AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **37** (2007), Pages 131–139

On perfect double dominating sets in grids, cylinders and tori

Abdollah Khodkar

Department of Mathematics
University of West Georgia Carrollton, GA 30118
U.S.A.
akhodkar@westga.edu

S.M. Sheikholeslami

Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz
Iran
s.m.sheikholeslami@azaruniv.edu

In memory of Professor M.A. Shahabi

Abstract

In a graph G, a vertex dominates itself and its neighbors. A subset $S \subseteq V(G)$ is a double dominating set of G if S dominates every vertex of G at least twice. A double dominating set S of a graph G is perfect if each vertex of G is dominated by exactly two vertices in S. In this note we study the existence and construction of perfect double dominating sets in grids, cylinders and tori.

1 Introduction

Let G be a graph with vertex set V. A vertex $v \in V$ is said to *dominate* all the vertices in its closed neighborhood $N[v]$. (For graph theory we follow the notation and terminology of [6].) A subset S of V is a *dominating set* of G if S dominates every vertex of G at least once [8]. When each vertex of G is dominated by exactly one element of S, then S is called a *perfect dominating set* (PDS) of G. Similarly, a subset S of V is a *double dominating set* of G if S dominates every vertex of G at least twice [7]. When each vertex of G is dominated by exactly two elements of S, then S is called a *perfect double dominating set* (PDDS) of G. The *double domination number* $dd(G)$ is the minimum cardinality among all double dominating sets of G. A *minimum double dominating set* of G is a double dominating set of cardinality $dd(G)$. In general [7], a subset S of V is a (perfect) k-*tuple* dominating set if each vertex of G is dominated by (exactly) at least k elements of S.

Determining whether an arbitrary graph has a dominating set of a given size is a well-known NP-complete problem [4, 10]. Straightforward proofs can be used to show that it is also NP-complete to decide if a graph has a perfect dominating set. In [2] it is proved that the existence of a perfect double dominating set is an NP-complete problem as well. Thus the general problem of determining if a graph has a dominating set, perfect dominating set or perfect double dominating set of a given size is quite hard, but for many significant classes of graphs it is manageable. Domination numbers of grid graphs, cylinders and tori have been studied by many researchers, but their domination numbers are known only for a few cases [3, 5, 9, 11]. Perfect dominating sets for certain graphs have also been investigated [1, 12, 13]. In particular, meshes and tori are studied in [12].

The authors of [2] study PDDSs in trees and in connected cubic graphs. In this paper we focus on PDDSs of grids, cylinders and tori. The paper is organized as follows: In Section 2 we completely characterize all grids that possess a PDDS and also specify the structure of the existing PDDSs. In Section 3 we determine which cylinders contain a PDDS and characterize the structure of their PDDSs. Section 4 is devoted to the determination of all tori which contain a PDDS and to the characterization of the structure of their PDDSs. We make use of the following result in this note.

Theorem 1 (See Corollary 6 in [2]) *Let G be a k-regular graph with vertex set V. If $\mathfrak{D}$ is a PDDS in G then $|\mathfrak{D}| = \frac{2|V|}{k+1}$.*

2 Grids

In this section we characterize all grids $P_n \times P_m$ that have perfect double dominating sets and determine the construction of those PDDSs which do exist. Throughout this section, $\mathfrak{D}$ is a perfect double dominating set in $P_n \times P_m$ when it exists. We assume that the vertices of the ith copy of P_n in $P_n \times P_m$ are $u_1^i, u_2^i, u_3^i, \ldots, u_n^i$ for $i = 1, 2, \ldots, m$. We also assume that $\mathfrak{D}$ has precisely c_i vertices in the ith copy of P_n in $P_n \times P_m$.

Lemma 1 (Proposition 11 in [2]) *$P_1 \times P_m$ has a PDDS if and only if $m \equiv 2$ (mod 3). If this holds the size of any such set is $2(m+1)/3$.*

Lemma 2 *$P_2 \times P_m$ has a PDDS if and only if m is odd. If this holds the size of any such set is $m+1$.*

