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The outer-connected domination number of a graph

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Abstract

For a given graph $G = (V, E)$, a set $D \subseteq V(G)$ is said to be an *outer-connected dominating set* if D is dominating and the graph $G - D$ is connected. The *outer-connected domination number* of a graph G, denoted by $\widetilde{\gamma}_c(G)$, is the cardinality of a minimum outer-connected dominating set of G. We study several properties of outer-connected dominating sets and give some bounds on the outer-connected domination number of a graph. We also show that the decision problem for the outer-connected domination number of a graph G is NP-complete even for bipartite graphs.

1 Introduction

Graph theory terminology not presented here can be found in [1, 5].

Let $G = (V, E)$ be a simple graph. The *neighbourhood* of a vertex v, denoted by $N_G(v)$, is the set of all vertices adjacent to v in G. If v is a vertex of G then the integer $\deg_G(v) = |N_G(v)|$ is said to be the *degree* of v in G. The *minimum* and *maximum degree* among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree one in a graph is called an *end-vertex*. A *support* is the unique neighbour of an end-vertex. Let $\Omega = \Omega(G)$ be the set of all end-vertices of G.

A set $D \subseteq V(G)$ is a *dominating set* in G if $N_G(v) \cap D \neq \emptyset$ for every vertex $v \in V(G) - D$. The *domination number* of a graph G, denoted $\gamma(G)$, is the cardinality of a minimum dominating set of G.

A set $D \subseteq V(G)$ is said to be an *outer-connected dominating set* of G if D is dominating and either $D = V(G)$ or $G - D$ is connected. The cardinality of a minimum outer-connected dominating set in G is called the *outer-connected domination number* of G and is denoted by $\widetilde{\gamma}_c(G)$. Observe that every graph G has an outer-connected dominating set, since the set of all vertices of G is an outer-connected dominating set in G.

2 Preliminary results

Let K_n, C_n and P_n denote the complete graph, the cycle and the path of order n, respectively. For positive integers $n_1, \ldots, n_t$ let $K_{n_1,\ldots,n_t}$ be the complete multipartite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_t$, where $|S_i| = n_i$ for $1 \leq i \leq t$. By a star of order n we mean the bipartite graph $K_{1,n-1}$ for $n \geq 2$.

In our first observation we present the outer-connected domination number of complete graphs, cycles, paths and complete multipartite graphs.

Observation 1

(i) $\widetilde{\gamma}_c(K_n) = 1$ *for* $n \geq 1$*;*

(ii) $\widetilde{\gamma}_c(C_n) = n - 2$ *for* $n \geq 3$*;*

(iii) $\widetilde{\gamma}_c(P_n) = \begin{cases} n-1, & n = 2,3, \\ n-2, & n \geq 4; \end{cases}$

(iv) If $t \geq 2$ *and* $n_1 \leq n_2 \leq \ldots \leq n_t$*, then*

$$\widetilde{\gamma}_c(K_{n_1,\ldots,n_t}) = \begin{cases} n_2 & if \quad t = 2 \quad and \quad n_1 = 1, \\ 1 & if \quad t \geq 3 \quad and \quad n_1 = 1, \\ 2 & if \quad t \geq 2 \quad and \quad n_1 > 1. \end{cases}$$

It follows from the next theorem and from its proof that outer-connected dominating sets and outer-connected domination numbers of a disconnected graph are determined by outer-connected dominating sets and outer-connected domination numbers of its components.

Theorem 1 *If* $G_1, \ldots, G_r$ *are the components of a graph* G*, then*

$$\widetilde{\gamma}_c(G) \;=\; |V(G)| \;-\; \max\{|V(G_i)| - \widetilde{\gamma}_c(G_i) \colon i = 1, \ldots, r\}.$$

Proof. Let $D_1, \ldots, D_r$ be minimum outer-connected dominating sets of $G_1, \ldots, G_r$ respectively. Then $V(G) - (V(G_1) - D_1), \ldots, V(G) - (V(G_r) - D_r)$ are outer-connected dominating sets of G and therefore

$$\begin{aligned} \widetilde{\gamma}_c(G) &\leq \min\{|V(G) - (V(G_i) - D_i)| \colon i = 1, \ldots, r\} \\ &= |V(G)| - \max\{|V(G_i) - D_i| \colon i = 1, \ldots, r\} \\ &= |V(G)| - \max\{|V(G_i)| - \widetilde{\gamma}_c(G_i) \colon i = 1, \ldots, r\}. \end{aligned}$$

