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Global forcing number of grid graphs

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Abstract

Let G be a simple connected graph with a perfect matching. Let $\mathcal{M}(G)$ denote the set of all perfect matchings in G, and $f : \mathcal{M}(G) \to \{0, 1\}^{|E(G)|}$ a characteristic function of perfect matchings of G. Any set $S \subseteq E(G)$ such that $f\,|_S$ is an injection is called a global forcing set in G, and the cardinality of smallest such S is called the global forcing number of G. In this paper we first establish lower bounds on this quantity and show how it can be computed for certain classes of composite graphs, and then we prove an explicit formula for global forcing number of the grid graph $R_{p,q} = P_p \times P_q$. We also briefly consider a vertex global forcing number of grid graphs and present an explicit formula for it. In the last section we give explicit formulas for global forcing numbers of complete graphs and discuss some further developments.

1 Introduction and motivation

The concept of **forcing set** has arisen in the context of the study of resonance structures in mathematical chemistry [3, 4] and has acquired a life of its own in purely graph-theoretical literature [1, 2, 6, 7, 11]. In all those works forcing sets were defined locally, as subsets of particular perfect matchings, and global results were obtained by taking minimum/maximum over the set of all perfect matchings in the graph. In this paper we study the forcing sets that are defined globally in a graph, i.e., without reference to a particular perfect matching. The motivation for such approach comes from the need to efficiently code and manipulate perfect matchings in large-scale computations [8, 9]. It turns out that for so defined forcing sets and their cardinalities a number of results can be established that are analogous to those from the local context. In particular, we prove an explicit formula for the global forcing number for rectangular grids and complete graphs.

The paper is organized as follows. In the next section we define the terms relevant for our subject and illustrate them by examples. That section also contains some lower bounds on the global forcing number and some results on global forcing number of two classes of composite graphs. The main result of the paper is established in Section 3. It consists of an explicit construction of a minimal global forcing set in a grid graph $P_p \times P_q$. The section also presents an explicit formula for vertex global forcing number of rectangular grids. The paper is concluded by a section concerned with global forcing sets and numbers of complete and complete bipartite graphs, and of some subsets of hexagonal networks. Possible directions of future research are briefly discussed at the end.

2 Mathematical preliminaries

All graphs in this paper are simple and connected. For all terms and notation not defined here we refer the reader to [5].

Let $G = (V(G), E(G))$ be a graph with a perfect matching. Denote by $\mathcal{M}(G)$ the set of all perfect matchings in G and consider a function $f : \mathcal{M}(G) \to \{0,1\}^{|E(G)|}$ defined by

$$[f(M)]_i = \begin{cases} 1 & , \quad e_i \in M \\ 0 & , \quad e_i \notin M. \end{cases}$$

The function f is the characteristic function of perfect matchings of G. Any set $S \subseteq E(G)$ such that the restriction of f on S is an injection is called a **global forcing set** of G. A global forcing set of the smallest cardinality is called a **minimal global forcing set**, and its cardinality is the **global forcing number** of G. For a given graph G we denote its global forcing number by $\varphi_g(G)$.

As an illustrative example we consider the complete graph K_4 shown in Fig. 1. It contains three different perfect matchings, $M_1 = \{e_1, e_5\}$, $M_2 = \{e_3, e_6\}$, and $M_3 = \{e_2, e_4\}$. It is easy to see that the restriction of $f : \{M_1, M_2, M_3\} \to \{0,1\}^6$ on the set $S = \{e_1, e_2\}$ is an injection. Hence, S is a global forcing set. Furthermore, it is obvious that no single edge of K_4 can be a global forcing set, and this makes S a minimal global forcing set. Hence, $\varphi_g(K_4) = 2$.

Clearly, the quantity $\varphi_g(G)$ must be related to the total number $\Phi(G)$ of perfect matchings in G, since $f(M)$ is, in fact, a code of M in the binary alphabet. Since the smallest number of binary digits necessary for representation of a number a in the binary notation is $\lceil \log_2 a \rceil$, we have the following result.

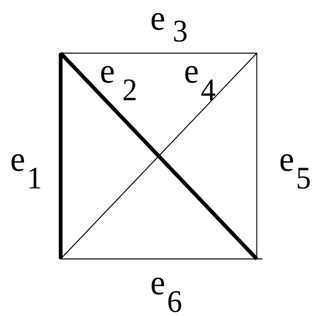

Figure 1: A minimal global forcing set in K_4.

Proposition 1
Let G be a graph with $\Phi(G)$ perfect matchings. Then $\varphi_g(G) \geq \lceil \log_2 \Phi(G) \rceil$. ■

By considering a cycle on $2n$ vertices one can see that the lower bound of Proposition 1 is actually attained for each even number of vertices. Furthermore, for any given $r \in \mathbb{N}$ one can construct a graph G_r such that $\varphi_g(G_r) = r = \lceil \log_2 \Phi(G_r) \rceil$. An example is shown in Fig. 2. The result is also exact for all graphs with unique perfect matching, and it can be made meaningful for graphs without perfect matchings by putting $\varphi_g(G) = -\infty$.

