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Simple acyclic graphoidal covers in a graph*

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Abstract

A simple acyclic graphoidal cover of a graph G is a collection ψ of paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ, every edge of G is in exactly one path in ψ and any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple acyclic graphoidal cover of G is called the simple acyclic graphoidal covering number of G and is denoted by $\eta_{as}(G)$ or simply η_{as}. In this paper we determine the value of η_{as} for several families of graphs. We also obtain several bounds for η_{as} and characterize graphs attaining the bounds.

1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [6]. All graphs in this paper are assumed to be connected and non-trivial.

If $P = (v_0, v_1, v_2, \ldots, v_n)$ is a path or a cycle in a graph G, then $v_1, v_2, \ldots, v_{n-1}$ are called internal vertices of P and v_0, v_n are called external vertices of P. If $P = (v_0, v_1, v_2, \ldots, v_n)$ and $Q = (v_n = w_0, w_1, w_2, \ldots, w_m)$ are two paths in G, then the walk obtained by concatenating P and Q at v_n is denoted by $P \circ Q$ and the path $(v_n, v_{n-1}, \ldots, v_2, v_1, v_0)$ is denoted by P^{-1}. For any subset V_1 of V, the subgraph of G induced by V_1 is denoted by $\langle V_1 \rangle$. For a unicyclic graph G with cycle C, if w is a vertex of degree greater than 2 on C with $deg\ w = k$, let $e_1, e_2 \ldots, e_{k-2}$ be the edges of $E(G) - E(C)$ incident with w. Let T_i, $1 \leq i \leq k-2$, be the maximal subtree of G such that T_i contains the edge e_i and w is a pendant vertex of T_i. Then $T_1, T_2, \ldots, T_{k-2}$ are called the *branches* of G at w. Also the maximal subtree T of G such that $V(T) \cap V(C) = \{w\}$ is called the *subtree* rooted at w.

The concepts of graphoidal cover and acyclic graphoidal cover were introduced by Acharya et al. [1] and Arumugam et al. [4].

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Definition 1.1. [1] *A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions.*

(i) Every path in ψ has at least two vertices.

(ii) Every vertex of G is an internal vertex of at most one path in ψ.

(iii) Every edge of G is in exactly one path in ψ.

If further no member of ψ is a cycle in G, then ψ is called an acyclic graphoidal cover of G. The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$. Similarly we define the acyclic graphoidal covering number $\eta_a(G)$.

An elaborate review of results in graphoidal covers with several interesting applications and a large collection of unsolved problems is given in Arumugam et al. [2]. Pakkiam and Arumugam [7, 8] determined the graphoidal covering number of several families of graphs.

Theorem 1.2. [7] *Let T be a tree with n pendant vertices. Then $\eta(T) = n - 1$.*

Definition 1.3. *Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to be an interior vertex of ψ if it is an internal vertex of some path in ψ, otherwise it is said to be an exterior vertex of ψ.*

Theorem 1.4. [8] *For any graphoidal cover ψ of G, let t_ψ denote the number of exterior vertices of ψ. Let $t = \min t_\psi$, where the minimum is taken over all graphoidal covers ψ of G. Then $\eta = q - p + t$.*

Corollary 1.5. *For any graph G, $\eta \geq q - p$. Moreover, the following are equivalent.*

(i) $\eta = q - p$.

(ii) There exists a graphoidal cover without exterior vertices.

(iii) There exists a set of internally disjoint and edge disjoint paths without exterior vertices. (From such a set of paths required graphoidal cover can be obtained by adding the edges which are not covered by the paths of this set.)

Corollary 1.6. *If there exists a graphoidal cover ψ of G such that every vertex v of G with $\deg v > 1$ is an internal vertex of a path in ψ, then ψ is a minimum graphoidal cover of G and $\eta(G) = q - p + n$, where n is the number of pendant vertices of G.*

Remark 1.7. *It has been proved in [4] that the results analogous to Theorems 1.2, 1.4, Corollaries 1.5 and 1.6 are true for the acyclic graphoidal covering number η_a also.*

Theorem 1.8. [4] *For any graph G with $\delta \geq 3$, we have $\eta_a = q - p$.*

Theorem 1.9. [4] *Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and m denote the number of vertices of degree greater than 2 on C. Then*

$$\eta_a(G) = \begin{cases} 2 & \text{if } m = 0 \\ n+1 & \text{if } m = 1 \\ n & \text{otherwise.} \end{cases}$$

Theorem 1.10. [4] *$\eta_a(K_{2,n}) = n-1$ if $n \geq 3$ and $\eta_a(K_{m,n}) = mn - m - n$ if $m, n > 2$.*

Remark 1.11. [4] *For any acyclic graphoidal cover ψ of G, $|\psi| \geq \Delta - 1$ and hence $\eta_a \geq \Delta - 1$.*

Definition 1.12. *A family $\{A_i : i \in I\}$ of subsets of a set A is said to satisfy the Helly property if whenever $J \subseteq I$ and $A_i \cap A_j \neq \phi$ for every $i, j \in J$, then $\bigcap_{j \in J} A_j \neq \phi$.*

Theorem 1.13. ([5], page 80) *Every family of subtrees of a tree satisfies the Helly property.*

If $G = (V, E)$ is a graph, then $\psi = E(G)$ is trivially a graphoidal cover and has the interesting property that any two paths in ψ have at most one vertex in common. Motivated by this observation we introduced the concept of simple graphoidal covers in a graph [3].

