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Closed-neighborhood anti-Sperner graphs

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Abstract

For a simple graph G let $N_G[u]$ denote the closed-neighborhood of vertex $u \in V(G)$. Then G is closed-neighborhood anti-Sperner (CNAS) if for every u there is a $v \in V(G)\backslash\{u\}$ with $N_G[u] \subseteq N_G[v]$; and a graph H is closed-neighborhood distinct (CND) if every closed-neighborhood is distinct, *i.e.*, if $N_H[u] \neq N_H[v]$ when $u \neq v$, for all u and $v \in V(H)$.

In this paper we are mainly concerned with proving some simple properties of CNAS graphs, and constructing CNAS graphs. We construct a family of connected CNAS graphs with n vertices for each fixed $n \geq 2$. We classify all connected CNAS graphs with ≤ 6 vertices using these families, and find the smallest connected CNAS graph that lies outside these families. We indicate how some CNAS graphs can be constructed from a related type of graph, called a NAS graph. Finally, we present an algorithm to construct all CNAS graphs on a fixed number of vertices from labelled CND graphs on fewer vertices.

1 Closed-Neighborhood anti-Sperner Graphs

Let $\mathcal{F} = \{N_1, N_2, \ldots\}$ be a family of sets. Then $\mathcal{F}$ is *Sperner* if no member of $\mathcal{F}$ is a subset of another member; and $\mathcal{F}$ is *anti-Sperner* if *every* member of $\mathcal{F}$ is a subset of another member.

Let G be a simple graph with a finite number of vertices. For each $u \in V(G)$ let $N_G[u]$ denote the closed-neighborhood of u, *i.e.*, vertex u together with the set of vertices to which u is adjacent.

Let $\mathcal{F}(G) = \{N_G[u] \mid u \in V(G)\}$ be the family of closed-neighborhoods of G. Then if $\mathcal{F}(G)$ is anti-Sperner we say that G is a *closed-neighborhood anti-Sperner (CNAS) graph*, *i.e.*, for every $u \in V(G)$ there is a $u^p \in V(G)\backslash\{u\}$ with $N_G[u] \subseteq N_G[u^p]$. Vertex u^p is a *closed-parent* of vertex u; so a CNAS graph is a graph in which every vertex has a closed-parent. We note that u and u^p are adjacent.

A graph H is *closed-neighborhood distinct (CND)* if every closed-neighborhood is distinct, *i.e.*, if $N_H[u] \neq N_H[v]$ when $u \neq v$, for all u and $v \in V(H)$.

If we replace the word 'closed' by 'open' in the first definition above then we have an open-neighborhood anti-Sperner graph, which we call a NAS graph. These graphs were introduced by Porter in [6], and studied further in Porter and Yucas [7], and in McSorley [4]. Our CNAS graphs are a natural variation of these graphs.

Both NAS and CNAS graphs are interesting because their definitions are quite natural; they also have connections with graphs which have been previously studied in Sumner [11] and Lim [3], see Section 4 of this paper. Many natural questions concerning extremal properties, girth, chromatic number– indeed almost any graph parameter or property– can be asked of these graphs; some such questions will be considered in future research. There is also an interesting connection with Cayley graphs (see McSorley [5]).

In this paper we are mainly concerned with proving some simple properties of CNAS graphs, and constructing CNAS graphs:

In Section 2 we construct a family of connected CNAS graphs with n vertices for each fixed $n \geq 2$. We classify all connected CNAS graphs with ≤ 6 vertices using these families, and find the smallest connected CNAS graph that lies outside these families.

In Section 3 we return to NAS graphs and indicate how some, but not all, CNAS graphs on a fixed number of $n \geq 2$ vertices can be constructed from a suitable NAS graph also on n vertices, thus establishing a link between the two different types of graphs.

Section 4 contains preparatory material for Section 5, in which we present an algorithm to construct all CNAS graphs on a fixed number of $n \geq 2$ vertices from labelled CND graphs on $\leq n-1$ vertices. This is similar to an algorithm that constructs NAS graphs from labelled ND (neighborhood distinct) graphs in McSorley [4].

Standard definitions of graph theory are from West [12].

2 Properties of CNAS graphs, families of CNAS graphs, small CNAS graphs

We first show some elementary properties of connected CNAS graphs; these properties are similar to the properties of NAS graphs proved in Sections 1 and 2 of [7]. Here $\delta(G)$ denotes the minimum degree of G, and $g(G)$ denotes the girth of G.

Theorem 2.1 *Let G be a connected CNAS graph on $n \geq 3$ vertices. Then $\delta(G) \geq 2$.*

Proof. Suppose that $\delta(G) = 1$ and let $u \in V(G)$ have degree 1. Let v be the unique neighbor of u. Now v is adjacent to its closed-parent v^p, and, since $n \geq 3$, then $v^p \neq u$. So $\{u, v\} \subseteq N_G[v^p]$, *i.e.*, $uv^p \in E(G)$, a contradiction since the degree of u is 1. ∎

Theorem 2.2 *Let G be a connected CNAS graph on $n \geq 3$ vertices. Then $g(G) = 3$, i.e., G contains a triangle.*