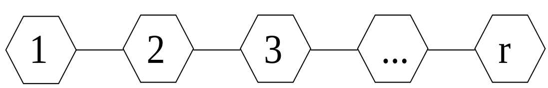

Figure 2: A graph with $\varphi_g(G) = \lceil \log_2 \Phi(G) \rceil$.

That the result of Proposition 1 is indeed a lower bound and not an exact result one can see by considering the graph from Fig. 3 with $\Phi(G) = 8$. The four edges shown

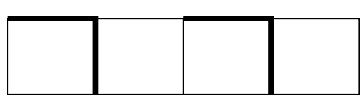

Figure 3: A graph with $\varphi_g(G) > \lceil \log_2 \Phi(G) \rceil$.

in bold are a global forcing set, and by a direct (but tedious) calculation one can verify that no set of three edges is a global forcing set. Hence, $\varphi_g(G) = 4 > 3$.

The main problem with the lower bound of Proposition 1 is that the quantity $\Phi(G)$ often depends on the basic parameters of the graph G in a very intricate and/or non-transparent way. It can be NP-hard to compute even for bipartite graphs ([5], p. 307). Hence, it is of interest to relate the quantity $\varphi_g(G)$ to some other, less derived, properties of the graph G. An alternative approach is to seek for the ways to express the global forcing number of composite graphs in terms of global forcing numbers of their components. We first briefly explore this possibility before turning our attention to the graphs that will yield to the first approach.

An edge e of a graph G is **allowed** if it appears in some perfect matching of G. Otherwise, the edge e is **forbidden**. It is clear that the forbidden edges carry no information about $\varphi_g(G)$. Hence, we have the following result.

Proposition 2
Let G be a graph with a perfect matching and $F \subset E(G)$ the set of forbidden edges of G. Then $\varphi_g(G) = \varphi_g(G - F)$. ■

From the definition of $\varphi_g(G)$ it follows that for a disconnected graph G we have $\varphi_g(G) = \sum_i \varphi_g(G_i)$, where G_i are connected components of G. Combining this fact with Proposition 2 one can express the global forcing number for members of some classes of composite graphs in terms of the global forcing numbers of their simpler constituents.

Let G_1 and G_2 be two connected graphs with disjoint vertex sets. A **link** of G_1 and G_2 anchored at the vertices $v \in V(G_1)$ and $w \in V(G_2)$ is a graph obtained by connecting the vertices v and w by an edge. We denote this graph by $G_1 \sim G_2$. A **splice** of G_1 and G_2 is a graph obtained by selecting a vertex $v \in V(G_1)$ and identifying it with some vertex from $V(G_2)$. A splice of two graphs we denote by $G_1 \cdot G_2$. Examples are shown schematically in Fig. 4.

Figure 4: A link (left) and a splice (right) of two graphs.

Corollary 3
Let G_1 and G_2 be two simple and connected graphs and $G_1 \sim G_2$ their link anchored at the vertices $v \in V(G_1)$ and $w \in V(G_2)$. Then

$$\varphi_g(G_1 \sim G_2) = \begin{cases} \varphi_g(G_1) + \varphi_g(G_2) & , \quad if \quad |V(G_1)| \quad is \quad even \\ \varphi_g(G_1 - v) + \varphi_g(G_2 - w) & , \quad if \quad |V(G_1)| \quad is \quad odd. \end{cases}$$

■

Corollary 4
Let G_1 and G_2 be two simple and connected graphs and $G_1 \cdot G_2$ their splice at v. Let $|V(G_1)|$ be even. Then

$$\varphi_g(G_1 \cdot G_2) = \varphi_g(G_1) + \varphi_g(G_2 - v).$$

■

Another important class of composite graphs are Cartesian products. For general graphs G_1 and G_2 there seem to be no simple formulas of the above type, but if both

G_1 and G_2 are paths, and if at least one of them has an even number of vertices, then the global forcing number of their Cartesian product can be explicitly expressed in terms of their sizes.

3 Global forcing number of a rectangular grid

Let G_1 and G_2 be two simple and connected graphs. A **Cartesian product** of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph with the vertex set $V(G_1) \times V(G_2)$, where the vertices (v_1, w_1) and (v_2, w_2) are connected by an edge if and only if $[v_1 = v_2$ and $\{w_1, w_2\} \in E(G_2)]$ or $[w_1 = w_2$ and $\{v_1, v_2\} \in E(G_1)]$.

Let P_n denote a path on n vertices, i.e., of length $n-1$, and $R_{p,q}$ denote a Cartesian product of two paths, $R_{p,q} = P_p \times P_q$. We take that at least one of the positive integers p and q is even.