Proof. Suppose that $\mathfrak{D}$ is a PDDS of $P_2 \times P_m$. We claim that $\mathfrak{D}$ contains both u_1^1 and u_2^1. It is clear that $\mathfrak{D}$ includes at least one of u_1^1 or u_2^1. Without loss of generality let $u_1^1 \in \mathfrak{D}$ and $u_2^1 \notin \mathfrak{D}$. This forces $u_1^2, u_2^2 \in \mathfrak{D}$ which is a contradiction since u_1^2 is dominated by at least three vertices of $\mathfrak{D}$. So $u_1^1, u_2^1 \in \mathfrak{D}$. Similarly, $u_1^m, u_2^m \in \mathfrak{D}$. This implies $u_1^2, u_2^2 \notin \mathfrak{D}$ and $u_1^3, u_2^3 \in \mathfrak{D}$ since $\mathfrak{D}$ is a PDDS. An induction argument shows that $u_1^{2k-1}, u_2^{2k-1} \in \mathfrak{D}$ and $u_1^{2k}, u_2^{2k} \notin \mathfrak{D}$ for each positive integer $1 \leq k \leq \lfloor (m+1)/2 \rfloor$. So m must be odd since $u_1^m, u_2^m \in \mathfrak{D}$. Moreover, the construction shows that the existing PDDS is unique. $\square$

Lemma 3 *Let $m \geq 3$. Then $P_3 \times P_m$ has a PDDS if and only if $m = 5$. If this holds the size of any such set is 8.*

Proof. Let $\mathfrak{D}$ be a PDDS in $P_3 \times P_m$. We claim that $u_2^1 \in \mathfrak{D}$. If $u_2^1 \notin \mathfrak{D}$ then we must have u_1^1, u_3^1, u_1^2 and $u_3^2 \in \mathfrak{D}$. This forces $c_3 = 0$. But then u_2^3 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction. So $u_2^1 \in \mathfrak{D}$. Therefore, $\mathfrak{D}$ contains exactly one of u_1^1 or u_3^1. Without loss of generality we can assume $u_1^1 \in \mathfrak{D}$. This implies $c_2 = 1$ and $u_3^2 \in \mathfrak{D}$. So $c_3 = 2$ and $u_1^3, u_3^3 \in \mathfrak{D}$. This forces $u_1^4 \in \mathfrak{D}$ which implies $c_5 = 2$ and $u_2^5, u_3^5 \in \mathfrak{D}$. If $m = 5$ then obviously $\mathfrak{D}$ is a PDDS for $P_3 \times P_5$. If $m \geq 6$ then $c_6 = 0$. This leads to the contradiction that $\mathfrak{D}$ cannot dominate u_1^6 twice. This completes the proof. $\square$

Lemma 4 *$P_4 \times P_m$ has a PDDS if and only if $m = 4$. If this holds the size of any such set is 8.*

Proof. By Lemmas 1, 2 and 3 we can assume $m \geq 4$.

First, let $u_1^1 \notin \mathfrak{D}$ then $u_1^2, u_2^1 \in \mathfrak{D}$. This forces $u_2^2 \notin \mathfrak{D}$ which implies $u_1^3, u_3^1 \in \mathfrak{D}$. So we must have $u_4^2 \in \mathfrak{D}$ which implies $u_4^3 \in \mathfrak{D}$. This forces $u_2^3, u_3^3, u_1^4, u_4^4 \notin \mathfrak{D}$ which implies $u_2^4, u_3^4 \in \mathfrak{D}$. We now consider two cases.

Case 1 $m = 4$. Then $\mathfrak{D}$ is a PDDS for $P_4 \times P_m$.

Case 2 $m \geq 5$. Then we must have $c_5 = 0$ which forces the vertex u_1^5 to be dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction.

Secondly, let $u_1^1 \in \mathfrak{D}$. We consider two cases.

Case 1 $u_2^1 \in \mathfrak{D}$. This forces $u_3^1, u_1^2, u_2^2 \notin \mathfrak{D}$ which implies $u_4^1, u_4^2, u_1^3, u_2^3 \in \mathfrak{D}$. Then u_3^2 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction.

Case 2 $u_1^2 \in \mathfrak{D}$. This forces $u_4^1 \in \mathfrak{D}$, otherwise u_3^1 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction. If $u_3^1 \notin \mathfrak{D}$ then $u_3^2, u_4^2 \in \mathfrak{D}$ which is a contradiction. So $u_3^1 \in \mathfrak{D}$. This forces $u_2^3, u_3^3, u_4^3 \in \mathfrak{D}$ which is also a contradiction. This completes the proof. $\square$

Lemma 5 *If $m, n \geq 5$ then $P_n \times P_m$ does not have a PDDS.*

Proof. In order to prove this theorem we consider two cases.