Now let D be a minimum outer-connected dominating set of G. Then $|D| = \widetilde{\gamma}_c(G)$ and in addition $G - D$ is connected. Hence $V(G) - D \subseteq V(G_l)$ for some $l \in \{1, \ldots, r\}$ and from the minimality of D it follows that $D \cap V(G_l)$ is a minimum outer-connected dominating set of G_l. Thus $D \cap V(G_l) = \widetilde{\gamma}_c(G_l)$ and $|V(G)| - \widetilde{\gamma}_c(G) = |V(G) - D| = |V(G_l) - (D \cap V(G_l))| = |V(G_l)| - \widetilde{\gamma}_c(G_l) \leq \max\{|V(G_i)| - \widetilde{\gamma}_c(G_i) \colon i = 1, \ldots, r\}$ and therefore $\widetilde{\gamma}_c(G) \geq |V(G)| - \max\{|V(G_i)| - \widetilde{\gamma}_c(G_i) \colon i = 1, \ldots, r\}$ which completes the proof. □

3 Bounds

It is obvious that if G is a graph of order n, then $1 \leq \widetilde{\gamma}_c(G) \leq n$. In addition, $\widetilde{\gamma}_c(G) = 1$ if and only if $G = K_1 + H$, where H is a connected graph of order $n-1$, while $\widetilde{\gamma}_c(G) = n$ if and only if $G = \overline{K_n}$. Hence $\widetilde{\gamma}_c(G) \leq n-1$ if G has at least one edge. Moreover, $\widetilde{\gamma}_c(G) \leq n-2$ if and only if G has at least one edge which is not an end-edge. In general, $\widetilde{\gamma}_c(G) \leq n-k$ if and only if there exists a proper connected subgraph H of G such that $|V(H)| = k$ and every vertex of H has a neighbour which belongs to $V(G) - V(H)$.

A characterization of graphs G of order n for which $\widetilde{\gamma}_c(G) = n-1$ is given in the following observation.

Observation 2 *If G is a connected graph on $n \geq 2$ vertices, then $\widetilde{\gamma}_c(G) = n-1$ if and only if G is a star.*

Sampathkumar and Walikar [7] have proved that $\frac{n(G)}{\Delta(G)+1} \leq \gamma_c(G) \leq 2m(G) - n(G)$ for a connected graph G. Now we present similar inequalities for the outer-connected domination number.

Let $\mathcal{A}$ be the family of graphs defined as follows: a graph G belongs to $\mathcal{A}$ if and only if there exists an outer-connected dominating set A of G such that $|PN_G[v, A]| = \Delta(G) + 1$ for every vertex v belonging to A, where $PN_G[v, A]$ is the private neighbourhood of v with respect to A, i.e. $PN_G[v, A] = N_G[v] - N_G[A - \{v\}]$.

Theorem 2 *If G is a connected graph with $n(G) \geq 2$, then*

$$\frac{n(G)}{\Delta(G)+1} \leq \widetilde{\gamma}_c(G) \leq 2m(G) - n(G) + 1.$$

In addition, $\widetilde{\gamma}_c(G) = \frac{n(G)}{\Delta(G)+1}$ if and only if G belongs to the family $\mathcal{A}$, while $\widetilde{\gamma}_c(G) = 2m(G) - n(G) + 1$ if and only if G is a star.

Proof. Since $\frac{n(G)}{\Delta(G)+1} \leq \gamma(G)$ (see [8]) and $\gamma(G) \leq \widetilde{\gamma}_c(G)$, we certainly have $\frac{n(G)}{\Delta(G)+1} \leq \widetilde{\gamma}_c(G)$. Moreover, since G is connected and has at least two vertices, we have $m(G) \geq n(G) - 1$ and $\widetilde{\gamma}_c(G) \leq n(G) - 1$. Consequently, $\widetilde{\gamma}_c(G) \leq 2m(G) - n(G) + 1$.

If G belongs to the family $\mathcal{A}$, then there exists an outer-connected dominating set A of G for which $\widetilde{\gamma}_c(G) \leq |A| = \frac{n(G)}{\Delta(G)+1} \leq \widetilde{\gamma}_c(G)$. Now assume that $\widetilde{\gamma}_c(G) = \frac{n(G)}{\Delta(G)+1}$ and let D be a minimum outer-connected dominating set in G. Then $|D| = \frac{n(G)}{\Delta(G)+1}$ and this forces each vertex of D to dominate exactly $\Delta(G)+1$ vertices and moreover $|PN_G[v, D]| = \Delta(G) + 1$. Consequently $G \in \mathcal{A}$.

If G is a star, then $\widetilde{\gamma}_c(G) = n(G) - 1 = 2m(G) - n(G) + 1$. If $\widetilde{\gamma}_c(G) = 2m(G) - n(G) + 1$, then, since G has at least one edge, $2m(G) - n(G) + 1 = \widetilde{\gamma}_c(G) \leq n(G) - 1$. Thus $m(G) \leq n(G) - 1$ and by the connectivity of G, $m(G) = n(G) - 1$. Consequently, $\widetilde{\gamma}_c(G) = n(G) - 1$ and, according to Observation 2, G is a star. $\square$

Before stating the next theorem, we describe a family $\mathcal{S}$ of graphs, which are the extremal graphs of the theorem.