Theorem 5

$$\varphi_g(R_{p,q}) = (p-1)(q-1) - \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor .$$

Proof

We introduce a coordinate system in $R_{p,q}$ so that the lower left corner (i.e., the lower left vertex) has the coordinates $(1,1)$, and the upper right corner has the coordinates (p,q).

Let us consider the set $T \subset E(R_{p,q})$ defined by

$$\begin{aligned} T \;=\; & \left\{ \{(2i,j),(2i,j+1)\}, i = 1,\ldots,\left\lfloor \frac{p}{2} \right\rfloor, j = 1,\ldots,q-1 \right\} \cup \\ & \left\{ \{(2i-1,2j),(2i,2j)\}, i = 1,\ldots,\left\lfloor \frac{p-1}{2} \right\rfloor, j = 1,\ldots,\left\lfloor \frac{q}{2} \right\rfloor \right\}. \end{aligned}$$

One can easily verify by a direct computation that the cardinality of T is given by $|T| = (p-1)(q-1) - \lfloor \frac{p-1}{2} \rfloor \lfloor \frac{q-1}{2} \rfloor$. For example, if p is even and q is odd, then $p = 2r$, $q = 2s+1$, $\lfloor \frac{p-1}{2} \rfloor = r-1$, $\lfloor \frac{q-1}{2} \rfloor = s$ and

$$\begin{aligned} |T| &= (q-1)\left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor = 2sr + (r-1)s = 3rs - s \\ &= (2r-1)2s + 2s - rs - s = (p-1)(q-1) + (1-r)s \\ &= (p-1)(q-1) - \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor . \end{aligned}$$

The claim follows similarly for other two possible combinations of the parities of p and q. An example of such a set in $R_{9,8}$ is shown in bold in Fig. 5. We claim that so defined set T is a global forcing set, i.e., that $f|_T$ is an injection. Let us suppose that there are two perfect matchings $M_1 \neq M_2 \in \mathcal{M}(G)$ such that $f|_T(M_1) = f|_T(M_2)$. Consider the graph G' induced by $M_1 \Delta M_2$, where Δ denotes the symmetric difference of two sets. The graph G' is 2-regular and it contains no edges of T. Hence, G' is a 2-regular subgraph of $R_{p,q} - T$ made of alternating cycles. Each such alternating cycle

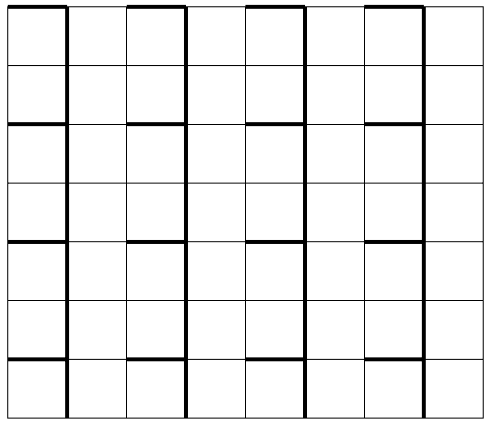

Figure 5: A global forcing set in $R_{9,8}$.

C has an odd number of vertices in its interior. This can be verified by induction on the number of squares of area 4 such as the one defined by the vertices $(1,1)$ and $(3,3)$ in the lower left corner of Fig. 6. Since all vertices of C are matched by the

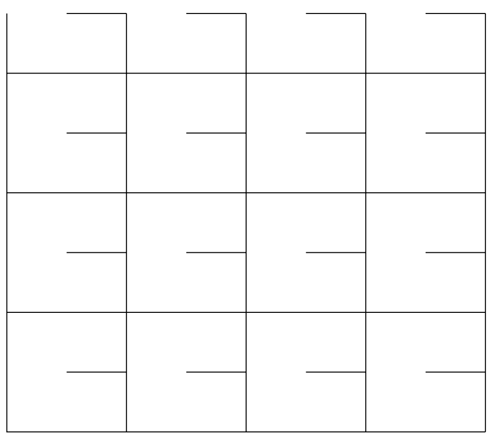

Figure 6: $R_{9,8} - T$.

edges of M_1, it follows that M_1 cannot cover all vertices in the interior of C, contrary to our assumption that M_1 is a perfect matching. Hence, $f|_T$ is an injection, and $\varphi_g(R_{p,q}) \leq |T|$.