Proof. Since G is connected and $\delta(G) \geq 2$ then G must have a cycle of length ≥ 3. Suppose $g(G) = \mathrm{g} \geq 4$, and let $u_1 u_2 \ldots u_{\mathrm{g}}$ be a g-cycle. Now either u_2^p lies on this cycle, or it doesn't. In the first case, without loss of generality let $u_2^p = u_1$, then $u_1 u_3 \in E(G)$, *i.e.*, $u_1 u_2 u_3$ is a 3-cycle, a contradiction. In the second case u_2^p lies off this cycle. But then both $u_2 u_2^p \in E(G)$ and $u_1 u_2^p \in E(G)$, and so $u_1 u_2 u_2^p$ is a 3-cycle, a contradiction. Hence $g(G) = 3$. ∎

Theorem 2.3 *Let G be a connected CNAS graph on $n \geq 2$ vertices. Then G contains no cut-vertices, i.e., G is 2-connected.*

Proof. Let u be a cut-vertex with closed parent u^p. Let C_1 and C_2 be 2 components of $G - u$ with $v_1 \in C_1$ and $v_2 \in C_2$, and with $uv_1, uv_2 \in E(G)$. Now either $u^p \in \{v_1, v_2\}$ or $u^p \notin \{v_1, v_2\}$. First, say $u^p = v_1$, then $v_1 v_2$ is a $v_1 - v_2$ path in $G - u$, a contradiction. Similarly in the second case $v_1 u^p v_2$ is a $v_1 - v_2$ path in $G - u$, again a contradiction. Hence G contains no cut-vertices. ∎

For $n \geq 1$ let K_n denote the complete graph on n vertices. For $m \geq 2$ let S_m be a connected or disconnected graph on m vertices, with no isolates. For $n \geq 2$ and $2 \leq m \leq n$ let $K_n \backslash S_m = K_n - E(S_m)$ denote the complete graph K_n with the edges of S_m removed. Finally, in any graph on $n \geq 2$ vertices, call a vertex *full* if it has degree $n - 1$.

We are primarily interested in connected CNAS graphs, since each component in a disconnected CNAS graph must itself be CNAS.

Theorem 2.4 *Let G be an arbitrary graph on $n \geq 2$ vertices with at least two full vertices. Then G is a connected CNAS graph.*

Proof. Clearly G is connected. Let u and $v \in V(G)$ be two full vertices then $N_G[u] = N_G[v] = V(G)$, the whole vertex set of G. So vertex u is a closed-parent of all vertices in $V(G) \backslash \{u\}$, and v is a closed-parent of u. Hence every vertex in $V(G)$ has a closed-parent, and so G is CNAS. ∎

In particular, for $n \geq 2$, the complete graph K_n is CNAS. Furthermore, we may preserve the CNAS property by removing edges from K_n provided that we always leave at least two full vertices:

Corollary 2.5 *For any $n \geq 2$ and any m with $2 \leq m \leq n-2$ let S_m be a graph on m vertices with no isolates. Then $K_n \backslash S_m$ is a connected CNAS graph.* ■

Indeed, we can classify incomplete connected CNAS graphs on n vertices with at least two full vertices:

Theorem 2.6 *For $n \geq 2$ let $G \neq K_n$ be a connected CNAS graph on n vertices with at least two full vertices. Then there is a graph S_m on m vertices with no isolates where $2 \leq m \leq n-2$ such that $G = K_n \backslash S_m$.*

Proof. Let $\{u_1, u_2, \ldots, u_m\}$ be the non-full vertices of G, since $G \neq K_n$ then $m \geq 2$. And let $\{u_{m+1}, \ldots, u_n\}$ be the full vertices of G, the number of these is $n - m \geq 2$, so $m \leq n-2$. Consider a copy of K_m with vertex set $\{u_1, u_2, \ldots, u_m\}$, and the K_n with vertex set $\{u_1, u_2, \ldots, u_n\}$. Let $S_m = \overline{G[u_1, u_2, \ldots, u_m]}$, where the complement is taken in the above K_m. Now if u_i is an isolate in S_m then in $G[u_1, u_2, \ldots, u_m]$ it has (full) degree $m-1$ and so in G it is full, a contradiction. Hence S_m has no isolates. Then $G = K_n \backslash S_m$, and $2 \leq m \leq n-2$. ■

Let

$$\mathcal{K}_n \backslash \mathcal{S} = \{K_n\} \cup \{K_n \backslash S_m \mid S_m \text{ has } m \text{ vertices and no isolates, } 2 \leq m \leq n-2\}.$$

So, for each $n \geq 2$, we have a family $\mathcal{K}_n \backslash \mathcal{S}$ of connected CNAS graphs, each member of which has at least two full vertices.

For those G outside these families we have:

Theorem 2.7 *Let G be a connected CNAS graph on $n \geq 2$ vertices without at least two full vertices. Then G has no full vertices.*

Proof. Clearly G cannot have exactly one full vertex, because this full vertex would not have a closed-parent; hence it has no full vertices. ■

Example 1 From checking in Read and Wilson [9] there are exactly 20 connected CNAS graphs G on $n \leq 6$ vertices. All except one, F, belongs to a family $\mathcal{K}_n \backslash \mathcal{S}$. The graph F is shown below. It is the smallest connected CNAS graph that lies outside these families, *i.e.*, without at least two full vertices. So, from Theorem 2.7, it has no full vertices, indeed it has 6 vertices and maximum degree 4.

