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Some new near resolvable BIBDs with $k = 7$ and resolvable BIBDs with $k = 5$

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Abstract

The existence of $(n, 7, 6)$ near resolvable BIBDs has been established for all $v \equiv 1 \pmod 7$ apart from five possible cases: 183, 246, 267, 274, 295. In this paper we remove these last 5 possible exceptions. We also obtain two new resolvable (v, k, λ) BIBDs for $(v, k, \lambda) = (90, 5, 4)$ and $(225, 5, 1)$.

1 Introduction

A design is a pair $(X, \mathcal{B})$ where X denotes a set of points of finite cardinality, v, and $\mathcal{B}$ is a family of subsets of X. The cardinalities of the subsets are called the block sizes.

A (K, λ) *group divisible design* (or GDD) is a design in which X is partitioned into groups with cardinalities in G. The block sizes have cardinalities in K. The design also satisfies the condition that every pair of points from distinct groups is contained in λ blocks, whilst no block contains a pair of points from the same group. Such a GDD is said to have group type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ if it has u_i of a groups of size g_i for $1 \le i \le s$.

A *pairwise balanced design* (or (v, K, λ)-PBD) is a GDD on v points with all groups of size 1 and all block sizes in the set K. If $K = \{k\}$, such a design is called a (v, k, λ) balanced incomplete block design (or BIBD).

A *parallel class* in a design is a set of blocks containing each point once, and a partial parallel class is a set of blocks containing each point at most once. A design is called *resolvable* if its blocks can be partitioned into parallel classes, and *near resolvable* if they can be partitioned into partial parallel classes each missing just a single point. Also, a GDD is called a *frame* if its blocks can be partitioned into partial parallel classes each missing just the points of a single group.

A necessary condition for a $(v, k, k-1)$ near resolvable BIBD to exist is that $v \equiv 1 \pmod k$. For $k = 7$, it was shown in [3] and [7], that this condition is sufficient except possibly for five cases, $v = 183$, 246, 267, 274, 295. In this paper, we provide direct constructions for these five unknown $(v, 7, 6)$ near resolvable BIBDs and for two new resolvable BIBDs with $k = 5$, namely for $(v, k, \lambda) = (225, 5, 1)$ and $(90, 5, 4)$. For most of our designs with $k = 7$, we construct a $(7, 6)$ frame using difference methods and then use the following fill in theorem from [7] to obtain the required near resolvable BIBD. This construction is not given in its most general form but is quite sufficient for what we require in this paper.

Lemma 1.1. *If there exists a (k, λ) frame of type $(g_1, g_2, \ldots, g_n)$ and a near resolvable (g_i+1, k, λ) BIBD for $i = 1, 2, \ldots, n$, then there exists a near resolvable (v, k, λ) BIBD for $v = (\sum_{i=1}^{n} g_i) + 1$.*

There is also a similar fill in construction to obtain a resolvable BIBD from a frame:

Lemma 1.2. *Suppose there exist (1) a (k, λ) frame of type $(g_1, g_2, \ldots, g_n)$ (2) a resolvable $(g_i + k, k, \lambda)$ BIBD with one block repeated λ times for $i = 1, 2, \ldots, n-1$, and a resolvable $(g_n + k, k, \lambda)$ BIBD. Then there exists a resolvable (v, k, λ) BIBD for $v = (\sum_{i=1}^{n} g_i) + k$.*

2 $(v, 7, 6)$ Near resolvable BIBDs

Our first new near resolvable BIBD is a cyclic one:

Lemma 2.1. *There exists a near resolvable* $(183, 7, 6)$ *BIBD.*

Proof. Take the point set as Z_{183}. Multiply each block below by 58^i for $0 \le i \le 4$ (except the first block which remains invariant under this multiplication). These 26 blocks form a near parallel class missing the point 0; develop them all (mod 183).

$(61, 122, 1, 58, 70, 34, 142)$, $(4, 8, 15, 32, 54, 121, 167)$, $(7, 22, 31, 41, 42, 73, 148)$,
$(17, 37, 45, 77, 96, 114, 164)$, $(18, 43, 53, 158, 171, 172, 176)$, $(51, 60, 62, 69, 123, 127, 140)$. □

Lemma 2.2. *There exist near resolvable $(v, 7, 6)$ BIBDs for $v = 246, 267, 274, 295$.*

Proof. For $v = 246$, 274, we construct $(7, 6)$ frames of types $7^p 28^1$ for $p = 31$ and 35; the required near resolvable BIBDs can then be obtained by Lemma 1.1, noting that $(v, 7, 6)$ near resolvable BIBDs exist for $v = 8, 29$ [7].