It remains to show that a set of a cardinality strictly smaller than $|T|$ cannot be a global forcing set in $R_{p,q}$. Let us suppose that there is a global forcing set Q with $|Q| < |T|$. Denote the graph $R_{p,q} - Q$ by G. The graph G is planar, and its number of faces $|F(G)|$ satisfies the following relation.

$$\begin{aligned}|F(G)| &= |E(G)| - |V(G)| + 2 \\ &> |E(R_{p,q})| - |T| - |V(R_{p,q})| + 2 \\ &= (p-1)q + p(q-1) - (p-1)(q-1) + \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor - pq + 2 \\ &= \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor + 1.\end{aligned}$$

Since exactly one face of G is unbounded, it follows that the set of bounded faces of G, denoted by $\mathcal{B}$, contains more than $\lfloor \frac{p-1}{2} \rfloor \lfloor \frac{q-1}{2} \rfloor$ faces.

Now we introduce some notation that will be useful in the rest of the proof. By $\mathcal{P}$ we denote the set of certain points with half-integer coordinates,

$$\mathcal{P} = \left\{ \left(2i + \frac{1}{2}, 2j + \frac{1}{2} \right), i = 1, \ldots, \left\lfloor \frac{p-1}{2} \right\rfloor, j = 1, \ldots, \left\lfloor \frac{q-1}{2} \right\rfloor \right\}.$$

$\mathcal{S}$ is the set of unit squares of the original $R_{p,q}$ grid that contain no point from $\mathcal{P}$ and share no edge with squares that contain points from $\mathcal{P}$. Finally, $\mathcal{E}$ is a set of edges of $R_{p,q}$ that are on the border of $R_{p,q}$ but not in the squares that contain points of $\mathcal{P}$. An example is shown in Fig. 7. The squares from $\mathcal{S}$ and the edges from $\mathcal{E}$ are shown in bold lines.

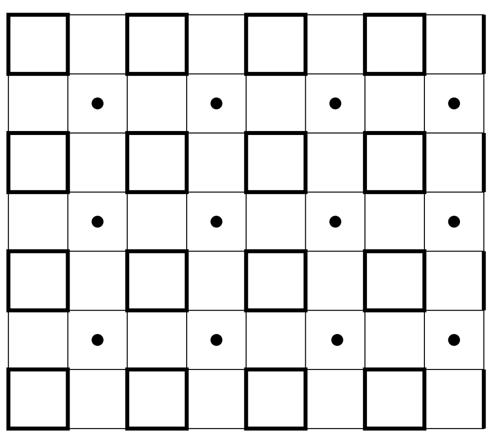

Figure 7: The sets $\mathcal{P}$, $\mathcal{S}$, and $\mathcal{E}$ in $R_{9,8}$.

Let Z_1 be the set of all bounded faces of G that contain no points from $\mathcal{P}$. Since there are more bounded faces of G than points in $\mathcal{P}$, it follows that the set Z_1 is non-empty. Let F be a face from Z_1 and $B = \partial F$ its boundary. Let us suppose that B is a cycle. It must be an even cycle, since $R_{p,q}$ is bipartite. Hence, the edges of B can be bipartitioned in two sets, B_1 and B_2, such that each of them is a perfect

matching of B. An edge $e \in \mathcal{E}$ is either whole in B, or none of its end-vertices are in B. We denote the set of edges from $\mathcal{E}$ that are not in B by $\mathcal{E}'$. Now we consider the squares from $\mathcal{S}$. Since B is a cycle, it follows that for a square $s \in \mathcal{S}$ we have one of the three possibilities:
(i) All four vertices of s are in B;
(ii) Two adjacent vertices of s are in B;
(iii) None of the vertices of s are in B.
Let $\mathcal{E}''$ be the set of edges that connect the two vertices not in B of the squares of type (ii), and $\mathcal{E}'''$ the set consisting of two independent edges from each square of the type (iii). Now the sets $M_1 = \mathcal{E}' \cup \mathcal{E}'' \cup \mathcal{E}''' \cup B_1$ and $M_2 = \mathcal{E}' \cup \mathcal{E}'' \cup \mathcal{E}''' \cup B_2$ are both perfect matchings in $R_{p,q}$. Since $B \subset R_{p,q} - Q$, one has $B_1 \cap Q = \emptyset$ and $B_2 \cap Q = \emptyset$, and hence $f|_Q(M_1) = f|_Q(M_2)$. This is a contradiction with the assumed injectivity of f on Q. Hence, no face from Z_1 has a boundary that is a cycle. By a similar reasoning one can prove that no face from Z_1 is bounded by two or more disjoint cycles. Since the interior of F must be connected, the only remaining possibility is that the boundary of F is made of a cycle with a self-intersection. An example is shown in Fig. 8.

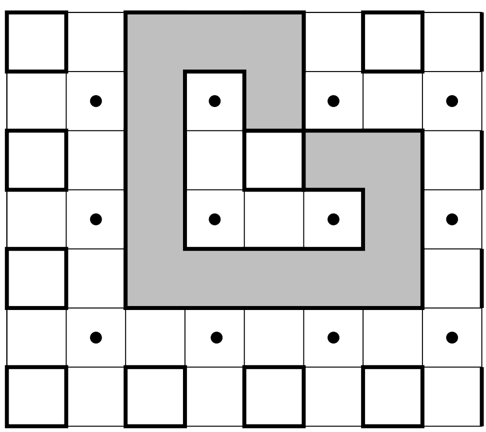

Figure 8: A face F from Z_1.