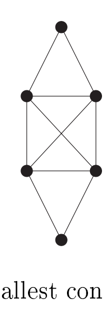

Fig. 1. F, the smallest connected CNAS graph without at least two full vertices

3 CNAS graphs and Neighborhood anti-Sperner graphs

In this Section we show how to construct some CNAS graphs with a fixed number of n vertices from NAS graphs with n vertices, thus establishing a connection between the two different types of graph.

Let $\mathcal{F}_\text{o}(H) = \{N_H(u) \,|\, u \in V(H)\}$ be the family of *open*-neighborhoods of a graph H. We always drop the prefix 'open' in open-neighborhood, open-parent, open-twin, etc.. Then if $\mathcal{F}_\text{o}(H)$ is anti-Sperner we say that H is a *neighborhood anti-Sperner (NAS) graph*. Hence, in a NAS graph H, for every $u \in V(H)$ there is a *parent* $u^{p_\text{o}} \in V(H)\backslash\{u\}$ such that $N_H(u) \subseteq N_H(u^{p_\text{o}})$.

NAS graphs have been studied in [4], [6], and [7]. Because the definition of a CNAS graph is similar to that of a NAS graph, we might sensibly ask whether we can construct CNAS graphs from NAS graphs. However it doesn't seem possible to construct *all* CNAS graphs of order n from NAS graphs of order n, but some CNAS graphs can be constructed:

For an arbitrary graph G, the set $P \subseteq V(G)$ is a *closed-parent-set* if it is closed under taking closed-parents, *i.e.*, if every $u \in P$ has a closed-parent $u^p \in P$. And a *closed-parent-set partition* of $V(G)$ is a partition of $V(G)$ into closed-parent-sets. Similarly, for an arbitrary H, the set $P_\text{o} \subseteq V(H)$ is a *parent-set* if it is closed under taking parents, *i.e.*, if every $u \in P_\text{o}$ has a parent $u^{p_\text{o}} \in P_\text{o}$. And a *parent-set partition* of $V(H)$ is a partition of $V(H)$ into parent-sets.

Theorem 3.1 *Let H be a NAS graph with $\{P_{\text{o},1}, P_{\text{o},2}, \ldots, P_{\text{o},d}\}$ a parent-set partition of $V(H)$. Let $G = H^+$ be the graph obtained from H by making $P_{\text{o},i}$ into a clique for each $1 \le i \le d$,* i.e.*, by making $G[P_{\text{o},i}] = K_{|P_{\text{o},i}|}$. Then G is a CNAS graph and $\{P_{\text{o},1}, P_{\text{o},2}, \ldots, P_{\text{o},d}\}$ is a closed-parent-set partition of $V(G)$.*

Proof. For an arbitrary vertex $u \in V(G) = V(H)$ let $u \in P_{\text{o},i}$ for some fixed i, so $N_G[u] = N_H(u) \cup P_{\text{o},i}$. Now, in H, let $u^{p_\text{o}} \in P_{\text{o},i}$ be a parent of $u \in P_{\text{o},i}$, so $N_G[u^{p_\text{o}}] = N_H(u^{p_\text{o}}) \cup P_{\text{o},i}$. Hence, since $N_H(u) \subseteq N_H(u^{p_\text{o}})$, then $N_G[u] \subseteq N_G[u^{p_\text{o}}]$, *i.e.*, $u^{p_\text{o}} \in P_{\text{o},i}$ is a closed-parent of u in G. Hence (in G) $P_{\text{o},i}$ is a closed-parent-set. Furthermore, since $u \in V(G)$ is arbitrary then G is CNAS. Each $P_{\text{o},i}$ is a closed-parent-set of $V(G)$, and so $\{P_{\text{o},1}, P_{\text{o},2}, \ldots, P_{\text{o},d}\}$ is a closed-parent-set partition of $V(G)$. ■

The null graph N_n is the graph with $n \ge 1$ vertices and no edges.

Example 2 See Fig. 2. We can construct many non-isomorphic CNAS graphs from a single NAS graph. Consider the NAS graph $H = N_1 \cup N_1 \cup K_{2,2}$ on 6 vertices. There are 5 different parent-set partitions of $V(H)$, yielding 4 non-isomorphic CNAS graphs $G = H^+$ on 6 vertices, 3 of which are connected.

The construction of Theorem 3.1 yields a CNAS graph $G = H^+$ with a closed-parent-set partition of $V(G)$ in which each closed-parent-set is a clique. However not all

CNAS graphs have such a partition, and those that do not cannot be obtained via this construction no matter which NAS graph H and which parent-set partition of $V(H)$ is used. The smallest CNAS graph without such a partition is $K_4 \backslash K_2$. Other constructions of CNAS graphs from NAS graphs do not seem to be available. But Theorem 3.1 is still useful for obtaining some CNAS graphs from NAS graphs, as illustrated in Example 2.