Definition 1.14. [3] *A simple graphoidal cover of a graph G is a graphoidal cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple graphoidal cover of G is called the simple graphoidal covering number of G and is denoted by $\eta_s(G)$ or simply η_s.*

In this paper we introduce the concept of simple acyclic graphoidal cover and simple acyclic graphoidal covering number η_{as} of a graph G and initiate a study of this parameter.

3 Main Results

Definition 3.1. *A simple acyclic graphoidal cover of a graph G is an acyclic graphoidal cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple acyclic graphoidal cover of G is called the simple acyclic graphoidal covering number of G and is denoted by $\eta_{as}(G)$ or simply η_{as}.*

Remark 3.2. *We observe that every path in a simple acyclic graphoidal cover of a graph is an induced path. Hence $\eta_{as}(G) = q$ if and only if G is complete.*

We first prove that results analogous to Theorem 1.4 and its Corollaries 1.5 and 1.6 are true for η_{as} as well.

Theorem 3.3. *For any simple acyclic graphoidal cover ψ of a graph G, let t_ψ denote the number of exterior vertices of ψ. Let $t = min\ t_\psi$, where the minimum is taken over all simple acyclic graphoidal covers ψ of G. Then $\eta_{as}(G) = q - p + t$.*

Proof. For any simple acyclic graphoidal cover ψ of G, we have

$$\begin{aligned} q &= \sum_{P\in\psi} |E(P)| \\ &= \sum_{P\in\psi} (t(P)+1) \quad (t(P) \text{ denotes the number of internal vertices of } P) \\ &= \sum_{P\in\psi} t(P) + |\psi| \\ &= p - t_\psi + |\psi|. \end{aligned}$$

Hence $|\psi| = q - p + t_\psi$ so that $\eta_{as}(G) = q - p + t$. □

Corollary 3.4. *For any graph G, $\eta_{as}(G) \geq q - p$. Moreover, the following are equivalent.*

(*i*) $\eta_{as}(G) = q - p$.

(*ii*) *There exists a simple acyclic graphoidal cover of G without exterior vertices.*

(*iii*) *There exists a set $\mathcal{P}$ of internally disjoint and edge disjoint induced paths without exterior vertices such that any two paths in $\mathcal{P}$ have at most one vertex in common. (From such a set $\mathcal{P}$ of paths the required simple acyclic graphoidal cover can be obtained by adding the edges which are not covered by the paths in $\mathcal{P}$).*

Corollary 3.5. *If there exists a simple acyclic graphoidal cover ψ of a graph G such that every vertex of G with degree at least two is interior to ψ, then ψ is a minimum simple acyclic graphoidal cover of G and $\eta_{as}(G) = q - p + n$, where n is the number of pendant vertices of G.*

Obviously for any tree T, we have $\eta = \eta_a = \eta_s = \eta_{as} = n - 1$, where n is the number of pendant vertices of T. Also there exist graphs which are not trees for which the above equations are valid as shown in the following lemma.

Lemma 3.6. *Let G be a graph of order p and size q. Then $\eta(G') = \eta_a(G') = \eta_s(G') = \eta_{as}(G') = p + q$, where G' is the graph obtained from G by attaching two pendant edges to every vertex of G.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$.

Let u_i and $w_i, 1 \leq i \leq p$, be the pendant vertices of G' adjacent to v_i. Then $\psi = \{(u_i, v_i, w_i) : 1 \leq i \leq p\} \cup E(G)$ is a graphoidal cover of G' which is simple as well as acyclic and every vertex of degree greater than 1 is interior to ψ. Hence $\eta(G') = \eta_a(G') = \eta_s(G') = \eta_{as}(G') = |\psi| = p + q$. □

The above lemma leads to the following problem.

Problem 3.7. *Characterize the class of graphs for which* $\eta = \eta_a = \eta_s = \eta_{as}$.

In the following theorems we determine the value of η_{as} for unicyclic graphs, wheels and complete bipartite graphs.

