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An improved lower bound for domination numbers of the queen's graph

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Abstract

The queen's graph Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are adjacent if they are in the same row, column, or diagonal. Let $\gamma(Q_n)$ be the minimum size of a dominating set of Q_n. It has been proved that $\gamma(Q_n) \geq (n-1)/2$ for all n. Known dominating sets imply that $\gamma(Q_n) = (n-1)/2$ for $n = 3, 11$. We show that $\gamma(Q_n) = (n-1)/2$ only for $n = 3, 11$, and thus that $\gamma(Q_n) \geq \lceil n/2 \rceil$ for all other positive integers n.

The queen's graph Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are adjacent if they are in the same row, column, or diagonal. A set D of squares of Q_n is a *dominating set* for Q_n if every square of Q_n is either in D or adjacent to a square in D. If no two squares of a set I are adjacent then I is an *independent set*. Let $\gamma(Q_n)$ denote the minimum size of a dominating set for Q_n; a dominating set of this size is a *minimum* dominating set. Let $i(Q_n)$ denote the minimum size of an independent dominating set for Q_n.

The problems of finding values of $\gamma(Q_n)$ and of $i(Q_n)$ are given as Problem C18 in [7], and have interested mathematicians for well over a century. De Jaenisch [8] considered these problems in 1862. In 1892, Rouse Ball [12] gave dominating sets and independent dominating sets of Q_n for $n \leq 8$. Ahrens [1] extended this in 1910 to $n \leq 13$ and $n = 17$ for $\gamma(Q_n)$ and to $n \leq 12$ for $i(Q_n)$. In most cases, proof that these sets were minimum dominating sets required recent work on lower bounds.

Raghavan and Venketesan [11] and Spencer [5, 13] independently proved that

$$\gamma(Q_n) \geq (n-1)/2. \tag{1}$$

The only values for which equality was known to hold in (1) were $n = 3, 11$, so researchers sought better bounds. Weakley showed [13] that

$$\gamma(Q_{4k+1}) \geq 2k + 1. \tag{2}$$

A combined effort by several researchers [2, 4, 6, 9, 10, 13, 14] has shown that $\gamma(Q_{4k+1}) = 2k + 1$ for $k \leq 32$.

Burger and Mynhardt showed [3] that $\gamma(Q_{4k+3}) \geq 2k+2$ for $3 \leq k \leq 7$, with equality for $k = 4, 7$. Weakley further restricted [14] the possibilities for equality to hold in (1), as will be described below in preparation for our main result: $\gamma(Q_n) = (n-1)/2$ holds only for $n = 3, 11$, and thus $\gamma(Q_n) \geq \lceil n/2 \rceil$ for all other positive integers n.

For the remainder of the paper, let n be a positive integer such that $\gamma(Q_n) = (n-1)/2$, and let D be a minimum dominating set of Q_n. We write d for the size of D; that is, $d = (n-1)/2$. In [13, Theorem 2] it was established that $n \equiv 3 \pmod 4$, so d is odd, and that D is an independent set, a fact we will often use.

We will identify the $n \times n$ chessboard with a square of side length n in the Cartesian plane, having sides parallel to the coordinate axes. We place the board with its center at the origin of the coordinate system, and refer to board squares by the coordinates of their centers. The square (x,y) is in *column* x and *row* y. (As n is odd, x and y are integers.) Columns and rows will be referred to collectively as *orthogonals*. The *difference diagonal* (respectively *sum diagonal*) through square (x,y) is the set of all board squares with centers on the line of slope $+1$ (respectively -1) through the point (x,y). The value of $y - x$ is the same for each square (x,y) on a difference diagonal, and we will refer to the diagonal by this value. Similarly, the value of $x + y$ is the same for each square on a sum diagonal, and we associate this value to the diagonal.

By [14, Theorem 3], for each (x, y) in D, both x and y are even. It then follows from the independence of D that the numbers of the rows and columns occupied by D are

$$0, \pm 2, \pm 4, \ldots, \pm(d-1). \tag{3}$$

As shown in the proof of [14, Theorem 4], this implies that there is a non-negative integer e such that the numbers of the sum diagonals and difference diagonals occupied by D are

$$0, \pm 2, \pm 4, \ldots, \pm 2e, \pm(2e+4), \pm(2e+8), \ldots, \pm(2d-2e-2). \tag{4}$$