For a given $F \in Z_1$ we denote by $in(F)$ the area completely surrounded by F, and by $out(F)$ the area that surrounds F. By $nin(F)$ we denote the number of faces from G contained in $in(F)$, and by $npin(F)$ the number of points from $\mathcal{P}$ contained in $in(F)$. Let Z_2 denote the set of faces of G that are not contained in $in(H)$ for any $H \in Z_1$. Then we have

$$\sum_{F \in Z_2} (nin(F) + 1) + \left[\left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor - \sum_{F \in Z_2} npin(F) \right] \geq |\mathcal{B}| > \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor .$$

Hence $\sum_{F \in Z_2} (nin(F) + 1 - npin(F)) > 0$, and there is at least one face $F \in Z_2$ such that $nin(F) \geq npin(F)$.

We denote the area (in unit squares) of a set $x \subset R_{p,q}$ by $A(x)$ and proceed by proving the following claim.

Claim $A(in(F)) < 4 \cdot npin(F)$.
Proof (of the claim) First we note that in a situation shown in Fig. 9 it is impossible to have $x \in in(F)$ and $a, b \in out(F)$, due to connectedness of interior of F. Denote

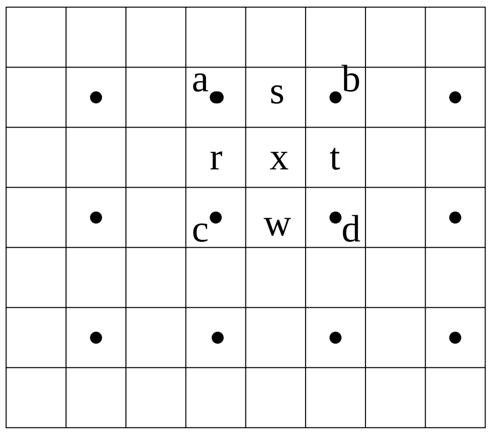

Figure 9: With the case analysis in the proof of the Claim.

by $\mathcal{F}$ the set of points from $\mathcal{P}$ that are contained in $in(F)$. Let each point from $\mathcal{F}$ be the center of a square with the side 2, i.e., with area equal to 4. Denote the union of all such squares by U. It suffices to show that $A(U) > A(in(F))$.

For each unit square $x \subseteq in(F)$ we define the set $K(x)$ in the following way:
1. If x contains a point from $\mathcal{F}$, then $K(x) = x$;
2. If x is not adjacent to F, then $K(x) = x$;
3. Otherwise, $K(x)$ is the union of x and the nearest half of each unit square from F adjacent to x.

An example is shown in Fig. 10. It suffices to show that $A(K(x) \cap U) \geq 1$ for each $x \in in(F)$. There are, up to rotation, four possible situations.

<u>Case 1</u> If a square $x \in in(F)$ is either vertically or horizontally sandwiched between two squares that contain each a point from $\mathcal{F}$, then those two squares are both in $in(F)$. Hence, $x \subseteq U$ and $A(K(x) \cap U) \geq 1$.

<u>Case 2</u> Unit squares a, b, c, and d from Fig. 9 are all contained in $in(F)$. Then again $x \subseteq U$ and $A(K(x) \cap U) \geq 1$.

<u>Case 3</u> Unit squares a, b, and c from Fig. 9 are contained in $in(F)$, while the square d is not. Then the squares t and w are contained in the interior of F, and $K(x)$ consists of x, the left half of t and the upper half of w. Each of those two half-squares contributes $1/4$ to the intersection with U, and x itself contributes $3/4$. All together, we have $A(K(x) \cap U) = 5/4 > 1$.

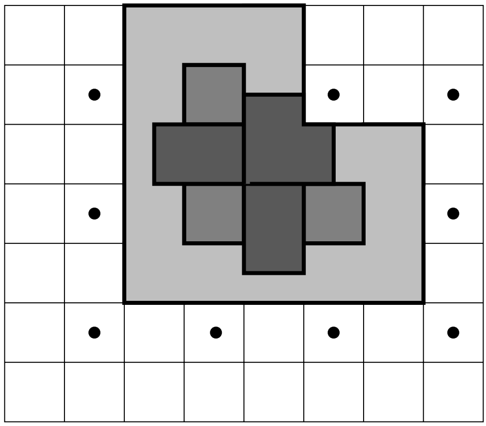

Figure 10: With the definition of $K(x)$.