Theorem 3.8. *Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and let m denote the number of vertices of degree greater than 2 on C. Then*

$$\eta_{as}(G) = \begin{cases} 3 & \text{if } m = 0 \\ n+2 & \text{if } m = 1 \\ n+1 & \text{if } m = 2 \\ n & \text{if } m \geq 3 \end{cases}$$

Proof. Let $C = (v_1, v_2, \ldots, v_k, v_1)$.
Case 1. $m = 0$

Then $G = C$ and $\eta_{as}(G) = 3$.
Case 2. $m = 1$.

Let v_1 be the unique vertex of degree greater than 2 on C. Let $T_i, 1 \leq i \leq (deg\ v_1) - 2$, be the branches of G at v_1. Let ψ_i, be a minimum simple acyclic graphoidal cover of the branch T_i. Let P_1 be the path in ψ_1 having v_1 as a terminal vertex. Let

$Q_1 = P_1 \circ (v_1, v_2)$
$Q_2 = (v_2, v_3, \ldots, v_k)$ and
$Q_3 = (v_k, v_1)$. Then

$$\psi = \left\{ \left(\bigcup_{i=1}^{(deg\ v_1)-2} \psi_i \right) - \{P_1\} \right\} \cup \{Q_1, Q_2, Q_3\}$$

is a simple acyclic graphoidal cover of G and the number of vertices exterior to ψ is $n + 2$. Hence $\eta_{as}(G) \leq n + 2$. Further, for any simple acyclic graphoidal cover ψ of G, the n pendant vertices of G and at least two vertices on C are exterior to ψ so that $t \geq n + 2$. Hence $\eta_{as}(G) \geq n + 2$. Thus $\eta_{as}(G) = n + 2$.
Case 3. $m = 2$.

Let v_1 and v_r, where $1 < r \leq k$, be the vertices of degree greater than 2 on C. Let S_1 and S_2 denote respectively the (v_1, v_r)-section and (v_r, v_1)-section of the cycle C and let v_s be an internal vertex of S_1(say). Let R_1 and R_2 denote the (v_1, v_s)-section of S_1 and (v_s, v_r)-section of S_1 respectively. Let ψ_i and ψ'_j, where $1 \leq i \leq (deg\, v_1) - 2$, $1 \leq j \leq (deg\ v_r) - 2$, be minimum simple acyclic graphoidal covers of the branches T_i and T'_j of G at v_1 and v_r respectively . Let P_1 and P'_1 denote respectively the paths in ψ_1 and ψ'_1 having the vertices v_1 and v_r as terminal vertices. Let

$Q_1 = P_1 \circ R_1$
$Q_2 = P'_1 \circ R_2^{-1}$ and
$Q_3 = S_2$. Then

$$\psi = \left\{ \left(\bigcup_{i=1}^{(deg\ v_1)-2} \psi_i \right) \cup \left(\bigcup_{j=1}^{(deg\ v_r)-2} \psi'_j \right) - \{P_1, P'_1\} \right\} \cup \{Q_1, Q_2, Q_3\}$$

is a simple acyclic graphoidal cover of G and the number of vertices exterior to ψ is $n+1$. Hence $\eta_{as}(G) \leq n+1$. Further, for any simple acyclic graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex on C are exterior to ψ so that $t \geq n+1$. Hence $\eta_{as}(G) \geq n+1$. Thus $\eta_{as}(G) = n+1$.

Case 4. $m \geq 3$.

Let $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$, where $1 \leq i_1 < i_2 < \ldots, < i_r \leq k$, be the vertices of degree greater than 2 on C. Let $\psi_{j_s}, 1 \leq j \leq r$ and $1 \leq s \leq (deg\ v_{i_j}) - 2$, be minimum simple acyclic graphoidal covers of the branches T_{j_s} of G at v_{i_j}. Let P_1, P_2 and P_3 respectively denote the paths in ψ_{1_1}, ψ_{2_1} and ψ_{3_1} having v_{i_1}, v_{i_2} and v_{i_3} as terminal vertices. Let

$Q_1 = P_1 \circ (v_{i_1}, v_{i_1+1}, \ldots, v_{i_2})$
$Q_2 = P_2 \circ (v_{i_2}, v_{i_2+1}, \ldots, v_{i_3})$ and
$Q_3 = P_3 \circ (v_{i_3}, v_{i_3+1}, \ldots, v_{i_1})$. Then

$$\psi = \left\{ \left(\bigcup_{j=1}^{r} \left(\bigcup_{s=1}^{(deg\ v_{i_j})-2} \psi_{j_s} \right) \right) - \{P_1, P_2, P_3\} \right\} \cup \{Q_1, Q_2, Q_3\}$$

is a simple acyclic graphoidal cover of G such that every vertex of degree greater than 1 is interior to ψ and hence $\eta_{as}(G) = n$. □

Corollary 3.9. *Let G be as in Theorem 2.8. Then $\eta_{as}(G) = \eta_a(G)$ if and only if $m \geq 3$.*

Proof. Follows from Theorem 1.9 and Theorem 2.8. □

Theorem 3.10. *For the wheel $W_n = K_1 + C_{n-1}$, we have*

$$\eta_{as}(W_n) = \begin{cases} 6 & \text{if } n = 4 \\ n+1 & \text{if } n \geq 5 \end{cases}$$

Proof. Let $V(W_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0v_i : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-2\} \cup \{v_1v_{n-1}\}$.