For any square (x, y), the Parallelogram Law asserts $2x^2 + 2y^2 = (x+y)^2 + (y-x)^2$, relating the numbers of the orthogonals of (x, y) to those of its diagonals. Thus $2\sum_{(x,y)\in D}(x^2 + y^2) = \sum_{(x,y)\in D}((x+y)^2 + (y-x)^2)$. Putting the numbers from (3) and (4) into this equation and simplifying gives

$$d^2 - 3(d-2e)^2 = -2. \tag{5}$$

Set $X = d$ and $Y = d - 2e$. Then $Y - 1$ is the number of terms after $\pm 2e$ in (4), so $X, Y > 0$ and from (5) we have $X^2 - 3Y^2 = -2$. The positive integer solutions of this Pell's equation can be found by standard methods; the corresponding values of d and e are given by the recursion

$$(d_1, e_1) = (1, 0), (d_2, e_2) = (5, 1), \text{ and} \\ (d_i, e_i) = 4(d_{i-1}, e_{i-1}) - (d_{i-2}, e_{i-2}) \text{ for } i > 2. \tag{6}$$

Then using $n_i = 2d_i + 1$ we get a sequence $(n_i)_{i=1}^{\infty}$ of integers defined by

$$n_1 = 3, n_2 = 11, \text{ and } n_i = 4n_{i-1} - n_{i-2} - 2 \text{ for } i > 2. \tag{7}$$

Since $\gamma(Q_n) = (n-1)/2$, $n = n_i$ for some positive integer i. We have now summarized what we need from [14].

From (4) we see that each number t of an occupied diagonal satisfies either $t \equiv 2e - 2 \pmod 4$ or $t \equiv 2e \pmod 4$, and that there are e occupied difference diagonals of the former kind and $d - e$ occupied difference diagonals of the latter kind; similarly for sum diagonals. Following [3], we refer to the former as *core* diagonals (both difference and sum), and the latter as *body* diagonals. Since all numbers of occupied rows and columns are even, a square of D is on a core difference diagonal if and only if it is on a core sum diagonal.

Definition. Say that (x, y) in D is a *core square* if its diagonals are core diagonals; otherwise (x, y) is a *body square.*

There are e core squares and $d - e$ body squares in D.

Definition. Let S be the sum of $|x| + |y|$ over (x, y) in D.

Using the list (3) of numbers of orthogonals occupied by D, we have

$$S = \sum_{(x,y) \in D} (|x| + |y|) = 4 \sum_{j=1}^{(d-1)/2} 2j = d^2 - 1.$$

It is useful to look at the terms in the sum S in another way. For any square (x, y), we have

$$|x| + |y| = \max\{|x + y|, |y - x|\}. \tag{8}$$

We now use (8) and the list (4) of the numbers of the diagonals occupied by D to construct a sum $S_{\max}$ that is an upper bound for S. The terms in this sum are the largest integers that could occur as absolute values of diagonal numbers of D. The *multiplicity* in $S_{\max}$ of a non-negative integer is the number of times it occurs as a term in the sum. Since D is independent, the multiplicity of each term is at most four.

As n is a member of the sequence $(n_i)_{i=1}^{\infty}$ defined by (7), it is convenient to consider the construction of $S_{\max}$ in four cases, depending on the residue of i modulo 4. We describe the first in detail; the others are similar.

Case: $i \equiv 1 \pmod 4$. Here (6) implies $d \equiv 1 \pmod 4$ and $e \equiv 0 \pmod 4$.

We look first at the e core squares of D. From (4), the core diagonals with numbers of largest absolute value are sum diagonals $\pm(2e-2)$ and difference diagonals $\pm(2e-2)$. Thus $2e-2$ has multiplicity four in $S_{\max}$. The next largest absolute values are all $2e-6$, again with multiplicity four, and the terms continue to decrease by four, each with multiplicity four, down to $e+2$.

There are $d-e$ body squares, and here $d-e \equiv 1 \pmod 4$, so the terms in $S_{\max}$ for body diagonals are $2d-2e-2$ with multiplicity four, followed by $2d-2e-6, \ldots, d-e+3$, each with multiplicity four, and a single $d-e-1$. So in this case

$$S_{\max} = \sum_{j=1}^{e/4} 4(2e-2-4(j-1)) + \sum_{j=1}^{(d-e-1)/4} 4(2d-2e-2-4(j-1)) + (d-e-1).$$