Let $\mathcal{S}$ be the family of graphs, where a graph G belongs to $\mathcal{S}$ if and only if there exists an independent set I in G such that $G-I$ is a tree and every vertex of $G-I$ is adjacent to exactly one vertex of I.

Theorem 3 *If G is a graph, then*

$$\widetilde{\gamma}_c(G) \geq n(G) - \frac{m(G)+1}{2}. \tag{1}$$

In addition, $\widetilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$ if and only if G belongs to the family $\mathcal{S}$.

Proof. Let D be a minimum outer-connected dominating set in G and let $m_G(D)$ denote the number of edges joining D and $V(G)-D$ in G. Then

$$m(G-D) \geq n(G) - \widetilde{\gamma}_c(G) - 1, \tag{2}$$

and

$$m_G(D) \geq n(G) - \widetilde{\gamma}_c(G). \tag{3}$$

Hence

$$m(G) \geq m(G-D) + m_G(D) \geq 2n(G) - 2\widetilde{\gamma}_c(G) - 1 \tag{4}$$

and thus $\widetilde{\gamma}_c(G) \geq n(G) - \frac{m(G)+1}{2}$.

We now prove that $\widetilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$ if and only if G belongs to the family $\mathcal{S}$.

Assume first that G belongs to $\mathcal{S}$. Then there exists an independent set I in G such that $G-I$ is a tree and every vertex of $G-I$ is adjacent to exactly one vertex of I. The set I is an outer-connected dominating set in G. Thus $\widetilde{\gamma}_c(G) \leq |I| = n(G) - n(G-I)$ and because $n(G-I) = m_G(I) = m(G) - m(G-I) = m(G) - n(G-I) + 1$ (and therefore $n(G-I) = \frac{m(G)+1}{2}$),

$$\widetilde{\gamma}_c(G) \leq n(G) - n(G-I) = n(G) - \frac{m(G)+1}{2}.$$

Consequently, by (1), $\widetilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$.

Assume now that $\widetilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$. Let D be a minimum outer-connected dominating set of G. Then by (4) we have

$$\begin{aligned} m(G) &\geq m(G-D) + m_G(D) \\ &\geq 2n(G) - 2\widetilde{\gamma}_c(G) - 1 \\ &= 2n(G) - 2\left(n(G) - \tfrac{m(G)+1}{2}\right) - 1 = m(G). \end{aligned} \tag{5}$$

Hence, and by (2) and (3), it follows that

$$m(G-D) = n(G) - \widetilde{\gamma}_c(G) - 1 \tag{6}$$

and

$$m_G(D) = n(G) - \widetilde{\gamma}_c(G). \tag{7}$$

Since $G-D$ is connected (by the choice of D) and has $n(G-D)-1$ edges, $G-D$ is a tree. Moreover, since $m(G) = m(G-D) + m_G(D) + m(G[D])$ and $m(G) = m(G-D)+m_G(D)$ by (5), $m(G[D]) = 0$ and D is an independent set. Now, since D is dominating, it follows from (7) that each vertex of $G-D$ has exactly one neighbour in D. This completes the proof of the fact that G belongs to the family S. □

Probably, the statement of the next lemma is well-known, but since we have not seen such a result anywhere, we state it here with a short proof.

Lemma 4 *In a graph G with $\delta(G) \geq 2$ there is a cycle of length at least $\delta(G)+1$.*

Proof. Let $(v_0, v_1, \ldots, v_l)$ be a longest path in G. Then $N_G(v_0) \subseteq \{v_1, v_2, \ldots, v_l\}$ and therefore $v_k \in N_G(v_0) \cap \{v_1, v_2, \ldots, v_l\}$ for some $k \geq \deg_G(v_0) \geq \delta(G)$. Consequently $(v_0, v_1, \ldots, v_k, v_0)$ is a required cycle. □

Theorem 5 *If G is a connected graph of order n, then*

$$\widetilde{\gamma}_c(G) \leq n - \delta(G).$$

Proof. The result is obvious if $\delta(G) \leq 2$. Now assume that $\delta(G) \geq 3$. By Lemma 4 there exists a cycle of length at least $\delta(G)+1$ in G. Let $C = (v_0, v_1 \ldots, v_l, v_0)$ be a shortest cycle in G of length at least $\delta(G)$. We claim that the set $D = V(G) - V(C)$ is an outer-connected dominating set of G. Certainly $G - D = G[V(C)]$ is connected. Suppose D is not dominating. Then $N_G(v) \cap D = \emptyset$ for some vertex $v \in V(C)$. We may assume, without loss of generality, that $v = v_0$ and $\deg_G(v_0) = r$. Then $N_G(v_0) = \{v_1, v_{i_1}, v_{i_2}, \ldots, v_{i_{r-2}}, v_l\}$ where $1 < i_1 < i_2 < \ldots < i_{r-2} < l$. Now $(v_0, v_{i_1}, v_{i_1+1}, \ldots, v_l, v_0)$ is a cycle of length at least $\delta(G)$ which is shorter than C, a contradiction. □

In the next observation we describe the main properties of minimum outer-connected dominating sets of a graph.