If $n = 4$, then $W_n = K_4$ and hence $\eta_{as}(W_n) = 6$. Now, suppose $n \geq 5$. Let $P_1 = (v_1, v_2, \ldots, v_{n-2})$ and $P_2 = (v_{n-3}, v_0, v_{n-1})$. Then $\psi = \{P_1, P_2\} \cup S$, where S is the set of edges of W_n not covered by P_1 and P_2 is a simple acyclic graphoidal cover of W_n and $|\psi| = n+1$. Hence $\eta_{as}(G) \leq n+1$. Further, for any simple acyclic graphoidal cover ψ of W_n, at least three vertices on $C = (v_1, v_2, \ldots, v_{n-1}, v_1)$ are exterior to ψ so that $t \geq 3$. Hence $\eta_{as}(W_n) \geq q - p + 3 = n+1$. Thus $\eta_{as}(W_n) = n+1$. □

Corollary 3.11. $\eta_{as}(W_n) \neq \eta_a(W_n)$ *for all* $n \geq 4$.

Proof. It follows from Theorem 1.8 that $\eta_a(W_n) = q - p = n-2$ and hence $\eta_{as}(W_n) \neq \eta_a(W_n)$. □

Theorem 3.12.

(*i*) $\eta_{as}(K_{1,n}) = n-1$, *for all* $n \geq 2$.

(ii)

$$\eta_{as}(K_{2,n}) = \begin{cases} 3 & \text{if } n = 2 \\ 4 & \text{if } n = 3 \\ 2n-3 & \text{if } n \geq 4 \end{cases}$$

(iii)

$$\eta_{as}(K_{3,n}) = \begin{cases} 5 & \text{if } n = 3 \\ 3(n-2) & \text{if } n \geq 4 \end{cases}$$

(iv) Let m and n be integers with $n \geq m \geq 4$. Then

$$\eta_{as}(K_{m,n}) = \begin{cases} mn - m - n & \text{if } n \leq \binom{m}{2} \\ mn - m - n + r & \text{if } n = \binom{m}{2} + r, r > 0 \end{cases}$$

Proof. We observe that, for any simple acyclic graphoidal cover ψ of $K_{m,n}$ any path in ψ is either a path of length 2 or an edge.
(i) Since $K_{1,n}$ is a tree with n pendant vertices, we have $\eta_{as}(K_{1,n}) = n - 1$.

(ii) Since $K_{2,2} = C_4$, we have $\eta_{as}(K_{2,2}) = 3$.

Now, let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of $K_{2,n}$.

If $n = 3$, then $\psi = \{(y_1, x_1, y_2), (y_1, x_2, y_3), (y_2, x_2), (y_3, x_1)\}$ is a simple acyclic graphoidal cover of $K_{2,3}$, so that $\eta_{as}(K_{2,3}) \leq 4$. Further, for any simple acyclic graphoidal cover ψ of $K_{2,3}$, the number of vertices interior to ψ is at most 2 so that $t \geq 3$. Hence $\eta_{as}(K_{2,3}) = q - p + 3 = 4$.

Now, suppose $n \geq 4$. Let $P_1 = (x_1, y_1, x_2), P_2 = (y_2, x_1, y_3)$ and $P_3 = (y_2, x_2, y_4)$. Then $\psi = \{P_1, P_2, P_3\} \cup S$, where S is the set of edges of $K_{2,n}$ not covered by P_1, P_2 and P_3 is a simple acyclic graphoidal cover of $K_{2,n}$ and $|\psi| = 2n - 3$. Hence $\eta_{as}(K_{2,n}) \leq 2n-3$. Further, for any simple acyclic graphoidal cover ψ of $K_{2,n}$ at most one vertex in Y is interior to ψ so that $t \geq n - 1$. Hence $\eta_{as}(K_{2,n}) \geq q - p + n - 1 = 2n - 3$. Thus $\eta_{as}(K_{2,n}) = 2n - 3$.

(iii) By a similar argument we can prove that $\eta_{as}(K_{3,3}) = q - p + 2 = 5$ and $\eta_{as}(K_{3,n}) = q - p + (n - 3) = 3(n - 2)$ for all $n \geq 4$.

(iv) Let m and n be integers with $n \geq m \geq 4$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of $K_{m,n}$.