For frames of types $7^p 28^1$ $(p = 31, 35)$, we take the point set as $Z_{7p} \cup \{\infty_1, \infty_2, \ldots, \infty_{28}\}$. One hole consists of the infinite points and the others are of the form $\{y, y + p, y + 2p, \ldots, y + 6p\}$ for $0 \le y \le p - 1$. The initial base blocks (which, together with their multiples, should be developed (mod $7p$)) are of three types. Multiply those of types 1 and 2 by 1 and -1 (mod $7p$) only. All points in any base block of type 1 are distinct (mod 7), therefore, adding $z, z + 7, z + 14, \ldots 7p + z - 7$ for any z $(0 \le z \le 6)$ produces a partial parallel class missing the infinite points. The base blocks of type 3 should be multiplied by $\pm 1, \pm w$, and $\pm w^2$ (where $w = 25$ for $p = 31$ and 116 for $p = 35$). The base blocks of types 2 and 3 (plus their multiples) form a partial parallel class missing the group $\{0, p, 2p, \ldots, 6p\}$; the 7 partial parallel classes missing this group are obtained by adding $0, p, 2p, \ldots, 6p$ to them. Finally when multiplying any base block by any value, each infinite point should be replaced by another not already used.

For $v = 295$, the construction is essentially the same except that here, we construct a frame of type $14^{19}28^1$ over $Z_{266} \cup \{\infty_1, \infty_2, \ldots, \infty_{28}\}$, and groups on the non-infinite points are $\{y, y+19, y+38, \ldots, y+247\}$ for $0 \le y \le 18$. Here the type 1 and type 2 blocks should be multiplied by 1, 113 and the type 3 blocks should be multiplied by w^i and $113w^i$ for $w = 11$ and $0 \le i \le 2$.

For $v = 267$, the construction is also similar; here we construct a frame of type $28^9 14^1$ over $(GF(4, x^2 = x+1) \times Z_{63}) \cup \{\infty_1, \infty_2, \ldots, \infty_{14}\}$. Here one group consists of the infinite points and the others are $GF(4) \times \{y, y+9, y+18, \ldots, y+54\}$ for $0 \le y \le 8$. This time, before developing blocks mod $(2^2, 63)$, the type 1 and type 2 blocks should be multiplied by $(1,1)$, and $(1,8)$ (mod $(2^2, 63)$), while the type 3 blocks should be multiplied by $(x^i, 4^i)$ and $(x^i, 8 \cdot 4^i)$ for $0 \le i \le 2$. As before, any block which is a multiple of a type 1 block generates 7 partial parallel classes missing the infinite points, while the blocks of types 2 and 3 (together with their multiples) form a partial parallel class missing the group $GF(4) \times \{0, 9, 18, \ldots, 54\}$.

$7^{31}28^1$:	Type 1:	$(0,1,25,191,33,174,10)$,	$(0,18,16,183,45,40,132)$,
	Type 2:	$(\infty_1,4,100,113,53,23,141)$,	$(\infty_3,11,58,148,111,171,152)$,
	Type 3:	$(73,77,107,118,184,196,198)$,	$(\infty_5,37,63,79,85,137,182)$,
		$(\infty_{11},9,48,75,81,96,190)$,	$(\infty_{17},32,70,87,143,158,165)$,
		$(\infty_{23},25,34,67,95,120,131)$.	

$7^{35}28^1$:	Type 1:	$(0,1,116,226,17,12,167)$,	$(0,92,137,212,66,61,216)$,
	Type 2:	$(49,16,141,186,68,48,178)$,	$(98,2,232,207,34,24,89)$,
		$(\infty_1,4,219,169,106,46,191)$,	$(\infty_3,11,51,36,8,193,93)$,
	Type 3:	$(42,56,66,73,75,95,142)$,	$(\infty_5,94,101,143,145,195,227)$,
		$(\infty_{11},6,81,87,127,154,228)$,	$(\infty_{17},20,74,77,79,113,235)$,
		$(\infty_{23},25,43,116,134,212,222)$.	