Observation 3 *Let G be a connected graph on at least 3 vertices. If D is a minimum outer-connected dominating set in G and Ω is the set of end-vertices of G, then*

(i) *$\Omega \subseteq D$ if G is not a star;*

(ii) *$D = \Omega$ or $|\Omega \cap D| = |\Omega| - 1$ if G is a star;*

(iii) *$\widetilde{\gamma}_c(G) \geq |\Omega|$;*

(iv) *$\widetilde{\gamma}_c(G) = |\Omega|$ if and only if every vertex of G is either a support or an end-vertex.*

Proof. Since (ii) is obvious and (iii) easily follows from (i) and Observation 2, we only prove (i) and (iv).
(i) Assume G is not a star and suppose to the contrary that $\Omega - D \neq \emptyset$. Then, since $G - D$ is connected, $\widetilde{\gamma}_c(G) = |V(G)| - 1$ and therefore, by Observation 2, G is a star, a contradiction.
(iv) The statement is trivial for stars. Thus assume G is not a star and $\widetilde{\gamma}_c(G) = |\Omega|$.

Then, by (i), Ω is a minimum outer-connected dominating set of G and this implies that every vertex belonging to $V(G) - \Omega$ is a support. Conversely, if every vertex of the graph G is a support or an end-vertex, then Ω is an outer-connected dominating set of G and, by (i), it is a minimum outer-connected dominating set of G and therefore $\widetilde{\gamma}_c(G) = |\Omega|$. □

A subdivision of an edge uv is obtained by inserting a new vertex w and replacing the edge uv with the edges uw and wv. A spider is the tree obtained from a star by subdividing all of its edges. A wounded spider is a tree obtained from a spider by removing at least one end-vertex. Certainly, a star is also a wounded spider.

The next theorem provides a lower bound for the outer-connected domination number of a tree.

Theorem 6 *If T is a tree of order $n \geq 3$, then*

$$\widetilde{\gamma}_c(T) \geq \Delta(T).$$

Furthermore, $\widetilde{\gamma}_c(T) = \Delta(T)$ if and only if T is a wounded spider.

Proof. The result is obvious if T is a star. Now let T be a tree of order $n \geq 3$ and assume T is not a star. Since T has at least $\Delta(T)$ end-vertices and since all end-vertices belong to every outer-connected dominating set of T we certainly have $\widetilde{\gamma}_c(T) \geq \Delta(T)$.

Clearly, if T is a wounded spider, then $\widetilde{\gamma}_c(T) = \Delta(T)$. Now assume T is a tree for which $\widetilde{\gamma}_c(T) = \Delta(T)$. Then since $\Delta(T) \leq |\Omega(T)| \leq \widetilde{\gamma}_c(T)$ we have $\Delta(T) = |\Omega(T)|$ (and $\widetilde{\gamma}_c(T) = |\Omega(T)|$). From the equality $\Delta(T) = |\Omega(T)|$, it follows that there exists a unique vertex, say u, of maximum degree, and $\Omega(T)$ is a minimum outer-connected dominating set of T. In addition, every inner vertex of a path joining u to an end-vertex of T (if any) is of degree 2. This and the fact that $\Omega(T)$ is dominating implies that every such a path is of length at most two and at least one of them is of length one. This proves that T is a wounded spider. □

Let $\mathcal{R}$, $\mathcal{R}'$, $\mathcal{R}''$ be families of trees on at least 3 vertices defined as follows: a tree T belongs to $\mathcal{R}$ if T is the corona of another tree, while a tree T belongs to $\mathcal{R}'$ or $\mathcal{R}''$, respectively, if T is obtained from a tree S belonging to $\mathcal{R}$ by adding a new vertex and joining it to an end-vertex of S or to an inner vertex of S, respectively.

Theorem 7 *If T is a tree of order $n \geq 3$, then*

$$\widetilde{\gamma}_c(T) \geq \left\lceil \frac{n}{2} \right\rceil$$

with equality $\widetilde{\gamma}_c(T) = \lceil \frac{n}{2} \rceil$ if and only if T belongs to $\mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''$.