If $m = n = 4$, then $\psi = \{(x_1, y_1, x_2), (x_1, y_2, x_3), (x_2, y_3, x_4), (x_3, y_4, x_4), (y_3, x_1, y_4), (y_2, x_2, x_4), (y_1, x_3, y_3), (y_1, x_4, y_2)\}$ is a simple acyclic graphoidal cover of $K_{4,4}$ without exterior vertices and hence $\eta_{as}(K_{4,4}) = q - p = 8$.

Suppose $m \geq 4$ and $n \geq 5$.

Case 1. $n \leq \binom{m}{2}$.

Let $J = \{\{i, j\} : 1 \leq i < j \leq m\}$. Clearly $|J| = \binom{m}{2}$. We define a relation " $<$ " on J by $(i, j) < (k, l)$ if either $i < k$ or $i = k$ and $j < l$. We now index the elements of Y by the set I of the first n elements of J. Thus $Y = \{y_{\{i,j\}} : \{i, j\} \in I\}$.
Let $P_{i,j} = (x_i, y_{\{i,j\}}, x_j)$, for all $\{i, j\} \in I$.

$Q_1 = (y_{\{2,3\}}, x_1, y_{\{2,4\}})$

$Q_2 = (y_{\{1,3\}}, x_2, y_{\{1,4\}})$
$Q_i = (y_{\{1,2\}}, x_i, y_{\{1,i+1\}})$, for all i, where $3 \leq i \leq m-1$.
$Q_m = (y_{\{1,2\}}, x_m, y_{\{2,3\}})$. Then
$\psi \ = \{P_{i,j} : 1 \leq i \leq m-1, 1 \leq j \leq m \text{ and } i < j\} \cup \{Q_1, Q_2, \ldots, Q_m\} \cup S$ (1)
where S is the set of edges not covered by any path $P_{i,j}$ or $\{Q_1, Q_2, \ldots, Q_m\}$ is a simple acyclic graphoidal cover of $K_{m,n}$ without exterior vertices. Hence $\eta_{as}(K_{m,n}) = q - p = mn - m - n$.

Case 2. $n > \binom{m}{2}$.

Let $n = \binom{m}{2} + r$, where $r > 0$.

Let $Y = \{y_{\{i,j\}} : \{i,j\} \in J\} \cup \{z_1, z_2, \ldots, z_r\}$.

Then the collection ψ given in (1) with $I = J$ is a simple acyclic graphoidal cover of $K_{m,n}$ in which the vertices $z_1, z_2, \ldots, z_r$ are exterior to ψ. Hence $\eta_{as}(K_{m,n}) \leq q-p+r = mn-m-n+r$. Further, for any simple acyclic graphoidal cover ψ of $K_{m,n}$, at least r vertices of Y are exterior to ψ so that $\eta_{as}(K_{m,n}) \geq q-p+r = mn-m-n+r$.

Hence $\eta_{as}(K_{m,n}) = mn - m - n + r$. □

Corollary 3.13. *Let $1 \leq m \leq n$. Then $\eta_{as}(K_{m,n}) = \eta_a(K_{m,n})$ if and only if $m = 1$ and $n \geq 1$ or $m \geq 4$ and $n \leq \binom{m}{2}$.*

Proof. Follows from Theorem 1.10 and Theorem 2.12. □

Corollary 3.14. *$\eta_{as}(K_{m,n}) = q - p$ if and only if $m \geq 4$ and $m \leq n \leq \binom{m}{2}$.*

For any graph G, $\eta_{as} \geq q - p$. The above corollary gives an infinite family of graphs for which this bound is attained. Hence we have the following.

Problem 3.15. *Characterize graphs for which $\eta_{as} = q - p$.*

We now proceed to obtain bounds for η_{as} and characterize graphs attaining the bounds.

Theorem 3.16. *Let G be a graph with diameter d. Then $\eta_{as}(G) \leq q - d + 1$. Further, equality holds if and only if for any diameter path $P = (u = v_1, v_2, \ldots, v_{d+1} = v)$ the following are satisfied.*

1. *Any two neighbours of each of u and v not on P are adjacent.*
2. *For any vertex w not on P,*
   (i) *$d(w, P) = 1$.*
   (ii) *$|N(w) \cap V(P)| \leq 3$.*
   (iii) *If $N(w) \cap V(P) = \{v_i, v_j, v_k\}$, where $i < j < k$, then $j = i + 1$ and $k = i + 2$.*
   (iv) *If $N(w) \cap V(P) = \{v_i, v_j\}$, where $i < j$, then $j = i + 1$ or $j = i + 2$.*
3. *Every component of $\langle V(G) - V(P)\rangle$ is complete.*
4. *If x and y are two adjacent vertices not on P, then $N(x) \cap V(P) = N(y) \cap V(P)$.*

5. *Suppose x and y are two non-adjacent vertices not on P. Then*

   (i) *If $N(x) \cap V(P) = \{v_i, v_{i+2}\}$ or $\{v_i, v_{i+1}, v_{i+2}\}$ and $N(y) \cap V(P) = \{v_j\}$ with $i \leq j$, then $j \neq i+1$.*