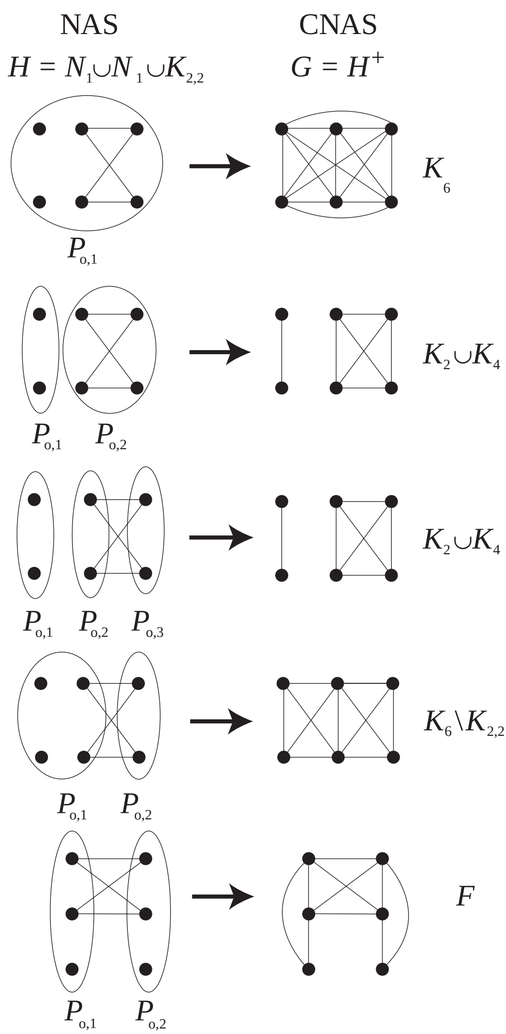

Fig. 2. The CNAS graphs which can be obtained from the NAS graph $H = N_1 \cup N_1 \cup K_{2,2}$; see Example 2

4 Closed-Neighborhood Distinct graphs

This Section contains preparatory material needed in Section 5.

The join $X \vee Y$ of two graphs X and Y with disjoint vertex sets is the graph with vertex set $V(X) \cup V(Y)$ and edge set $E(X) \cup E(Y) \cup \{xy \mid x \in V(X) \text{ and } y \in V(Y)\}$, *i.e.*, every vertex in $V(X)$ is joined to every vertex in $V(Y)$.

Recall that a graph H is closed-neighborhood distinct (CND) if every closed-neighborhood is distinct, *i.e.*, if $N_H[u] \neq N_H[v]$ when $u \neq v$, for all u and $v \in V(H)$. Sumner [11] called such graphs *point distinguishing* and they are also known as *supercompact*, see Lim [3]. See also Entringer and Gassman [2] for further properties of these graphs.

Sumner proved the following Theorem for graphs in which every neighborhood is distinct, which he called point determining. But he stated that there is a dual Theorem for CND graphs. We state his theorem using our notation, (see Theorem 2 of Sumner [11] and Theorem 2.1 of Chia and Lim [1]):

Let H be a CND graph with ≥ 2 vertices. Then there is a vertex $w \in V(H)$ such that $H - w$ is also CND.

We use Sumner's result in the following algorithm which constructs all CND graphs on t vertices from CND graphs on $t-1$ vertices:

Algorithm CND Graphs A four step algorithm to construct all CND graphs H on a fixed number of $t \geq 2$ vertices from all CND graphs on $t-1$ vertices.

(1) List all non-isomorphic CND graphs H_{t-1} on $t-1$ vertices.

(2) For each H_{t-1} list all subsets $S \subseteq V(H_{t-1})$ for which $S \neq N_{H_{t-1}}[u]$ for all $u \in V(H_{t-1})$, *i.e.*, S is distinct from all closed-neighborhoods of H_{t-1}. Note that $S = \emptyset$ is to be considered.

(3) Let $w \notin V(H_{t-1})$ be a new vertex. For each such H_{t-1} and S let H be the graph with vertices and edges as follows:

$$V(H) = V(H_{t-1}) \cup \{w\} \quad \text{and} \quad E(H) = E(H_{t-1}) \cup \{ws \mid s \in S\},$$

i.e., H is the graph obtained by joining w to S.

(4) Remove isomorphic copies from the graphs in (3).

We now have a complete list of CND graphs H with t vertices, with no repeated H. The CND graphs with ≤ 3 vertices are N_1, N_2, N_3, and P_3.

Example 3 We find all CND graphs on 4 vertices from the two CND graphs N_3 and P_3 on 3 vertices. For each such graph there are $2^3 - 3 = 5$ subsets S, yielding 10 CND graphs on 4 vertices. Removing isomorphic copies leaves the 5 non-isomorphic CND graphs below:

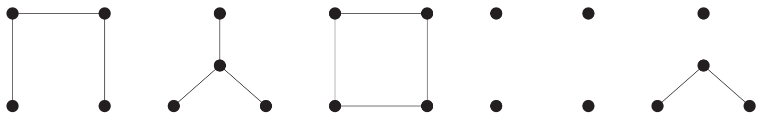

Fig. 3. All CND graphs on 4 vertices

Clearly we can then use these CND graphs on 4 vertices to construct all CND graphs on 5 vertices, and so on. Hence, for any $t \geq 1$, we can construct all CND graphs on $\leq t$ vertices.