Proof. Let $T = (V, E)$ be a tree and let D be a minimum outer-connected dominating set of T. Suppose, on the contrary, that $\widetilde{\gamma}_c(T) < \lceil \frac{n}{2} \rceil$. Then $\widetilde{\gamma}_c(T) < \frac{n}{2}$ and by the pigeon hole principle $|N_T(v) \cap (V - D)| \geq 2$ for some $v \in D$. But then any path joining two vertices of $N_T(v) \cap (V - D)$ in the connected graph $T - D$ together with v form a cycle in T, which is impossible.

Now we prove that $\widetilde{\gamma}_c(T) = \lceil \frac{n}{2} \rceil$ if and only if T belongs to $\mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''$.

If $T \in \mathcal{R}$, then T is a corona and $\Omega(T)$ is a minimum outer-connected dominating set of T and $\widetilde{\gamma}_c(T) = |\Omega(T)| = \frac{n}{2} = \lceil \frac{n}{2} \rceil$.

Assume $T \in \mathcal{R}' \cup \mathcal{R}''$. Then there exists an end-vertex v such that $T - v$ is a corona. If $T \in \mathcal{R}'$ and u is a neighbour of v, then $\Omega(T) \cup \{u\}$ is a minimum outer-connected dominating set of T and $\widetilde{\gamma}_c(T) = |\Omega(T)| + 1 = \frac{n-1}{2} + 1 = \lceil \frac{n}{2} \rceil$. Finally, if $T \in \mathcal{R}''$, then $\Omega(T)$ is a minimum outer-connected dominating set of T and $\widetilde{\gamma}_c(T) = |\Omega(T)| = \lceil \frac{n}{2} \rceil$.

Let T be a tree of order at least 3 such that $\widetilde{\gamma}_c(T) = \lceil \frac{n}{2} \rceil$. If $n = 3$, then certainly $T = P_3 \in \mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''$. Thus assume T has at least 4 vertices. Then $\lceil \frac{n}{2} \rceil < n - 1$ which implies that $\widetilde{\gamma}_c(T) < n - 1$, so T is not a star. Consequently, by Observation 3, $\Omega(T) \subseteq D$. If $D = \Omega(T)$, then $|\Omega(T)| = \lceil \frac{n}{2} \rceil$ and therefore every vertex belonging to $V - \Omega(T)$ is adjacent to exactly one vertex in $\Omega(T)$ or one of them is adjacent to two end-vertices and each of the other vertices is adjacent to exactly one end-vertex. This implies T belongs to $\mathcal{R}$ or $\mathcal{R}''$.

Finally assume $\Omega(T) \subsetneq D$. Then there exists a vertex $v \in D$ such that $\deg_T(v) \geq 2$. We shall prove that $\deg_T(v) = 2$ and v is the only such vertex. From the connectivity of $T - D$ it follows that $|N_T(v) \cap (V - D)| \leq 1$. We claim that $|N_T(v) \cap D| \leq 1$. Suppose, to the contrary, that two vertices x and y belong to $N_T(v) \cap D$. Since T is a tree we have $|N_T(\{x, y, v\}) \cap (V - D)| \leq 1$. This, and the fact that no two vertices in $V - D$ share common neighbour in D, imply that $|V - D| = |N_T(\{x, y, v\}) \cap (V - D)| + |N_T(D - \{x, y, v\}) \cap (V - D)| \leq 1 + |D| - 3 = |D| - 2$. Hence, $n - |D| \leq |D| - 2$ and $|D| \geq \frac{n}{2} + 1 > \lceil \frac{n}{2} \rceil$. Thus v has exactly one neighbour in D and exactly one neighbour in $V - D$. Suppose now that $|D - \Omega(T)| \geq 2$. Then there exist $u, v \in D$ such that $\deg_T(u) = \deg_T(v) = 2$. Denote by u_1 and v_1 the neighbour of u and v in D, respectively. Then $|V - D| = |N_T(\{u, u_1, v, v_1\}) \cap (V - D)| + |N_T(D - \{u, u_1, v, v_1\}) \cap (V - D)| \leq 2 + |D| - 4 = |D| - 2$. Hence, $n - |D| \leq |D| - 2$ and $\lceil \frac{n}{2} \rceil = |D| \geq \frac{n}{2} + 1 > \lceil \frac{n}{2} \rceil$, a contradiction. We obtain that v is the unique vertex of degree two in D and the vertex $x \in N_T(v) \cap D$ is an end-vertex of T. Thus, we conclude that T belongs to the family $\mathcal{R}'$. □

As an immediate consequence of this theorem and of Ore's theorem [6] we have the following corollary.