Case 4 Unit squares a and d are in $in(F)$, while b and c are not. Then all four of r, s, t, and w are contained in the interior of F, and $K(x)$ is made of whole x and four halves of the adjacent squares from the interior of F. The square of U with center in the point in a intersects $K(x)$ in a figure of area $3/4$, and the same is valid for the square of U centered at c. Hence, $A(K(x) \cap U) = 3/2 > 1$.

As all possible cases have been exhausted, we have proved the Claim. Therefore, $A(in(F)) < 4 \cdot npin(F)$. From $nin(F) \geq npin(F)$ it follows that there must be at least one "small" face of G contained in $in(F)$ (hence not on the border of $R_{p,q}$). Here by "small" face we mean a face that has the area smaller than four unit squares. We denote that face by H. The face H must be (up to rotation) one of the shapes shown in Fig 11. For any of those shapes it can be proved that $f|_Q$ is not an injection.

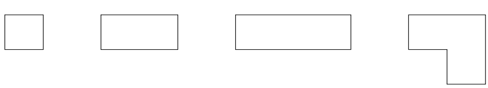

Figure 11: Possible shapes of small faces of G.

Since all four proofs are very similar, and the first three are simpler than the fourth one, we present only the proof for the L-shaped face of area 3.

Let us suppose that p is odd. Then we add 2 edges next to the L-shaped face in the way shown in Fig. 12 left. (If both p and q are even, then it does not matter if we add the edges to the left or below the face.) We denote those two edges by E_1. Then we proceed by adding $2(q-3)$ edges in the way shown in Fig. 12, middle. This set of edges is denoted by E_2. Finally, we add a set of $\frac{q}{2}(p-4)$ edges that cover the remaining vertices of $R_{p,q}$ in the way shown in Fig. 12, right, and denote it by E_3. Now, by partitioning the edges of ∂H in two sets, C_1 and C_2, so that the edges alternate between C_1 and C_2, we obtain two perfect matchings of $R_{p,q}$,

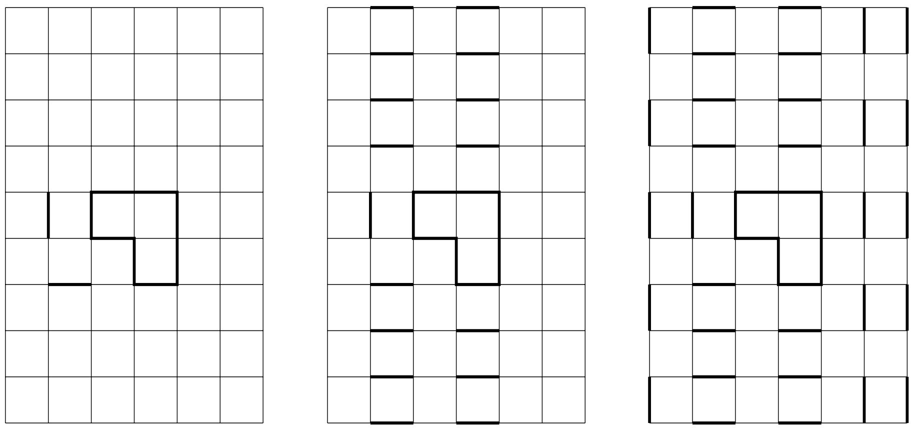

Figure 12: Completition of ∂H to a perfect matching in $R_{p,q}$.

$M_1 = C_1 \cup E_1 \cup E_2 \cup E_3$ and $M_2 = C_2 \cup E_1 \cup E_2 \cup E_3$, that differ only on ∂H. But the face H is in $R_{p,q} - Q$, and hence $f|_Q(M_1) = f|_Q(M_2)$. Thus we have arrived to a contradiction with our initial assumption that Q is a global forcing set. Therefore, there is no global forcing set with less than $(p-1)(q-1) - \lfloor \frac{p-1}{2} \rfloor \lfloor \frac{q-1}{2} \rfloor$ edges, and the Theorem is proven. ■

The exact result of Theorem 5 enables us to assess the quality of the lower bounds of Proposition 1. It is well known (see, e.g., [5], p. 329) that

$$\Phi(R_{p,q}) = 2^{pq/2} \prod_{k=1}^{p} \prod_{l=1}^{q} \left[\cos^2 \frac{k\pi}{p+1} + \cos^2 \frac{l\pi}{q+1} \right]^{1/4}.$$

The asymptotic behavior of $\Phi(R_{p,q})$ is given by $\Phi(R_{p,q}) = \exp(\frac{\mathcal{G}}{\pi} pq)$, where $\mathcal{G}$ is the Catalan constant, $\mathcal{G} = 0.915966....$ Proposition 1 then gives $\varphi_g(R_{p,q}) \geq \log_2 e^{0.29pq} \approx 0.42pq$, while the exact result behaves asymptotically as $0.75pq$.