Case: $i \equiv 2 \pmod 4$. Here $d \equiv 1 \pmod 4$ and $e \equiv 1 \pmod 4$, and

$$S_{\max} = \sum_{j=1}^{(e-1)/4} 4(2e-2-4(j-1)) + (e-1) + \sum_{j=1}^{(d-e)/4} 4(2d-2e-2-4(j-1)).$$

Case: $i \equiv 3 \pmod 4$. Here $d \equiv 3 \pmod 4$ and $e \equiv 0 \pmod 4$, and

$$S_{\max} = \sum_{j=1}^{e/4} 4(2e-2-4(j-1)) + \sum_{j=1}^{(d-e-3)/4} 4(2d-2e-2-4(j-1)) + 3(d-e+1).$$

Case: $i \equiv 0 \pmod 4$. Here $d \equiv 3 \pmod 4$ and $e \equiv 3 \pmod 4$, and

$$S_{\max} = \sum_{j=1}^{(e-3)/4} 4(2e-2-4(j-1)) + 3(e+1) + \sum_{j=1}^{(d-e)/4} 4(2d-2e-2-4(j-1)).$$

Lemma 1 *Let n be a positive integer such that $\gamma(Q_n) = (n-1)/2$ and let D be a minimum dominating set of Q_n. Then the $(n-1)/2$ terms in the sum S_{max} are the values of $|x|+|y|$ as (x,y) ranges through D.*

Proof. In each of the four cases just considered, simplification yields $S_{\max} = 3(d^2 - 2de + 2e^2 - 1)/2$, and then (5) implies $S_{\max} = d^2 - 1$, which is also the value of S. By the way in which the terms of $S_{\max}$ were chosen, there is an ordering $((x_i, y_i))_{i=1}^{(n-1)/2}$ of D such that $|x_i| + |y_i|$ does not exceed the ith term of $S_{\max}$. The conclusion then follows from $S_{\max} = S$. □

We now turn our attention to the squares of D that lie in row 0 or in column 0.

Definitions. Let R_0 denote the set of squares in row 0 of the $n \times n$ board, and let C_0 denote the set of squares in column 0.
For each non-negative integer k, let $A_k = \{(\pm k, 0), (0, \pm k)\}$. Note that A_0 contains only the board's center square, and for $k > 0$, A_k contains four squares that are pairwise adjacent in Q_n.

Lemma 2 *Let n be a positive integer such that $\gamma(Q_n) = (n-1)/2$ and let D be a minimum dominating set of Q_n. If the positive integer k does not occur in S_{max} or has multiplicity 4 in S_{max}, then D and A_k are disjoint. If m is the only integer that occurs in S_{max} with multiplicity less than 4, then $m = 0$ and $D \cap (R_0 \cup C_0) = \{(0,0)\}$.*

Proof. Since each (x, y) in A_k has $|x| + |y| = k$, it is clear from Lemma 1 that if k does not occur in the sum $S_{\max}$ then D and A_k are disjoint.

If k has multiplicity 4 in $S_{\max}$, then by Lemma 1 there are four distinct squares (x, y) in D satisfying $|x| + |y| = k$. The independence of D implies both that these four squares together occupy sum diagonals $\pm k$ and difference diagonals $\pm k$, and that no two of these diagonals can meet at a square of D. (Informally, no square of D can "use up" two of the diagonals having number of absolute value k, as the independence of D would then prevent $|x| + |y| = k$ from occurring at four squares of D.) Therefore no square of D is in A_k.

Suppose now that m is the only integer that occurs in the sum $S_{\max}$ with multiplicity less than 4. By the first part of this lemma we have $D \cap (R_0 \cup C_0) \subseteq A_m$. The inequality $|D \cap (R_0 \cup C_0)| \leq 1$ then clearly holds if $m = 0$, and it also holds if $m > 0$, as D is independent and any two of the four squares of A_m are adjacent in Q_n. But $|D \cap R_0| = 1$ and $|D \cap C_0| = 1$ since D is an independent set that by (3) meets R_0 and C_0. Thus $D \cap R_0 = D \cap C_0 = \{(0,0)\}$, so $m = 0$. □

Theorem 3 *The only integers n for which $\gamma(Q_n) = (n-1)/2$ are $n = 3, 11$.*

Proof. It is easily verified that $\{(0,0)\}$ dominates Q_3 and $\{(0,0), \pm(2,4), \pm(4,-2)\}$ dominates Q_{11}, so by (1) we have $\gamma(Q_3) = 1$ and $\gamma(Q_{11}) = 5$.