$14^{19}28^1$:	Type 1:	$(0,2,22,242,178,96,258)$,	$(0,6,66,194,151,65,183)$,
	Type 2:	$(\infty_1,23,253,123,48,262,222)$,	$(\infty_3,25,9,99,41,185,173)$,
	Type 3:	$(72,73,89,121,125,142,166)$,	$(22,35,44,94,108,197,238)$,
		$(\infty_5,16,17,37,60,134,194)$,	$(\infty_{11},29,43,50,63,189,264)$,
		$(\infty_{17},26,88,135,193,208,241)$,	$(\infty_{23},13,54,69,182,252,265)$.

$28^9 14^1$:	Type 1:	$((0,0),(0,2),(0,8),(0,32),(1,10),(x,40),(x+1,34))$,
	Type 2:	$(\infty_1,(1,25),(1,50),(x,37),(x,11),(x+1,22),(x+1,44))$,
		$((0,21),(1,17),(1,61),(x,5),(x,55),(x+1,20),(x+1,31))$,
	Type 3:	$((0,3),(0,35),(1,1),(1,7),(1,32),(1,42),(x+1,38))$,
		$((0,38),(0,55),(1,14),(1,30),(1,44),(x,33),(x,43))$,
		$((0,40),(1,39),(1,52),(x,8),(x,59),(x,60),(x+1,10))$,
		$((\infty_3,(0,1),(1,33),(1,43),(1,48),(x+1,5),(x+1,26))$,
		$((\infty_9,(0,37),(0,53),(0,60),(x,14),(x,47),(x+1,43))$.

□

Combining the results of this section with those from [3] and [7] we can now update the status of $(v, 7, 6)$ near resolvable BIBDs as follows:

Theorem 2.3. *The necessary conditions for existence of a* $(v, 7, 6)$ *near resolvable BIBD, i.e.* $v > 7$ *and* $v \equiv 1 \pmod 7$ *are sufficient.*

3 Some Resolvable BIBDs with $k = 5$

In this section we update the status of $(v, 5, \lambda)$ RBIBDs. First, we have two new direct constructions:

Lemma 3.1. *There exist a resolvable* $(225, 5, 1)$ *BIBD.*

Proof. We first construct a $(5, 1)$ frame of type 20^{11}. Its point set is $X = Z_{220}$, and groups are of the form $\{y, y + 11, y + 22, \ldots, y + 209\}$ for $0 \leq y \leq 10$. Now develop the following 10 blocks (mod 220):

$$\begin{array}{ll}
(1, 68, 113, 126, 6), & (4, 183, 96, 130, 32), \\
(3, 127, 9, 26, 18), & (1, 131, 156, 192, 52), \\
(9, 7, 27, 23, 175), & (3, 96, 193, 59, 13), \\
(5, 32, 147, 168, 8), & (9, 79, 40, 210, 72), \\
(4, 118, 177, 42, 189), & (5, 6, 54, 80, 183).
\end{array}$$

Note that the pair of base blocks in each row contains the 10 nonzero residues (mod 11) once; as a result, each pair generates 11 partial parallel classes, one missing each size 20 group of the frame. Filling in the groups of this frame using 5 extra points, a $(25, 5, 1)$ resolvable BIBD and Lemma 1.2 gives a resolvable $(225, 5, 1)$ BIBD. □

Our next construction is similar to the one in [2] for $v = 70$ (which was obtained from a resolvable $(5, 4)$-GDD of type 5^{14}):

Lemma 3.2. *There exists a resolvable* $(90, 5, 4)$ *BIBD.*

Proof. We construct a $(5, 4)$-RGDD of type 5^{18} over $Z_{85} \cup \{\infty_i : i = 1, 2, \ldots, 5\}$. The first 17 groups are of the form $\{x, x + 17, x + 34, x + 51, x + 68\}$ for $0 \leq x \leq 16$, and the 18th group consists of the infinite points. Develop (mod 85) the following base blocks which form a parallel class. Finally, form a $(5, 5, 4)$ BIBD on each group.