We conclude this section by a short digression about another type of forcing sets in rectangular grids. Let S be a subset of $V(R_{p,q})$ and $g : \mathcal{M}(R_{p,q}) \to \{L, R, D, U\}^{pq}$ a function that to each perfect matching $M \in \mathcal{M}(R_{p,q})$ assigns the directions of the edges that cover the vertices from $R_{p,q}$. A set $S \subseteq V(R_{p,q})$ such that $g|_S$ is an injection is called a vertex global forcing set, and the cardinality of smallest such set is called the **vertex global forcing number** of $R_{p,q}$. We denote this quantity by $\nu(R_{p,q})$.

Theorem 6

$$\nu(R_{p,q}) = \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor .$$

Proof
Let $W \subseteq V(R_{p,q})$ be a set defined by

$$W = \{(2i, 2j), i = 1, \ldots, \left\lfloor \frac{p-1}{2} \right\rfloor, j = 1, \ldots, \left\lfloor \frac{q-1}{2} \right\rfloor\}.$$

The value $g|_W(M)$ completely determines the value of $f|_T(M)$, where T is defined as in the proof of Theorem 5. Hence $g|_W$ is an injection and $\nu(R_{p,q}) \leq \lfloor \frac{p-1}{2} \rfloor \lfloor \frac{q-1}{2} \rfloor$.

Suppose that there is $R \subseteq V(R_{p,q})$ such that $g|_R$ is an injection and $|R| < \lfloor \frac{p-1}{2} \rfloor \lfloor \frac{q-1}{2} \rfloor$. Note that in $R_{p,q}$ there always are $\lfloor \frac{p-1}{2} \rfloor \lfloor \frac{q-1}{2} \rfloor$ disjoint unit squares, such as shown in bold in Fig. 13. Hence, at least one of them contains no vertex from R, and any two perfect matchings that differ only on that square are assigned the same value by

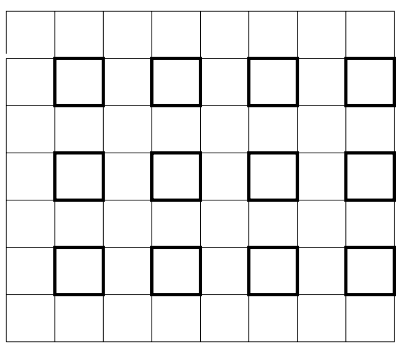

Figure 13: With the proof of Theorem 6.

$g|_R$. Therefore, $g|_R$ is not an injection, and any vertex global forcing set must have at least $\lfloor \frac{p-1}{2} \rfloor \lfloor \frac{q-1}{2} \rfloor$ elements. This proves the Theorem. ■

4 Concluding remarks

In this section we discuss some possible directions of future research and present some particular results. The most natural next step would be to extend the results of Theorem 5 to cylinders and tori, i.e., to Cartesian products $P_p \times C_q$ and $C_p \times C_q$ for pq even. Also, it might be fruitful to consider the case of hypercubes in the manner of references [1] and [7]. In both cases, one can expect results expressing the global forcing number of those graphs in terms of their size. Two families of graphs that yield to such approach are the complete bipartite graphs with equal bipartition classes and the complete graphs on even number of vertices. We denote such graphs by $K_{n,n}$ and K_{2n}, respectively.

Proposition 7

$$\varphi_g(K_{n,n}) = (n-1)^2.$$

Proof

Let S be a global forcing set in $K_{n,n}$. We will prove that the graph $G = K_{n,n} - S$ is a tree. Let us suppose that it is not. Then G must contain at least one cycle, and this cycle must be even. It also must have the same number of vertices in both classes of bipartition. Hence, we can partition the edges of that cycle C into two perfect matchings C_1 and C_2 of C. The graph $K_{n,n} - [C]$ is a complete bipartite graph with equal bipartition classes, and hence it contains a perfect matching. Denote it by M. Now $M \cup C_1$ and $M \cup C_2$ are both perfect matchings of $K_{n,n}$ that do not differ on S, a contradiction. Hence, G must be a tree, and the claim follows. ■

The following result will be useful in proving the result about K_{2n}.

Lemma 8

Let G be a graph on n vertices that contains no even cycles. Then $|E(G)| \leq \frac{3}{2}n - 2$ if n is even, and $|E(G)| \leq \frac{3}{2}(n-1)$ if n is odd.