Case 1 $u_1^1 \in \mathfrak{D}$. Without loss of generality we can assume $u_1^2 \in \mathfrak{D}$. This forces $u_2^1, u_2^2, u_1^3 \notin \mathfrak{D}$ and $u_3^1 \in \mathfrak{D}$. Since u_2^2 must be dominated by exactly two vertices of $\mathfrak{D}$, consider two subcases.

Subcase 1 $u_2^3 \in \mathfrak{D}$. This implies $u_3^2, u_1^4 \notin \mathfrak{D}$. So $u_4^1 \in \mathfrak{D}$, $u_4^2 \notin \mathfrak{D}$, $u_3^3 \in \mathfrak{D}$ and $u_2^4 \notin \mathfrak{D}$. Now u_1^4 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction.

Subcase 2 $u_3^2 \in \mathfrak{D}$. This implies $u_2^3, u_3^3 \notin \mathfrak{D}$. Now u_2^3 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction.

Case 2 $u_1^1 \notin \mathfrak{D}$. This forces $u_1^2, u_2^1 \in \mathfrak{D}$ and $u_2^2 \notin \mathfrak{D}$. So $u_1^3, u_3^1 \in \mathfrak{D}$ which forces $u_1^4, u_2^3, u_3^2, u_4^1 \notin \mathfrak{D}$. Since u_2^3 must be dominated by exactly two vertices of $\mathfrak{D}$, we consider two subcases.

Subcase 1 $u_2^4 \in \mathfrak{D}$. This forces $u_1^5, u_3^3 \notin \mathfrak{D}$ which implies $u_4^2, u_3^4, u_4^3 \in \mathfrak{D}$. Then we must have $u_2^5, u_3^5, u_4^5 \notin \mathfrak{D}$. But then u_1^5 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction.

Subcase 2 $u_3^3 \in \mathfrak{D}$. This forces $u_4^2 \notin \mathfrak{D}$ which implies $u_4^3, u_5^1, u_5^2 \in \mathfrak{D}$. Then we must have $u_2^4, u_3^4, u_4^4 \notin \mathfrak{D}$. But then u_2^4 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction. □

Now we state the main result of this section.

Theorem 2 *$P_n \times P_m$ has a perfect double dominating set if and only if*

$$\begin{cases} n = 1 \text{ and } m \equiv 2 \pmod 3 \\ n = 2 \text{ and } m \equiv 1 \pmod 2 \\ n = 3 \textit{ and } m = 5 \\ n = m = 4. \end{cases}$$

An argument similar to that described in this section leads to the following theorem. The proof of this theorem is left for the reader.

Theorem 3 *$P_n \times P_m$ has a perfect 3-tuple dominating set if and only if*

$$\begin{cases} n = m = 2 \\ n = 3 \text{ and } m = 4. \end{cases}$$

3 Cylinders

In this section we characterize cylinders which contain a perfect double dominating set (PDDS) and determined the structure of their PDDSs. We assume that $u_1^i, u_2^i, \ldots, u_n^i$ are the vertices of the ith copy of C_n in $C_n \times P_m$ for $i = 1, 2, \ldots, m$. We also assume that $\mathfrak{D}$ has precisely c_i vertices in the ith copy of C_n in $C_n \times P_m$. Throughout this section $n \geq 3$.

Lemma 6 (Proposition 10 in [2]) *$C_n \times P_1$ has a PDDS if and only if $n \equiv 0$ (mod 3). If this holds the size of any such set is $2n/3$.*

Lemma 7 *$C_n \times P_2$ has a PDDS if and only if n is even. If this holds the size of any such set is n.*

Proof. Let $\mathfrak{D}$ be a PDDS in $C_n \times P_2$. It is clear that $c_1 \geq 1$. So, without loss of generality, we can assume $u_1^1 \in \mathfrak{D}$. We consider two cases.

Case 1 $u_1^2 \in \mathfrak{D}$. This forces $u_2^1, u_2^2, u_n^1, u_n^2 \notin \mathfrak{D}$ which implies $u_3^1, u_3^2 \in \mathfrak{D}$. An induction argument shows that $u_{2k-1}^1, u_{2k-1}^2 \in \mathfrak{D}$ and $u_{2k}^1, u_{2k}^2 \notin \mathfrak{D}$ for each positive integer $1 \leq k \leq \lfloor (n+1)/2 \rfloor$. Thus, we must have $n = 2k$ for some k.