   (ii) *If $N(x) \cap V(P) = \{v_i, v_{i+2}\}$ or $\{v_i, v_{i+1}, v_{i+2}\}$ and $N(y) \cap V(P) = \{v_j, v_{j+1}\}$ with $i \leq j$, then $j \geq i+2$.*

   (iii) *If $N(x) \cap V(P) = \{v_i, v_{i+2}\}$ or $\{v_i, v_{i+1}, v_{i+2}\}$ and $N(y) \cap V(P) = \{v_j, v_{j+2}\}$ with $i \leq j$, then $j \neq i+1$.*

   (iv) *If $N(x) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\}$ and $N(y) \cap V(P) = \{v_j, v_{j+1}, v_{j+2}\}$ with $i \leq j$, then $j \geq i+2$.*

Proof. Let u and v be two vertices in G with $d(u, v) = d$ and let P be a shortest u-v path in G . Then $\psi = \{P\} \cup (E(G) - E(P))$ is a simple acyclic graphoidal cover of G and $|\psi| = q - d + 1$. Hence $\eta_{as}(G) \leq q - d + 1$.

Now, let G be a graph with diameter d and $\eta_{as}(G) = q-d+1$. Let $P = (v_1, v_2, \ldots, v_{d+1})$ be a diameter path in G.

Suppose there exists a vertex w not on P such that $d(w, P) \geq 2$. Let $P_1 = (v_i, u_1, u_2, \ldots, u_n = w)$, where $1 \leq i \leq d+1$ and $n \geq 2$ be a shortest v_i-w path in G. Then $\psi = \{P, P_1\} \cup S$, where S is the set of edges of G not covered by P and P_1 is a simple acyclic graphoidal cover of G such that $|\psi| < q - d + 1$, which is a contradiction. Thus $d(w, P) = 1$. This proves 2(i) of the theorem. Since P is a diameter path, conditions 2(ii), 2(iii) and 2(iv) follow immediately.

We now prove condition (1) of the theorem. Suppose there exist two non-adjacent vertices x and y not on P which are adjacent to u. Then $Q = (x, u, y)$ is an induced path in G with $|V(Q) \cap V(P)| = 1$. Now, $\psi = \{P, Q\} \cup S$, where S is the set of edges of G not covered by P and Q is a simple acyclic graphoidal cover of G such that $|\psi| < q - d + 1$, which is a contradiction. Hence any two neighbours of u not on P are adjacent. Similarly, any two neighbours of v not on P are adjacent. This proves condition (1) of the theorem.

We now prove (3). Suppose there exists a component H of $\langle V(G) - V(P)\rangle$ having two non-adjacent vertices, say x and y. Let Q be a shortest x-y path in H. Then $\psi = \{P, Q\} \cup S$, where S is the set of edges of G not covered by P and Q is a simple acyclic graphoidal cover of G such that $|\psi| < q - d + 1$, which is a contradiction. Hence every component of $\langle V(G) - V(P)\rangle$ is complete. This proves condition (3) of the theorem.

Now, let x and y be two adjacent vertices not on P. We claim that $N(x) \cap V(P) = N(y) \cap V(P)$. Suppose there exists a vertex v_i on P such that $v_i \in N(x) - N(y)$. Then $\psi = \{P, Q = (v_i, x, y)\} \cup S$, where S is the set of edges of G not covered by P and Q is a simple acyclic graphoidal cover of G such that $|\psi| < q - d + 1$, which is a contradiction. Hence $N(x) \cap V(P) \subseteq N(y) \cap V(P)$. Similarly, $N(y) \cap V(P) \subseteq N(x) \cap V(P)$. Thus $N(x) \cap V(P) = N(y) \cap V(P)$. This proves condition (4) of the theorem.

Now, let x and y be two non-adjacent vertices not on P. Suppose $N(x) \cap V(P) = \{v_i, v_{i+2}\}$ or $\{v_i, v_{i+1}, v_{i+2}\}$. If $N(y) \cap V(P) = \{v_j\}$ with $i \leq j$ and $j = i+1$, let

$P_1 = (u = v_1, v_2, \ldots, v_i, v_{i+1}, y)$, $P_2 = (v_{i+1}, v_{i+2}, \ldots, v_{d+1} = v)$ and $P_3 = (v_i, x, v_{i+2})$. Then $\psi = \{P_1, P_2, P_3\} \cup S$, where S is the set of edges of G not covered by P_1, P_2 and P_3 is a simple acyclic graphoidal cover of G and $|\psi| = q - d$, which is a contradiction. Hence $j \neq i + 1$. This proves condition 5(i) of the theorem. By a similar argument, conditions 5(ii), 5(iii) and 5(iv) can be easily proved.