Suppose then that n is a positive integer such that $\gamma(Q_n) = (n-1)/2$. Then there is a positive integer i such that $n = n_i$, with n_i as defined in (7). We continue the examination of the four cases considered earlier.

If $i \equiv 1 \pmod 4$ then only $d - e - 1$ occurs in $S_{\max}$ with multiplicity less than 4. By Lemma 2, $d - e - 1 = 0$, which by (6) occurs only when $e = 0$, $d = 1$, and $n = 3$.

If $i \equiv 2 \pmod 4$ then only $e - 1$ occurs in $S_{\max}$ with multiplicity less than 4. By Lemma 2, $e - 1 = 0$, so $e = 1$, $d = 5$, and $n = 11$.

If $i \equiv 3 \pmod 4$ then only $d - e + 1$ occurs in $S_{\max}$ with multiplicity less than 4. By Lemma 2, $d - e + 1 = 0$. However, (6) implies $d > e$, so this case does not occur.

If $i \equiv 0 \pmod 4$ then only $e + 1$ occurs in $S_{\max}$ with multiplicity less than 4. By Lemma 2, $e + 1 = 0$, but by definition $e \geq 0$, so this case does not occur. □

From (1) and Theorem 2, we have the following.

Corollary 4 *For all positive integers n other than 3 and 11, $\gamma(Q_n) \geq \lceil n/2 \rceil$.*

There is evidence that this lower bound is quite good. Work from [2, 4, 6, 9, 10, 13, 14] reported in [10] shows that for n from 1 to 120, excluding 3 and 11, we have $\lceil n/2 \rceil \leq \gamma(Q_n) \leq i(Q_n) \leq \lceil n/2 \rceil + 1$. In this range, $\gamma(Q_n) = \lceil n/2 \rceil$ is known for 46 values of n and $\gamma(Q_n) = \lceil n/2 \rceil + 1$ is known for $n = 8, 14, 15, 16$. See [10] for more information on specific values of $\gamma(Q_n)$ and $i(Q_n)$, and many dominating sets.

References

[1] W. Ahrens, *Mathematische Unterhaltungen und Spiele* (B.G. Teubner, Leipzig-Berlin, 1910).

[2] A. P. Burger and C. M. Mynhardt, Symmetry and domination in queens graphs, *Bull. Inst. Combin. Appl.* **29** (2000), 11–24.

[3] A. P. Burger and C. M. Mynhardt, Properties of dominating sets of the queen's graph Q_{4k+3}, *Utilitas Math.* **57** (2000), 237–253.

[4] A. P. Burger, C. M. Mynhardt and E. J. Cockayne, Domination numbers for the queen's graph, *Bull. Inst. Combin. Appl.* **10** (1994), 73–82.

[5] E. J. Cockayne, Chessboard domination problems, *Discrete Math.* **86** (1990), 13–20.

[6] P. B. Gibbons and J. Webb, Some new results for the queens domination problem, *Australas. J. Combin.* **15** (1997), 145–160.

[7] R. K. Guy, *Unsolved Problems in Number Theory, 2nd Edition* (Springer-Verlag, New York, 1994).

[8] C. F. de Jaenisch, *Traité des Applications de l'Analyse Mathématique au Jeu des Échecs*, Appendix, pages 244 ff (Imperial St. Petersburg Academy of Sciences, St. Petersburg, 1862).

[9] M. D. Kearse and P. B. Gibbons, Computational methods and new results for chessboard problems, *Australas. J. Combin.* **23** (2001), 253–284.

[10] P. R. J. Östergård and W. D. Weakley, Values of Domination Numbers of the Queen's Graph, *Electron. J. Combin.* **8** (2001), no. 1, Research Paper 29, 19 pp. (electronic).

[11] V. Raghavan and S. M. Venketesan, On bounds for a covering problem, *Inform. Process. Lett.* **25** (1987), 281–284.

[12] W. W. Rouse Ball, *Mathematical Recreations & Essays, 3rd Edition* (revision by H. S. M. Coxeter of the original 1892 edition), Chapter 6, Minimum pieces problem (Macmillan, London, 1939).

[13] W. D. Weakley, Domination in the queen's graph, in: *Graph Theory, Combinatorics, and Algorithms, Vol. 2 (Kalamazoo, MI 1992)*, Y. Alavi and A. Schwenk (Editors), 1223–1232 (Wiley, New York, 1995).

[14] W. D. Weakley, A lower bound for domination numbers of the queen's graph, *J. Combin. Math. Combin. Comput.* **43**(2002), 231–254.

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