Case 2 $u_2^1 \in \mathfrak{D}$. This implies $u_1^2, u_2^2, u_3^1, u_n^1 \notin \mathfrak{D}$ which forces $u_3^2, u_4^2 \in \mathfrak{D}$. An induction argument shows that $u_{2k-1}^1, u_{2k}^1 \in \mathfrak{D}$ when k is odd and $u_{2k-1}^2, u_{2k}^2 \in \mathfrak{D}$ when k is even. So when n is even $\mathfrak{D}$ is a PDDS for $C_n \times P_2$. If n is odd then u_n^2 is dominated only by one vertex of $\mathfrak{D}$ which is a contradiction. □

Lemma 8 *$C_n \times P_3$ has no PDDSs.*

Proof. It is easy to see that $C_3 \times P_3$ has no PDDSs. So let $n \geq 4$. Let $\mathfrak{D}$ be a PDDS in $C_n \times P_3$. Obviously, $c_1 \geq 1$. So, without loss of generality, we can assume $u_1^1 \in \mathfrak{D}$. We consider three cases.

Case 1 $u_1^2 \in \mathfrak{D}$. Then $u_2^1, u_2^2, u_n^1, u_n^2 \notin \mathfrak{D}$. Since u_1^3 must be dominated by two vertices of $\mathfrak{D}$ we must have either $u_2^3 \in \mathfrak{D}$ or $u_n^3 \in \mathfrak{D}$. If $u_2^3 \in \mathfrak{D}$ then u_n^3 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction. If $u_n^3 \in \mathfrak{D}$ then u_2^3 is dominated by at most one vertex of $\mathfrak{D}$ which is also a contradiction.

Case 2 $u_2^1 \in \mathfrak{D}$. This forces $u_n^1, u_3^1, u_1^2, u_2^2 \notin \mathfrak{D}$. We consider two subcases.

Subcase 1 $u_2^3 \in \mathfrak{D}$. This implies $u_3^2 \notin \mathfrak{D}$ which forces $u_3^3, u_4^1, u_4^2 \in \mathfrak{D}$. If $n = 4$ then we have $u_4^1 = u_n^1 \notin \mathfrak{D}$. This is a contradiction. If $n \geq 5$ then u_5^3 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction.

Subcase 2 $u_2^3 \notin \mathfrak{D}$. This forces $u_1^3, u_3^2, u_3^3 \in \mathfrak{D}$ and $u_4^2, u_4^3 \notin \mathfrak{D}$. So u_4^1 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction.

Case 3 $u_n^1 \in \mathfrak{D}$. An argument similar to that described in Case 2 verifies that this case is also impossible. □

Lemma 9 *$C_3 \times P_m$ has no PDDSs for $m \geq 2$.*

Proof. Let $\mathfrak{D}$ be a PDDS in $C_3 \times P_m$. Obviously, $c_1 \geq 1$. So, without loss of generality, we can assume $u_1^1 \in \mathfrak{D}$. If $u_1^2 \in \mathfrak{D}$ then u_2^1 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction. If $u_2^1 \in \mathfrak{D}$ then u_3^2 is dominated by at most one vertex of $\mathfrak{D}$ which is also a contradiction. Similarly, $u_3^1 \in \mathfrak{D}$ leads to a contradiction. This completes the proof. □

Lemma 10 *$C_4 \times P_m$ has no PDDSs for $m \geq 3$.*

Proof. By Lemma 8 we can assume $m \geq 4$. Let $\mathfrak{D}$ be a PDDS in $C_4 \times P_m$. Obviously, $c_1 \geq 1$. So, without loss of generality, we can assume $u_1^1 \in \mathfrak{D}$. We consider three cases.
Case 1 $u_1^2 \in \mathfrak{D}$. Then $u_2^1, u_2^2, u_1^3, u_4^1, u_4^2 \notin \mathfrak{D}$ which implies $u_3^1, u_3^2 \in \mathfrak{D}$. Then u_2^3 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction.
Case 2 $u_2^1 \in \mathfrak{D}$. This forces $u_1^2, u_2^2, u_3^1, u_4^1 \notin \mathfrak{D}$ which implies $u_3^2, u_4^2 \in \mathfrak{D}$. This forces $c_4 = 0$. But then u_1^3 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction.
Case 3 $u_4^1 \in \mathfrak{D}$. An argument similar to that described in Case 2 verifies that this case is also impossible. □

Lemma 11 *$C_n \times P_m$ has no PDDSs for any two integers $n \geq 5$ and $m \geq 4$.*

Proof. Let $\mathfrak{D}$ be a PDDS in $C_n \times P_m$. Since $c_1 \geq 1$, without loss of generality, we can assume u_1^1. We consider three cases.