Corollary 1 *For a tree $T \neq K_1$ we have $\widetilde{\gamma}_c(T) = \gamma(T)$ if and only if T is a corona.*

4 Edge subdivision and vertex removing

Now we examine the effects on $\widetilde{\gamma}_c(G)$ when G is modified by an edge subdivision. We start with some notation. If uv is an edge of G then by $G \oplus w_{uv}$ we denote the graph obtained from G by the subdivision of uv.

Theorem 8 *For every integer k there exist a graph G and an edge uv of G such that $\widetilde{\gamma}_c(G \oplus w_{uv}) - \widetilde{\gamma}_c(G) = k$.*

Proof. We consider three cases.

Case 1. If $k \leq -2$ then we construct graphs G and $G \oplus w_{uv}$ as follows. We begin with four spiders S_i with $|V(S_i)| = 2|k|-1$ and denote its centers by x_i, $i = 1, 2, 3, 4$. Next we add four end-vertices y_i and four edges $x_i y_i$. Finally, to obtain the graph G, we add vertices u, v and edges $uv, ux_1, ux_2, vx_3, vx_4$, x_1x_3 (see Fig. 1). It is easy to observe that $D = N_G[x_2] \cup \Omega(G)$ is a minimum outer-connected dominating set of G and thus $\widetilde{\gamma}_c(G) = 5|k| + 1$.

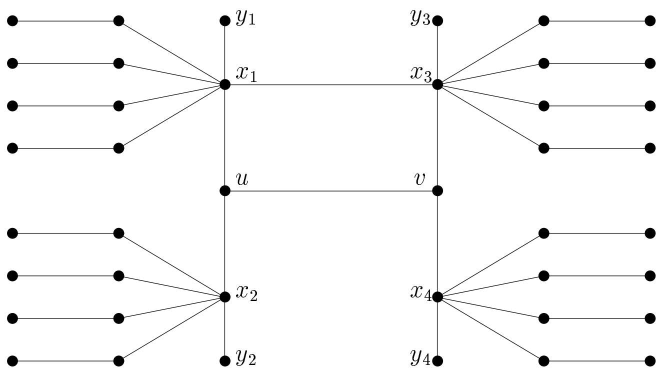

Figure 1: Graph G for $k = -5$

Let $G \oplus w_{uv}$ be a graph which results if the edge $e = uv$ is subdivided (see Fig. 1). Notice that $D = \{w\} \cup \Omega(G)$ is the minimum outer-connected dominating set of $G \oplus w_{uv}$ of cardinality $4|k|+1$. Thus $\widetilde{\gamma}_c(G \oplus w_{uv}) = 4|k|+1$ and $\widetilde{\gamma}_c(G \oplus w_{uv}) - \widetilde{\gamma}_c(G) = -|k| = k$.

Case 2. Define $A = \{u, v, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$. Then for $k = -1, 0, 1$ let H be the subgraph of G induced by A, $A - \{y_4\}$, $A - \{y_3, y_4\}$, respectively. The difference $\widetilde{\gamma}_c(H \oplus w_{uv}) - \widetilde{\gamma}_c(H) = k$ is easy to verify.

Case 3. If $k \geq 2$, then let G be the join $P_{3k} + K_1$, where $x_1, x_2, \ldots, x_{3k}$ are consecutive vertices of P_{3k} and x is the universal vertex of G. Then obviously $\{x\}$ is a minimum outer-connected dominating set of G, so $\widetilde{\gamma}_c(G) = 1$. It is also easy to see that $D = \{x, x_1, \ldots, x_k\}$ is a minimum outer-connected dominating set of $G \oplus w_{x_k x_{k+1}}$. Hence $\widetilde{\gamma}_c(G \oplus w_{x_k x_{k+1}}) = k + 1$ and the proof is complete. □

Now we investigate how removing a vertex influences an outer-connected domination number. We have the following two propositions.

Proposition 9 *For every connected graph G and a vertex $v \in V(G)$ such that $G - v$ is connected we have*

$$\widetilde{\gamma}_c(G) \leq \widetilde{\gamma}_c(G - v) + 1.$$

Proof. If D is a minimum outer-connected dominating set of $G - v$, then clearly $D \cup \{v\}$ is a minimum outer-connected dominating set in G and therefore $\widetilde{\gamma}_c(G) \leq \widetilde{\gamma}_c(G - v) + 1$. □

Proposition 10 *For every integer $k \geq -1$, there exists a graph G such that $\widetilde{\gamma}_c(G - v) - \widetilde{\gamma}_c(G) = k$.*

Proof. If $k \geq 1$, then let G be the graph which results if we add to a path P_{k+3} a vertex v and edges joining v to all vertices from the path. The vertex v is a universal non-cut vertex of G and thus we have $\widetilde{\gamma}_c(G) = 1$. Next we remove v with all edges incident to v. Notice that $G - v$ is a path on at least four vertices, so by Observation 1, $\widetilde{\gamma}_c(G - v) = k + 3 - 2$. Thus $\widetilde{\gamma}_c(G - v) - \widetilde{\gamma}_c(G) = k$. For $k = 0$ and $k = -1$ let G be a path P_2 and P_3, respectively, and let v be an end-vertex of G. It is easy to verify that $\widetilde{\gamma}_c(G - v) - \widetilde{\gamma}_c(G) = k$. □

5 Comparing $\widetilde{\gamma}_c$ to other types of domination numbers

In this section we investigate relations between the outer-connected domination number and other types of domination numbers. We begin with some definitions.