Conversely, suppose conditions (1),(2),(3),(4) and (5) of the theorem are satisfied for any diameter path $P = (u = v_1, \ldots, v_{d+1} = v)$. Let ψ be a minimum simple acyclic graphoidal cover of G.

Case 1. $P \in \psi$.

We claim that every vertex not on P is exterior to ψ. Let w be a vertex not on P. Let H be the component of $\langle V(G) - V(P) \rangle$ containing the vertex w. If $H = K_1$, then $N(w) \subseteq V(P)$ and hence w is exterior to ψ. If $|V(H)| \geq 2$, then it follows from conditions (3) and (4) that any path having w as an internal vertex is not an induced path and hence w is exterior to ψ. Thus every vertex not on P is exterior to ψ. Hence the number of vertices interior to ψ is exactly $d-1$ so that $t = p-(d-1) = p-d+1$. Thus $\eta_{as}(G) = q - d + 1$.

Case 2. $P \notin \psi$.

We claim that if there exists a vertex x not on P which is interior to ψ, then there exists a vertex v_j on P, where $2 \leq j \leq d$, which is exterior to ψ. Let Q be the path in ψ having x as an internal vertex. Then the two neighbours of x which are on Q are of the form $\{v_i, v_{i+2}\}$, for some i, where $1 \leq i \leq d - 1$. We now claim that the vertex v_{i+1} is exterior to ψ. This is obvious if $deg\ v_{i+1} = 2$. Let $deg\ v_{i+1} \geq 3$. We now consider the following cases.

Subcase 2.1. $|N(x) \cap V(P)| = 2$.

Then $N(x) \cap V(P) = \{v_i, v_{i+2}\}$. Let y be a vertex not on P which is adjacent to v_{i+1}. Now by condition (4), the vertices x and y are not adjacent. Also it follows from conditions 5(i)-5(iii) and the condition 2(ii) that $|N(y) \cap V(P)| = 3$. Now it follows from 2(iii) that $N(y) \cap V(P)$ is a set of three consecutive vertices of P. Since $v_{i+1} \in N(y) \cap V(P)$, it follows from 5(iii) that $N(y) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\}$. Thus for any two neighbours y and z of v_{i+1} not on P, we have $N(y) \cap V(P) = N(z) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\}$ and hence it follows from condition 5(iv) that y and z are adjacent. Hence any path having v_{i+1} as an internal vertex is not an induced path. Thus v_{i+1} is exterior to ψ.

Subcase 2.2. $|N(x) \cap V(P)| = 3$.

Then $N(x) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\}$. If $deg\ v_{i+1} = 3$, then v_{i+1} is exterior to ψ. Suppose $deg\ v_{i+1} \geq 4$. Let $y \neq x$ be a vertex not on P which is adjacent to v_{i+1}. It follows from conditions 5(i) to 5(iv) that the vertices x and y are adjacent and so by condition (4), we have $N(y) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\}$. Hence any path having v_{i+1} as an internal vertex is not an induced path. Thus v_{i+1} is exterior to ψ.

Thus for every vertex not on P which is interior to ψ, there exists a vertex v_j, where $2 \leq j \leq d$, on P which is exterior to ψ. Also it is clear that for any two distinct vertices not on P which are interior to ψ, their corresponding vertices on P

which are exterior to ψ are also distinct. Hence it follows from condition (1) that the number of vertices interior to ψ is at most $d-1$ so that $t \geq p-(d-1) = p-d+1$. Hence $\eta_{as}(G) \geq q-d+1$. Thus $\eta_{as}(G) = q-d+1$. □

Theorem 3.17. *For any graph G, $\eta_{as}(G) \geq \Delta - 1$. Further, equality holds if and only if G is homeomorphic to a star.*

Proof. Obviously $\eta_{as}(G) \geq \Delta - 1$. Suppose $\eta_{as}(G) = \Delta - 1$. Let $\psi = \{P_1, P_2, \ldots, P_{\Delta-1}\}$ be a minimum simple acyclic graphoidal cover of G. Let v be a vertex of G with *deg* $v = \Delta$. Then v is interior to ψ and v lies on each P_i. Since ψ is a simple acyclic graphoidal cover of G, we have $V(P_i) \cap V(P_j) = \{v\}$, for all $i \neq j$. Hence G is homeomorphic to a star. The converse is obvious. □

Theorem 3.18. *For any graph G, $\eta_{as}(G) \geq \binom{\omega}{2}$, where ω is the clique number of G. Further, if $\eta_{as}(G) = \binom{\omega}{2}$, then the following are satisfied.*

(i) There exists a unique maximum clique H in G.

(ii) If $v \in V(H)$, then deg $v = \omega$ or $\omega - 1$.

(iii) If $v \in V(G) - V(H)$, then deg $v \leq \lfloor \frac{\omega}{2} \rfloor + 1$.