Case 1 $u_1^2 \in \mathfrak{D}$. This forces $u_2^1, u_2^2, u_1^3, u_n^1, u_n^2 \notin \mathfrak{D}$ and $u_{n-1}^1, u_3^1 \in \mathfrak{D}$. Since u_2^2 must be dominated by two vertices of $\mathfrak{D}$ we consider two subcases.

Subcase 1 $u_3^2 \in \mathfrak{D}$. Then u_2^3 is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction.

Subcase 2 $u_2^3 \in \mathfrak{D}$. This implies $u_n^3 \notin \mathfrak{D}$ and $u_{n-1}^2, u_{n-1}^3 \in \mathfrak{D}$. Now u_{n-1}^2 is dominated by more than two vertices of $\mathfrak{D}$ which is a contradiction.

Case 2 $u_2^1 \in \mathfrak{D}$. This forces $u_n^1, u_1^2, u_2^2, u_3^1 \notin \mathfrak{D}$. Since u_2^2 must be dominated by two vertices of $\mathfrak{D}$ we consider two subcases.

Subcase 1 $u_2^3 \in \mathfrak{D}$. This forces $u_3^2 \notin \mathfrak{D}$ and $u_3^3, u_4^1, u_4^2 \in \mathfrak{D}$. Then we must have $u_4^3, u_5^1, u_5^2, u_5^3 \notin \mathfrak{D}$. Now if $n = 5$ then u_5^2 is dominated by only one vertex of $\mathfrak{D}$, namely u_4^2, which is a contradiction. If $n \geq 6$ then we must have $u_6^1, u_6^2, u_6^3 \in \mathfrak{D}$. Now u_6^2 is dominated by more than two vertices of $\mathfrak{D}$ which is also a contradiction.

Subcase 2 $u_3^2 \in \mathfrak{D}$. Then $u_4^1, u_2^3 \notin \mathfrak{D}$ and $u_4^2 \in \mathfrak{D}$. This forces $u_5^1 \in \mathfrak{D}$ and $u_5^2, u_3^3, u_4^3 \notin \mathfrak{D}$. So $u_1^3, u_2^4, u_3^4 \in \mathfrak{D}$ and $u_4^4 \notin \mathfrak{D}$. This implies $u_5^3 \in \mathfrak{D}$. Now u_5^2 is dominated by three vertices of $\mathfrak{D}$ which is a contradiction.

Case 3 $u_n^1 \in \mathfrak{D}$. An argument similar to that described in Case 2 verifies that this case is also impossible. □

Now we state the main result of this section.

Theorem 4 *$C_n \times P_m$ has a perfect double dominating set if and only if*

$$\begin{cases} n \equiv 0 \pmod{3} \text{ and } m = 1 \\ n \equiv 0 \pmod{2} \text{ and } m = 2. \end{cases}$$

4 Tori

In this section we characterize all tori $C_n \times C_m$, $m, n \geq 3$, that have perfect double dominating sets and determine the construction of those PDDSs which do exist. Since the tori are 4-regular, the complement of a PDDS of a torus is a perfect 3-tuple dominating set of that torus. Hence, we also characterize all tori $C_n \times C_m$, $m, n \geq 3$, that have perfect 3-tuple dominating sets. In this section we assume that $u_1^i, u_2^i, \ldots, u_n^i$ are the vertices of the ith copy of C_n in $C_n \times C_m$ for $i = 1, 2, \ldots, m$. We also assume that $\mathfrak{D}$ has precisely c_i vertices in the ith copy of C_n in $C_n \times C_m$.