$$\begin{array}{llll}
(0, 36, 56, 57, 63), & (20, 22, 44, 53, 62), & (15, 38, 40, 51, 52), & (1, 16, 19, 28, 55), \\
(14, 64, 73, 79, 83), & (11, 31, 70, 71, 84), & (5, 30, 33, 43, 76), & (42, 46, 49, 67, 72), \\
(9, 17, 25, 29, 75), & (24, 32, 39, 69, 80), & (8, 27, 48, 58, 74), & (2, 4, 7, 37, 81), \\
(21, 35, 77, 78, 82), & (3, 13, 34, 66, \infty_1), & (23, 45, 47, 60, \infty_2), & (10, 26, 54, 59, \infty_3), \\
(6, 50, 61, 68, \infty_4), & (12, 18, 41, 65, \infty_5). & &
\end{array}$$

□

There are a couple of other recent updates. Kaski and Östergård [8] showed no $(15, 5, 4)$ resolvable BIBD exists, and Abel, Bennett and Ge [2] constructed a

$(70, 5, 4)$ RBIBD (as a resolvable $(70, 5, 1)$ perfect Mendelsohn design). Using these results, we can now update the existence results in Theorem 6.1 of [4] for resolvable $(v, 5, \lambda)$ BIBDs with $\lambda \in \{1, 2, 4\}$ as follows:

Theorem 3.3. *Necessary conditions for existence of a $(v, 5, \lambda)$ RBIBD are $\lambda(v-1) \equiv 0 \pmod 4$ and $v \equiv 0 \pmod 5$. For $\lambda = 1, 2, 4$ these conditions are sufficient except for $(v, \lambda) \in \{(10, 4), (15, 2), (15, 4)\}$ and possibly for the following cases:*

1. *$\lambda = 1$, and $v \in \{45, 345, 465, 645\}$;*
2. *$\lambda = 2$, and $v \in \{45, 115, 135, 195, 215, 235, 295, 315, 335, 345, 395\}$;*
3. *$\lambda = 4$, and $v \in \{135, 160, 190, 195\}$.*

For other indices, if $m = \gcd(4, \lambda)$ then we can construct the design in most cases by simply taking λ/m copies of a (v, k, m) RBIBD. This will fail for designs in our exception lists in Theorem 3.3, but some further results are known here for $v \in \{10, 15, 45\}$. First, we note that no $(10, 5, 4c)$ resolvable BIBD exists for c odd by the following theorem:

Theorem 3.4. 1. *If a $(2k, k, k-1)$ BIBD exists, then a $(2k, k, t(k-1))$ resolvable BIBD exists for any even value of t.*

2. *If k and t are both odd, then no $(2k, k, t(k-1))$ resolvable BIBD exists.*

Proof. For (1), since a block and its complement form a parallel class, taking $t/2$ copies of the BIBD and its complement yields the resolvable BIBD.

For (2), suppose a $(2k, k, t(k-1))$ resolvable BIBD exists, and suppose that three points, x, y, z occur as a triple in exactly s blocks. Then x occurs with y and not z in another $t(k-1) - s$ blocks, and x occurs with z and not y in another $t(k-1) - s$ blocks, and finally x occurs with neither y nor z in $t + s$ blocks. Now considering these blocks and their complements, we see that y and z appear together in $s + 0 + 0 + (t + s)$ blocks, so $2s + t = t(k-1)$, and so $t(k-2)$ must be even, contradicting our assumption that both t, k are odd. Put another way, each triple appears in $s = t(k-2)/2$ blocks, i.e., any $(2k, k, t(k-1))$ resolvable RBIBD is also a 3-design. □

We stated and proved Theorem 3.4 for completeness. Both parts are well-known, and there are several non-existence proofs: we presented the rather nice proof given by Mathon and Rosa [5].

In our final theorem, we mention a few more known results for $(v, 5, \lambda)$ resolvable BIBDs with $v \in \{10, 15, 45\}$:

Theorem 3.5. *There exist $(v, 5, \lambda)$ RBIBDs in each of the following cases:*

1. *$v = 10$ and $\lambda \equiv 0 \pmod 8$. (This follows from the previous theorem, since a $(10, 5, 4)$ BIBD can be obtained by developing $(0, 1, 2, 4, 8)$ and $\{\infty, 0, 1, 4, 6)$ $\pmod 9$.)*

2. $v = 15$*,* $\lambda \equiv 0$ *(mod* 2*) and* $\lambda \geq 6$ *[1, 6];*

3. $v = 45$ *and* $\lambda \geq 3$ *[4].*

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