We will need *labelled* graphs H in which every vertex $u \in V(H)$ has been labelled with a positive integer $\ell(u) \geq 1$. We also need the concept of label-isomorphism:

Let H and H' be two arbitrary labelled graphs. Then H and H' are *label-isomorphic* if there is a bijection between $V(H)$ and $V(H')$ which is a graph isomorphism that preserves labels. So if in a label-isomorphism we have $u \in V(H) \leftrightarrow u' \in V(H')$, then $\ell(u) = \ell(u')$.

Finally, for an arbitrary graph G, if $N_G[u] = N_G[v]$ for two different vertices u and $v \in V(G)$ then u and v are *closed-twins*. We note that closed-twins are adjacent. We denote a closed-twin of u by u^*. If u has no closed-twin then it is *closed-twinless*. If H is CND then every vertex in $V(H)$ is closed-twinless.

5 Constructing CNAS Graphs from labelled CND Graphs

In this final Section we show how to construct all CNAS graphs G on a fixed number of $n \geq 2$ vertices from labelled CND graphs H on $\leq n-1$ vertices.

Let G be an arbitrary graph. Consider the following equivalence relation $\equiv$ on $V(G)$: $u \equiv u'$ if and only if $N_G[u] = N_G[u']$. The equivalence class containing u is $U = \{u' \in V(G) \,|\, N_G[u] = N_G[u']\} \neq \emptyset$. Here every vertex u' is a closed-twin of u, which we normally write as u^*, provided that it is distinct from u. We let t denote the number of equivalence classes under $\equiv$ of $V(G)$ (or of G); and denote the classes themselves by $U_1, U_2, \ldots, U_t$, where $|U_i| = \ell_i$ for each $i = 1, 2, \ldots, t$.

Theorem 5.1 *Let G be an arbitrary graph with equivalence relation $\equiv$. Let U and V be two distinct equivalence classes with $u \in U$ and $v \in V$ arbitrary. Then*

(i) the induced subgraph $G[U] = K_{|U|}$,

(ii) $uv \in E(G)$ if and only if $G[U \cup V] = G[U] \vee G[V] = K_{|U|} \vee K_{|V|}$,

(iii) $uv \notin E(G)$ if and only if $G[U \cup V] = K_{|U|} \cup K_{|V|}$.

Proof. (i) If $|U| = 1$ then clearly $G[U] = K_{|U|}$. So assume that $|U| \geq 2$ and let u and u^* be two arbitrary distinct vertices in U. Then $u \in N_G[u] = N_G[u^*]$, *i.e.*, $uu^* \in E(G)$. Since u and u^* are arbitrary, then $G[U] = K_{|U|}$.

(ii) Now $u \in U$ and $v \in V$ are arbitrary, let $u' \in U$ and $v' \in V$ also be arbitrary, (so $u = u'$ and/or $v = v'$ is allowed). Since $uv \in E(G)$, so $v \in N_G[u] = N_G[u']$, so $u' \in N_G[v] = N_G[v']$, and then $u'v' \in E(G)$. Hence $G[U \cup V] = G[U] \vee G[V] = K_{|U|} \vee K_{|V|}$, using (i). The converse is clear.

(iii) Similar to (ii), using (i) again. ∎

So, in any graph G and for any two distinct equivalence classes U and V, either $G[U \cup V] = K_{|U|} \vee K_{|V|}$ or $G[U \cup V] = K_{|U|} \cup K_{|V|}$. This suggests the following two constructions:

Construction $\mathbf{G}_\equiv$ Let G be an arbitrary graph, with equivalence relation $\equiv$ and equivalence classes $U_1, U_2, \ldots, U_t$, where $|U_i| = \ell_i$ for each $i = 1, 2, \ldots, t$. Construct a labelled graph $G_\equiv$ with t vertices, and edges as follows:

$$V(G_\equiv) = \{U_1, U_2, \ldots, U_t\} \text{ and } E(G_\equiv) = \{U_iU_j \mid G[U_i \cup U_j] = K_{|U_i|} \vee K_{|U_j|}\},$$

where vertex U_i has been labelled with ℓ_i for each i. We call $G_\equiv$ the *closed-reduced* graph of G. See [3] where an unlabelled version of this graph is called $S(G)$. An unlabelled version is also known as the *Roberts reduct*, see Roberts [10], and Section 10.6 of Prisner [8]. Note that $|V(G)| = \sum_{i=1}^{t} \ell_i$.

Construction $\mathbf{H}^\uparrow$ Let H be an arbitrary labelled graph, so every $u \in V(H)$ has been labelled with a positive integer $\ell(u) \geq 1$. Construct a new graph $H^\uparrow$ from H by replacing each vertex u with the $\ell(u)$ vertices from $exp(u) = \{x_1, x_2, \ldots, x_{\ell(u)}\}$, the *expansion set* of u, where $H^\uparrow[exp(u)] = K_{\ell(u)}$ is a clique. Similarly, replace v by $exp(v) = \{y_1, y_2, \ldots, y_{\ell(v)}\}$, etc.. If $uv \in E(H)$ then let $H^\uparrow[exp(u), exp(v)] = K_{\ell(u)} \vee K_{\ell(v)}$, and if $uv \notin E(H)$ then let $H^\uparrow[exp(u), exp(v)] = K_{\ell(u)} \cup K_{\ell(v)}$.