Proof

For $n \leq 4$ the claim can be easily verified directly. We proceed by induction on n. Let G be a graph on $n \geq 5$ vertices without even cycles. We note that no two vertices of G can be connected by three internally disjoint paths. If they were connected by three such paths, at least two of them would be of the same parity, and their union would make an even cycle. Now take a cycle C of length k in G. By deleting the edges of C all the vertices of C end in different components of $G - C$. Hence, the graph disintegrates into l components with $v_1, \ldots, v_l$ vertices and $e_1, \ldots, e_l$ edges, where $l \geq k$. Now, by the inductive hypothesis,

$$|E(G)| = \sum_{i=1}^{l} e_i + k \leq \frac{3}{2}\sum_{i=1}^{l} v_i - \frac{3}{2}l + k.$$

Since $l \geq k$ it further means that $|E(G)| \leq \frac{3}{2}n - k/2$, and this, in turn, implies $|E(G)| \leq \frac{3}{2}n - 3/2$, since $k \geq 3$. This proves the Theorem for odd n, and for even n, the claim follows from the integrality of $|E(G)|$. ■

The lower bound from Lemma 8 is sharp. For odd n it is attained for the windmill graph with $\frac{n-1}{2}$ wings, and for even n we add one more vertex and the edge connecting it to the center of the windmill.

Proposition 9

$$\varphi_g(K_{2n}) = 2(n-1)^2.$$

Proof

By the same reasoning as in Proposition 7 we prove that the graph $K_{2n} - S$ cannot contain an even cycle if S is a global forcing set. From Lemma 8 it follows that $|E(K_{2n} - S)| \leq 3n - 2$. Hence, $|S| \geq |E(K_{2n})| - 3n + 2 = 2(n-1)^2$. The claim of the proposition will follow if we exhibit a global forcing set of cardinality $2(n-1)^2$. Let us label the vertices of K_{2n} with labels $1, 2, \ldots, 2n$ and consider the subgraph T

of K_{2n} induced by the edges of the form $\{i, i+1\}$ for all $i = 1, \ldots, 2n-1$ and of the form $\{i, i+2\}$ for all odd $i = 1, \ldots, 2n-3$. Since T contains exactly one perfect matching, the edges of K_{2n} not in $E(T)$ must form a global forcing set, and there are exactly $2(n-1)^2$ of them. ■

Here again one can compare the exact results with the lower bounds from Proposition 1. Using the formulas $\Phi(K_{n,n}) = n!$ and $\Phi(K_{2n}) = \frac{(2n)!}{2^n n!}$ one can see that in both cases the logarithm of the number of perfect matchings behaves asymptotically as $n \log n$, while the exact results are quadratic in n.

Another natural step is to consider other types of grids. We have mentioned in the introduction of this paper the chemical roots of its subject. Hence, we consider it proper to conclude it by a result of some chemical relevance, concerning a subset of a hexagonal grid. For some results about triangular grids, we refer the reader to [10].

Let L_n be a linear polyacene, i.e., a graph with n hexagons shown in Fig. 14.

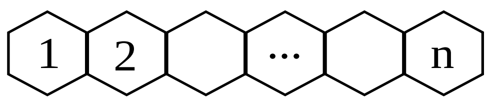

Figure 14: A linear polyacene with n hexagons.

Proposition 10

$$\varphi_g(L_n) = n.$$

Proof
It follows from a simple parity argument that each perfect matching of L_n must contain one and only one vertical edge of L_n. Hence, any set of n vertical edges of L_n is a global forcing set, and $\varphi_g(L_n) \leq n$. It is easy to see that the set of $n-1$ vertical edges that are shared between two hexagons cannot be a global forcing set, since it cannot distinguish between the perfect matching containing the leftmost vertical edge, and the one containing the rightmost vertical edge.

Let us now suppose that there is a global forcing set $T \subseteq E(L_n)$ with $|T| \leq n-1$. The inner dual of L_n is a path on n vertices. (The inner dual of a planar graph G is obtained from the ordinary dual G' by deleting the vertex corresponding to the unbounded face, together with all incident edges.) If a vertical edge from L_n is in T, we mark the corresponding edge in the inner dual P_n, and if a hexagon from L_n contains non-vertical edges of T, we mark the corresponding vertex in P_n. Let x denote the number of marked vertices in P_n and y the number of marked edges. Then $x + y \leq |T| \leq n-1$, since a hexagon may contain more than one edge from T. Let us delete all marked vertices and unmarked edges from P_n. The remaining graph has $n - x$ vertices and y edges. Since $n - x \geq y + 1$, it is non-empty, i.e., it contains at least one component, and this component must be a path, say on $k \geq 1$ vertices. This path corresponds to a sequence of k consecutive hexagons that are collectively incident to $k-1$ edges from T and those edges must necessarily be vertical. But we have already proved that such a set of edges cannot be a global forcing set, and hence T cannot be a global forcing set. ■

Another direction would be to seek for results expressing the global forcing number of composite graphs in terms of global forcing numbers of their building blocks. It would be interesting to explore if anything better than the rather obvious relation $\varphi_g(G_1 \times G_2) \geq \varphi_g(G_1) + \varphi_g(G_2)$ could be obtained for Cartesian products.

One could also try to step out of the context of graphs with perfect matchings and consider the concepts analogous to the global forcing sets and numbers working with maximum and/or inclusion-wise maximal matchings instead of the perfect ones.

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