A set $D \subseteq V(G)$ is a *connected dominating set* of G if it is dominating and the induced subgraph $G[D]$ is connected. The cardinality of a minimum connected dominating set of G is the *connected domination number* and is denoted by $\gamma_c(G)$.

We say that a set $D \subseteq V(G)$ is a *doubly connected dominating set* of G if it is dominating and the induced subgraphs $G[D]$ and $G[V(G) - D]$ are connected. The cardinality of a minimum doubly connected dominating set in G is a *doubly connected domination number* and is denoted by $\gamma_{cc}(G)$. Properties of the doubly connected domination number of a graph are studied in [2].

A set $D \subseteq V(G)$ is a *restrained dominating set* if every vertex in $V(G) - D$ is adjacent to a vertex in D and to another vertex in $V(G) - D$. By $\gamma_r(G)$ we denote the size of a smallest restrained dominating set of G. This type of domination was studied for example in [3].

Since for an arbitrary graph G every connected dominating set is a dominating set and every doubly connected dominating set is a connected dominating set, we have the following inequality chain

$$\gamma(G) \leq \widetilde{\gamma}_c(G) \leq \gamma_{cc}(G).$$

However, each of the differences $\widetilde{\gamma}_c(G) - \gamma(G)$ and $\gamma_{cc}(G) - \widetilde{\gamma}_c(G)$ may be arbitrarily large.

Proposition 11 *For any non-negative integers r and t, there exists a graph G such that $\widetilde{\gamma}_c(G) - \gamma(G) = r$ and $\gamma_{cc}(G) - \widetilde{\gamma}_c(G) = t$.*

Proof. Let G be the graph obtained from the star $K_{1,r+t+1}$ by subdividing t of its edges. It easy to verify that $\gamma(G) = 1+t$, $\widetilde{\gamma}_c(G) = r+t+1$ and $\gamma_{cc}(G) = |V(G)|-1 = r + 2t + 1$. □

In the next proposition we prove that the numbers $\widetilde{\gamma_c}(G)$ and $\gamma_c(G)$ are incomparable.

Proposition 12 *For every positive integer r there exist graphs G_1 and G_2 such that $\widetilde{\gamma_c}(G_1) - \gamma_c(G_1) = r$ and $\gamma_c(G_2) - \widetilde{\gamma_c}(G_2) = r$.*

Proof. Let G_1 be a star of order $r + 2$ and let G_2 be a graph pictured in Figure 2. It is straightforward to verify that $\widetilde{\gamma_c}(G_1) = r + 1$, $\gamma_c(G_1) = 1$ and $\widetilde{\gamma_c}(G_2) = r + 2$, $\gamma_c(G_2) = 2r + 2$. Therefore, $\widetilde{\gamma_c}(G_1) - \gamma_c(G_1) = r$ and $\gamma_c(G_2) - \widetilde{\gamma_c}(G_2) = r$. □

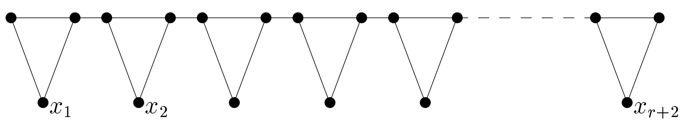

Figure 2: Graph G_2

Theorem 13 *For any connected graph G with $n(G) > 1$,*

(i) $\gamma_r(G) \leq \widetilde{\gamma_c}(G) + 1$;

(ii) $\gamma_r(G) = \widetilde{\gamma_c}(G) + 1$ *if and only if G is a star;*

(iii) *For any non-negative integer k there exists a graph G such that $\widetilde{\gamma_c}(G) - \gamma_r(G) = k$.*

Proof.

(i) If $\widetilde{\gamma_c}(G) \leq n(G) - 2$, then every outer-connected dominating set of G is a restrained dominating set of G and therefore $\gamma_r(G) \leq \widetilde{\gamma_c}(G) \leq \widetilde{\gamma_c}(G) + 1$. Otherwise $\widetilde{\gamma_c}(G) = n(G) - 1$ and, by Observation 2, G is a star, so $\gamma_r(G) = n(G) = \widetilde{\gamma_c}(G) + 1$.