We illustrate these constructions with F below. The equivalence classes of F under $\equiv$ are: $U_1 = \{x_1\}$, $U_2 = \{y_1, y_2\}$, $U_3 = \{z_1, z_2\}$, and $U_4 = \{t_1\}$.

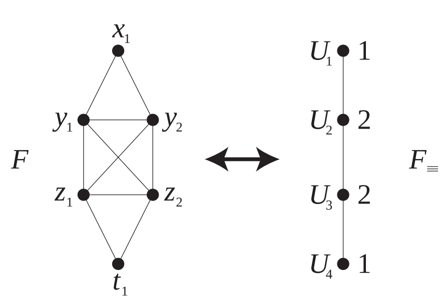

Fig. 4. Illustrating constructions $G_\equiv$ and $H^\uparrow$

From the above two constructions we have:

Theorem 5.2 *Let G be an arbitrary graph. Then $G = (G_{\equiv})^{\uparrow}$.* ■

Given an arbitrary graph G, as we closed-reduce to $G_{\equiv}$ we identity vertices with the same closed-neighborhood, so $G_{\equiv}$ should be CND (Theorem 3.1 of [3]):

Theorem 5.3 *Let G be an arbitrary graph. Then $G_{\equiv}$ is CND.*

Proof. Let U and V be two distinct vertices in $V(G_{\equiv})$. Suppose that $G_{\equiv}$ is not CND and $N_{G_{\equiv}}[U] = N_{G_{\equiv}}[V] = \{U, V, U_1, U_2, \ldots, U_d\}$ for some $d \geq 1$, or $N_{G_{\equiv}}[U] = N_{G_{\equiv}}[V] = \{U, V\}$.

In the first case let $u \in V(G)$ lie in equivalence class U, then, since $N_{G_{\equiv}}[U]$ is a clique, we have $N_G[u] = U \cup V \cup (\bigcup_{k=1}^{d} U_k)$. Similarly, if $v \in V$ then $N_G[v] = V \cup U \cup (\bigcup_{k=1}^{d} U_k)$. Hence $N_G[u] = N_G[v]$ so $u \equiv v$, a contradiction since $u \in U$ and $v \in V$ and $U \neq V$. Thus $G_{\equiv}$ is CND. The proof is similar when $N_{G_{\equiv}}[U] = N_{G_{\equiv}}[V] = \{U, V\}$. ■

The following two technical Lemmas are required before our main results:

Lemma 5.4 *Let H be an arbitrary labelled CND graph with $t \geq 1$ vertices. Then $H^{\uparrow}$ has t equivalence classes under $\equiv$.*

Proof. Let $H^{\uparrow}$ have s equivalence classes under $\equiv$, we will show that $s = t$.

Let each vertex $u \in V(H)$ be labelled with $\ell(u) \geq 1$. The Lemma is clearly true if $t = 1$. So assume that $t \geq 2$ and let u and v be distinct vertices in $V(H)$. In the construction of $H^{\uparrow}$ from H we replace u by the $\ell(u)$ vertices from $exp(u) = \{x_1, x_2, \ldots, x_{\ell(u)}\}$, and we replace v by the $\ell(v)$ vertices from $exp(v) = \{y_1, y_2, \ldots, y_{\ell(v)}\}$. Let $x_i \in exp(u)$ and $y_j \in exp(v)$ be arbitrary. Now, since H is CND, we have $N_H[u] \neq N_H[v]$. Without loss of generality let $w \in N_H[u] \backslash N_H[v]$ and let $exp(w) = \{z_1, z_2, \ldots, z_{\ell(w)}\}$, ($w = u$ is allowed). If $w \neq u$ then, in $H^{\uparrow}$, we have $x_i \in N_{H^{\uparrow}}[z_1]$ but $y_j \notin N_{H^{\uparrow}}[z_1]$. So $N_{H^{\uparrow}}[x_i] \neq N_{H^{\uparrow}}[y_j]$, and so $x_i \not\equiv y_j$ in $H^{\uparrow}$. So x_i and y_j are in distinct equivalence classes of $H^{\uparrow}$. Now let $V(H) = \{u_1, u_2, \ldots, u_t\}$. We can apply the above argument to every distinct pair u_a and $u_b \in V(H)$, showing that $exp(u_a)$ and $exp(u_b)$ are contained in distinct equivalence classes of $H^{\uparrow}$. Hence $t \leq s$. A slight modification of this argument is required if $w = u$.

Suppose $s > t$. Let $e_1, e_2, \ldots, e_s$ be representatives of the s equivalence classes under $\equiv$ in $H^{\uparrow}$, one from each class. Then, by the pigeon hole principle, there must be some vertex $u \in V(H)$ whose expansion set $exp(u)$ contains two of $e_1, e_2, \ldots, e_s$. Suppose that e_a and $e_b \in exp(u)$ where $1 \leq a < b \leq s$, then $N_{H^{\uparrow}}[e_a] = N_{H^{\uparrow}}[e_b]$, *i.e.*, $e_a \equiv e_b$ in $H^{\uparrow}$, a contradiction. Hence $s \leq t$. And so $s = t$. ■

In an arbitrary graph G we say that vertex $u \in V(G)$ is *closed-parentless* if u does not have a closed-parent. And if u does have a closed-parent u^p with $N_G[u] \subset N_G[u^p]$ then we call u^p a *proper closed-parent* of u.

