|$ for each vertex w. Then Lemma 3.1 implies that there is a vertex v such that $M_4(v) = V(G)$. This means that $r \geq \frac{1}{3}|V(G)|$.

Thus the condition $r \geq \frac{1}{3}|V(G)|$ is equivalent to the condition $r \geq \frac{1}{3}|M_4(w)|$ for each vertex w.

Using similar arguments we can show that Theorem 3.8 is equivalent to the following proposition:

Proposition 3.7. *Let G be a 2-connected r-regular graph where $r \geq \frac{1}{3}(|M_4(w)| - 1)$ for each vertex w. Then either G is hamiltonian or G is the Petersen graph.*

Remark 3.1. If we replace in Propositions 3.3–3.7 the set $M_4(w)$ by the set $M_3(w)$ and the graph $G_4(w)$ by the graph $G_3(w)$ we obtain new propositions such that the classes of graphs given by them are larger than the classes of graphs given by the corresponding original theorems. (Consider, for example, the $(3p-1)$-regular graph $G = G(p, 2n)$ defined in Remark 2.1 with $n \geq 7$ and $p \geq 5$ which has $\kappa(G) = 2p$, $\kappa(G_4(w)) = p$, $|M_4(w)| = 9p$ and $|M_3(w)| = 7p$ for every vertex w.)

Conjecture. A 2-connected r-regular graph G is hamiltonian if $r \geq \frac{1}{3}|M_3(w)|$ for each vertex w in G.

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