(ii) The result follows immediately from (i).

(iii) Let $G = (k+1)K_2 + K_1$. It is an easy exercise to verify that $\widetilde{\gamma_c}(G) - \gamma_r(G) = k$. □

6 Complexity issues for $\widetilde{\gamma_c}$

In this section we consider the decision problem of the OUTER-CONNECTED DOMINATING SET as follows

OUTER-CONNECTED DOMINATING SET (OCDS)
INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
QUESTION: Does G have an outer-connected dominating set of cardinality at most k?

The decision problem of OCDS stays NP–complete even when restricted to connected bipartite graphs.

To prove that the decision problem for arbitrary graphs is NP-complete, we need to use a well-known NP-completeness result, called Exact Three Cover (X3C), which is defined as follows.

EXACT COVER BY 3-SETS (X3C)
INSTANCE: A finite set X with $|X| = 3q$ and a collection $\mathcal{C}$ of 3-element subsets of X.
QUESTION: Does $\mathcal{C}$ contain an exact cover for X, that is, a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that every element of X occurs in exactly one member of $\mathcal{C}'$?

Garey and Johnson in [4] proved that X3C is NP–complete.

Theorem 14 *OCDS for bipartite graphs is NP–complete.*

Proof. We know that the OCDS problem for bipartite graphs is in the NP class of decision problems as it is easy to verify in polynomial time whether a given subset of vertices of G is an outer-connected dominating set of G. To show that OCDS is an NP-complete problem, we will establish a polynomial transformation from X3C. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of X3C.

We will construct a bipartite graph G and a positive integer k such that this instance of X3C will have an exact three cover if and only if G has an outer-connected dominating set of cardinality at most k.

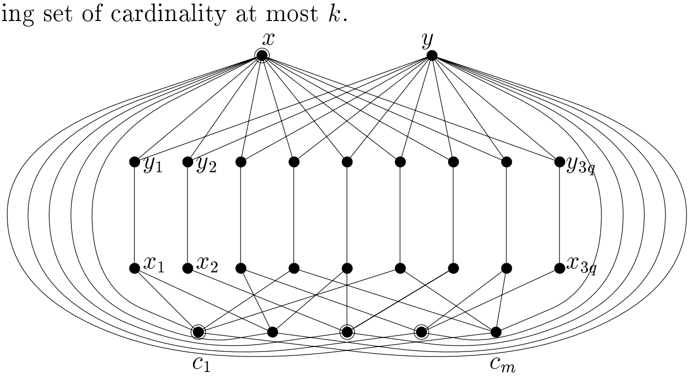

Figure 3: Reduction from X3C to OCDS

Now we describe the construction of G and k as follows:

$$\begin{aligned}
V(G) &= \{x_1, x_2, \ldots, x_{3q}\} \cup \{y_1, y_2, \ldots, y_{3q}\} \cup \{c_1, c_2, \ldots, c_m\} \cup \{x, y\},\\
E(G) &= \{x_i y_i : i \in \{1, 2, \ldots, 3q\}\}\\
&\quad \cup \{x_i c_j : x_i \in C_j,\ i \in \{1, 2, \ldots, 3q\},\ j \in \{1, 2, \ldots, m\}\}\\
&\quad \cup \{x y_i, y y_i : i \in \{1, 2, \ldots, 3q\}\}\\
&\quad \cup \{x c_j, y c_j : j \in \{1, 2, \ldots, m\}\}\\
k &= q + 1.
\end{aligned}$$

The graph G so obtained is connected and bipartite.

Assume first that $\mathcal{C}$ has an exact 3-cover, say $\mathcal{C}'$. Then $\{c_j : C_j \in \mathcal{C}'\} \cup \{x\}$ is an outer-connected dominating set of cardinality $q+1$.

Now assume that D is an outer-connected dominating set of cardinality at most $q+1$. If x and y do not belong to D, then since D is dominating, at least $3q$ vertices of G belong to D to dominate y_i, $i = 1, 2, \ldots, 3q$, so $|D| \geq 3q$, a contradiction. Hence, at least one of x and y, say x, belongs to D. Notice that $N[x] \cap X = \emptyset$. Moreover, for each vertex u belonging to $\{x_1, \ldots, x_{3q}, y_1, \ldots, y_{3q}\}$, $|N_G(u) \cap X| = 1$ and for every vertex v belonging to $\{c_1, \ldots, c_m\}$, $|N_G(v) \cap X| = 3$. Hence $y \notin D$ and exactly q vertices of $\{c_1, \ldots, c_m\}$, say vertices $c_{j_1}, \ldots, c_{j_q}$, must belong to D in such a way that the corresponding set $\{C_{j_1}, \ldots, C_{j_q}\}$ is an exact cover of X. □

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