In the following Lemma, as usual, we denote the equivalence class under $\equiv$ containing u by U, and the equivalence class containing u^p by U^p.

Lemma 5.5

(i) In an arbitrary graph G let u^p be a proper closed-parent of u. Then, in $G_\equiv$, U^p is a proper closed-parent of U.

(ii) For a CNAS graph G let $W \in V(G_\equiv)$ be closed-parentless. Then $\ell(W) \geq 2$.

Proof. (i) In G since u^p is a proper closed-parent of u then $N_G[u^p] \neq N_G[u]$, and so $U^p \neq U$, *i.e.*, in $G_\equiv$ the vertices U^p and U are distinct, and $U^pU \in E(G_\equiv)$.

We first show that U^p is a closed-parent of U. If not, then there exists a vertex V with $V \in N_{G_\equiv}[U]$ but $V \notin N_{G_\equiv}[U^p]$. Now $V \neq U$ since $U^pU \in E(G_\equiv)$ and so $U \in N_{G_\equiv}[U^p]$, and $V \neq U^p$ since $U^p \in N_{G_\equiv}[U^p]$. Let $v \in V(G)$ be in equivalence class V. Then $v \in N_G[u]$ but $v \notin N_G[u^p]$, a contradiction since u^p is a (proper) closed-parent of u. So, in $G_\equiv$, U^p is a closed-parent of U. Now $G_\equiv$ is CND so U^p cannot be a closed-twin of U, but it is a closed-parent of U, so it must be a proper closed-parent of U.

(ii) Let $W \in V(G_\equiv)$ be closed-parentless, then W has no proper closed-parents in $G_\equiv$. Let $w \in V(G)$ lie in equivalence class W, so, by (i), w has no proper closed-parents in G. But G is CNAS so w must have a closed-parent which must be a closed-twin w^*, so $|W| \geq 2$, *i.e.*, $\ell(W) \geq 2$. ∎

The following main result deals with both connected and disconnected CNAS graphs.

Theorem 5.6 *Let G be an arbitrary graph. Then G is a CNAS graph with t equivalence classes under $\equiv$ if and only if $G_\equiv$ is a labelled t vertex CND graph in which all closed-parentless vertices have label ≥ 2.*

Proof. First let G be a CNAS graph with t equivalence classes under $\equiv$ given by $U_1, U_2, \ldots, U_t$, where $|U_i| = \ell_i$ for each $i = 1, 2, \ldots, t$. Then the construction of $G_\equiv$ from G and Theorem 5.3 shows that $G_\equiv$ is a labelled t vertex CND graph. From Lemma 5.5(ii) all closed-parentless vertices in $G_\equiv$ have label ≥ 2.

Conversely suppose that $G_\equiv$ is a labelled t vertex CND graph in which all closed-parentless vertices have label ≥ 2. From Theorem 5.2 we have $G = (G_\equiv)^\dagger$. Now any vertex $u \in V(G)$ is a $u_j \in exp(U) = \{u_1, u_2, \ldots, u_{\ell(U)}\}$ for some $U \in V(G_\equiv)$, and the closed-neighborhoods $N_G[u_j]$ for $j = 1, 2, \ldots, \ell(U)$ are all equal. Either $\ell(U) = 1$ or $\ell(U) \geq 2$. If $\ell(U) = 1$ then U is not closed-parentless and so U has a closed-parent U^p, and then u_1 has a closed-parent in $exp(U^p)$. If $\ell(U) \geq 2$, then each u_j has a closed-twin, which is a closed-parent. Hence, in either case, $u = u_j$ has a closed-parent, and so G is CNAS. Furthermore, since $G_\equiv$ is a t vertex CND graph then, from Lemma 5.4, the graph $G = (G_\equiv)^\dagger$ has t equivalence classes under $\equiv$. ∎

For connected graphs we have:

Lemma 5.7 *Let G be an arbitrary graph. Then G is connected if and only if $G_\equiv$ is connected.*

Proof. Let G be connected. To see that $G_\equiv$ is connected, let U and V be two different vertices of $G_\equiv$, and let $u \in U$ and $v \in V$ in G. Then, since G is connected, there is a path $u = w_1 w_2 \cdots w_d = v$ between u and v in G, but then $U = W_1 W_2 \cdots W_d = V$ is a walk between U and V in $G_\equiv$, and so $G_\equiv$ is connected. The converse is proved similarly. ∎

Using Lemma 5.7 we have the following 'connected' version of Theorem 5.6:

Theorem 5.8 *Let G be an arbitrary graph. Then G is a connected CNAS graph with t equivalence classes under $\equiv$ if and only if $G_\equiv$ is a connected labelled t vertex CND graph in which all closed-parentless vertices have label ≥ 2.*

∎

We need another definition: Let $n \geq 2$ be a positive integer. A *partition* of n is a set $\mathcal{P} = \{\ell_1, \ell_2, \ldots, \ell_t\}$ of $t \geq 1$ integers that satisfy $1 \leq \ell_1 \leq \ell_2 \cdots \leq \ell_t$ and $\sum_{i=1}^{t} \ell_i = n$. Partition $\mathcal{P}$ has t parts.