Proof. Let H be a maximum clique in G so that $|E(H)| = \binom{\omega}{2}$. Let ψ be a simple acyclic graphoidal cover of G. Since any path in ψ covers at most one edge of H, it follows that $\eta_{as}(G) \geq \binom{\omega}{2}$.

Now, let G be a graph with $\eta_{as}(G) = \binom{\omega}{2}$. Let ψ be a minimum simple acyclic graphoidal cover of G.

Suppose there exists a vertex $v \in V(H)$ with *deg* $v > \omega$. Let x and y be two vertices not on H which are adjacent to v. Let P and Q be paths in ψ covering the edges xv and yv respectively. Since $\eta_{as}(G) = \binom{\omega}{2}$, each of the paths P and Q covers exactly one edge of H and both of them are induced paths. Hence it follows that $P \neq Q$ and v is interior to both P and Q, which is a contradiction. Hence *deg* $v = \omega$ or *deg* $v = \omega - 1$. This proves condition (ii) of the theorem.

Now, let $v \in V(G) - V(H)$. Since any path in ψ which contains v covers exactly two vertices of H and v is an internal vertex of at most one path in ψ, it follows that *deg* $v \leq \lfloor \frac{\omega}{2} \rfloor + 1$. This proves condition (iii) of the theorem. Now, it follows from (iii) that H is the unique maximum clique in G. □

We now proceed to investigate the structure of graphs which admit a (minimum) simple acyclic graphoidal cover satisfying the Helly property.

Theorem 3.19. *A graph G has a simple acyclic graphoidal cover satisfying the Helly property if and only if G is triangle-free.*

Proof. Suppose G is triangle-free. Then $\psi = E(G)$ is a simple acyclic graphoidal cover of G satisfying the Helly property.

Conversely, suppose G has a triangle, say $C = (u, v, w, u)$. Let ψ be any simple acyclic graphoidal cover of G. Then the edges uv, vw and uw lie on three different paths in ψ, say P_1, P_2 and P_3 respectively. Clearly $\{P_1, P_2, P_3\}$ is a pairwise intersecting family of paths in ψ. If there exists a vertex x which is common to the paths P_1, P_2 and P_3, then the vertices x and v are common to both P_1 and P_2, which is a contradiction. Hence $V(P_1) \cap V(P_2) \cap V(P_3) = \phi$. Thus ψ does not satisfy the Helly property. □

Theorem 3.20. *Every simple acyclic graphoidal cover of a graph G satisfies the Helly property if and only if G is a tree.*

Proof. Suppose G is a graph in which every simple acyclic graphoidal cover satisfies the Helly property. Suppose G contains a cycle, say $C = (v_1, v_2, \ldots, v_k, v_1)$, where $k \geq 3$. Let $P_1 = (v_1, v_2)$, $P_2 = (v_2, v_3, \ldots, v_k)$ and $P_3 = (v_k, v_1)$. Then $\psi = \{P_1, P_2, P_3\} \cup (E(G) - E(C))$ is a simple acyclic graphoidal cover of G. Clearly $\{P_1, P_2, P_3\}$ is pairwise intersecting family of paths in ψ, whereas there exists no vertex in G common to the paths P_1, P_2 and P_3. Hence ψ does not satisfy the Helly property, which is a contradiction. Hence G is a tree.

The converse follows from Theorem 1.13. □

We now construct some classes of graphs with a minimum simple acyclic graphoidal cover satisfying the Helly property.

Theorem 3.21. *Let C be a cycle of length greater than 3. Then the graph G obtained from C by attaching a pendant edge to every vertex of C has a minimum simple acyclic graphoidal cover satisfying the Helly property.*

Proof. Let $C = (v_1, v_2, \ldots, v_n, v_1)$, where $n \geq 4$. Let $u_1, u_2, \ldots, u_n$ be the pendant vertices of G which are adjacent to $v_1, v_2, \ldots, v_n$ respectively. Then $\psi = \{(u_i, v_i, v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_n, v_n, v_1)\}$ is a minimum simple acyclic graphoidal cover of G. Clearly any pairwise intersecting family of paths in ψ contains at most two paths and hence ψ satisfies the Helly property. □

Theorem 3.22. *Let G be a graph. Then the graph G' obtained from G by attaching two pendant edges to every vertex of G has a minimum simple acyclic graphoidal cover satisfying the Helly property.*

Proof. Let ψ be the minimum simple acyclic graphoidal cover of G' given in Lemma 2.6. Then any pairwise intersecting family of paths in ψ has at most two paths and hence ψ satisfies the Helly property. □

The above results lead to the following problems.

Problem 3.23. *Characterize graphs which admit a minimum simple acyclic graphoidal cover satisfying the Helly property.*

Problem 3.24. *Characterize graphs in which every minimum simple acyclic graphoidal cover satisfies the Helly property.*

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