We now present an algorithm to construct (connected) CNAS graphs G from (connected) labelled CND graphs H. It uses Theorems 5.6 and 5.8 where we denote $G_\equiv$ by H, and consider all possible (connected) labelled CND graphs H, and then construct all possible (connected) CNAS graphs G by using $G = H^\uparrow$.

Let the labels on the t vertices of $H = G_\equiv$ be $\{\ell_1, \ell_2, \ldots, \ell_t\}$, where each $\ell_i \geq 1$. If G is CNAS with $n \geq 2$ vertices then a vertex $u \in V(G)$ of maximum degree must have a closed-twin, so $|U| \geq 2$. So some $\ell_i \geq 2$, and since $n = \sum_{i=1}^{t} \ell_i$, then $t \leq n - 1$.

Algorithm (Connected) CNAS Graphs A four step algorithm to construct all (connected) CNAS graphs G on a fixed number of $n \geq 2$ vertices from all (connected) labelled CND graphs H on $1 \leq t \leq n-1$ vertices.

`For each fixed` $t = 1, 2, \ldots, n-1$:

(1) `By repeated use of Algorithm CND Graphs, (suitably modified to generate connected CND graphs if required), list all non-isomorphic (connected) CND graphs` H_t `on` t `vertices.`

(2) `List all partitions` $\mathcal{P}_t$ `of` n `with` t `parts.`

(3) `For each (connected) graph` H_t `and partition` $\mathcal{P}_t = \{\ell_1, \ell_2, \ldots, \ell_t\}$`, label its` t `vertices with` $\{\ell_1, \ell_2, \ldots, \ell_t\}$ `in all possible non-label isomorphic ways, ensuring that all closed-parentless vertices have label` ≥ 2.

(4) `For each (connected) labelled graph` H_t `construct` $G = {H_t}^\uparrow$.

Because of Theorems 5.6 and 5.8 we have a complete list of (connected) CNAS graphs G with n vertices, with no repeated G. We also note that because of the extensive computation required in Steps (1), (2), and (3) the above algorithm is not an efficient way to construct all CNAS graphs on a fixed large number n of vertices.

Example 4 See Fig. 5. We illustrate Algorithm Connected CNAS Graphs for $n = 5$. From the Algorithm we need to consider all connected CND graphs H_t on $1 \le t \le 4$ vertices. These graphs H_t are shown below, suitably labelled. Two such H_t cannot be labelled since closed-parentless vertices, indicated by $\overline{cp}$, require a label of greater than or equal to 2, thus forcing the sum of all labels to be greater than 5.

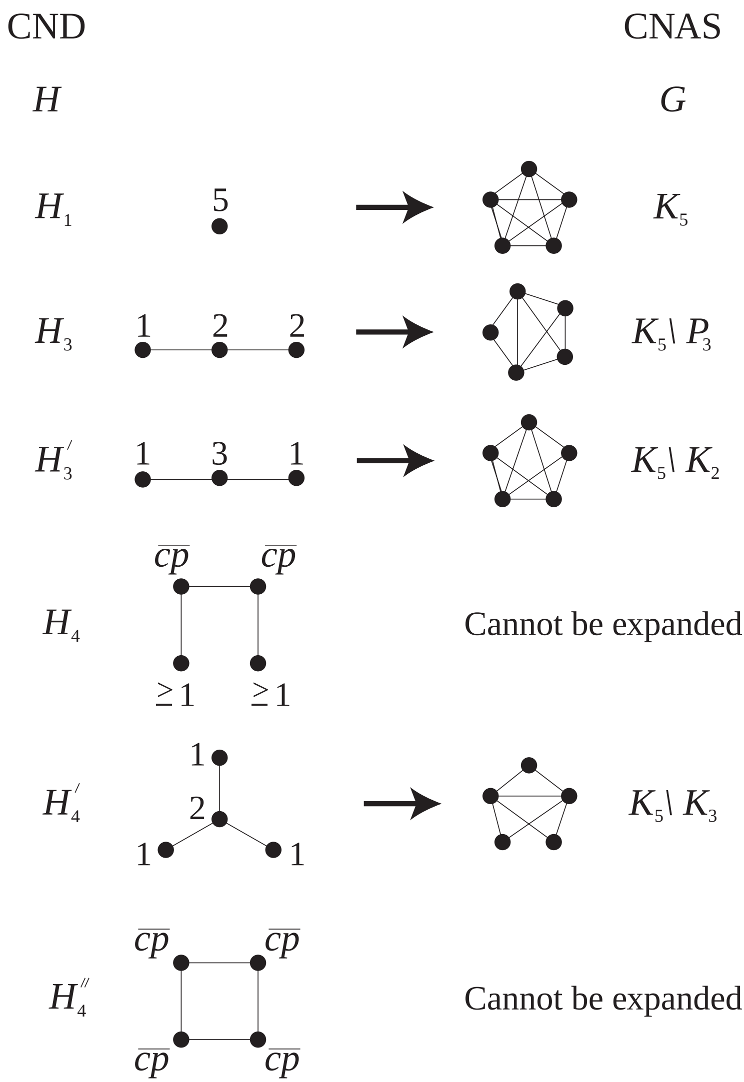

Fig. 5. The connected CNAS graphs on 5 vertices, illustrating Algorithm Connected CNAS Graphs for $n = 5$, see Example 4

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