Lemma 12 *$C_3 \times C_m$ has no PDDSs for any positive integer m.*

Proof. Let $\mathfrak{D}$ be a perfect double dominating set in $C_3 \times C_m$. Since $C_3 \times C_m$ is a 4-regular graph we have $|\mathfrak{D}| = \frac{6m}{5}$ by Theorem 1. So $m = 5k$ and $|\mathfrak{D}| = 6k$ for some positive integer k. Now since $|\mathfrak{D}| > m$ there is a copy of C_3 in $C_3 \times C_m$, say the i-th copy, such that $c_i \geq 2$. Without loss of generality we can assume $u_1^i, u_2^i \in \mathfrak{D}$. This implies $u_3^i, u_1^{i+1}, u_2^{i+1}, u_3^{i+1} \notin \mathfrak{D}$. Now u_3^{i+1} is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction. This completes the proof. □

Lemma 13 *$C_4 \times C_m$ has no PDDSs for any positive integer m.*

Proof. By Lemma 12 we can assume $m \geq 4$. Let $\mathfrak{D}$ be a perfect double dominating set in $C_4 \times C_m$. Since $C_4 \times C_m$ is a 4-regular graph we have $|\mathfrak{D}| = \frac{8m}{5}$ by Theorem 1. So $m = 5k$ and $|\mathfrak{D}| = 8k$ for some positive integer k. Now since $|\mathfrak{D}| > m$ there is a copy of C_4 in $C_4 \times C_m$, say i-th copy, such that $c_i \geq 2$. We consider two cases.

Case 1 $\mathfrak{D}$ contains two adjacent vertices of the ith copy of C_4. Without loss of generality we can assume $u_1^i, u_2^i \in \mathfrak{D}$. Then $u_1^{i-1}, u_2^{i-1}, u_3^i, u_4^i \notin \mathfrak{D}$. If $u_3^{i+1} \in \mathfrak{D}$ then $u_3^{i-1} \notin \mathfrak{D}$ and $u_4^{i-1}, u_2^{i-2}, u_3^{i-2} \in \mathfrak{D}$. But now u_4^{i-1} is dominated by at most one vertex of $\mathfrak{D}$ which is a contradiction. Similarly, it is impossible to have $u_3^{i-1} \in \mathfrak{D}$. So u_3^i is dominated by at most one vertex of $\mathfrak{D}$ which is impossible.

Case 2 $\mathfrak{D}$ contains two non-adjacent vertices of the ith copy of C_4. Without loss of generality we can assume $u_1^i, u_3^i \in \mathfrak{D}$. Now there are two subcases (up to isomorphism) to consider.

Subcase 1 $u_1^{i+1}, u_3^{i+1} \in \mathfrak{D}$. This forces $c_{i+2} = 0$. This is a contradiction since u_2^{i+2} is dominated by at most one vertex of $\mathfrak{D}$.

Subcase 2 $u_1^{i+1}, u_3^{i-1} \in \mathfrak{D}$. This implies $u_2^{i+2}, u_3^{i+2}, u_4^{i+2} \in \mathfrak{D}$. But then u_3^{i+2} is dominated by more than two vertices of $\mathfrak{D}$ which is a contradiction. □

Lemma 14 *Let 5 divide m and n. Then $C_n \times C_m$ has a perfect double dominating set.*

Proof. It is easy to observe that if $G \times C_k$ has a PDDS, then so does $G \times C_{ak}$ for all $a \geq 1$. On the other hand

$$D = \{u_1^2, u_1^5, u_2^2, u_2^4, u_3^1, u_3^4, u_4^1, u_4^3, u_5^3, u_5^5\}$$

is a PDDS of $C_5 \times C_5$. Now the result follows. □

Lemma 15 *Let $m, n \geq 3$. If $C_n \times C_m$ has a PDDS then both n and m are multiples of 5.*

Proof. Let $\mathfrak{D}$ be a PDDS in G. Without loss of generality, we can assume $u_1^2, u_2^2 \in \mathfrak{D}$. This forces $u_1^1, u_2^1, u_3^2, u_n^2, u_1^3, u_2^3 \notin \mathfrak{D}$. Consider two cases.

Case 1 $u_3^1 \in \mathfrak{D}$. This implies $u_4^2, u_3^3 \notin \mathfrak{D}$ and $u_4^3, u_2^4, u_3^4 \in \mathfrak{D}$. Then we must have $u_1^4, u_4^4, u_5^4, u_2^5, u_3^5, u_4^5 \notin \mathfrak{D}$ and $u_n^3, u_5^3, u_5^5, u_3^6, u_4^6 \in \mathfrak{D}$. Thus, $u_5^2, u_6^3, u_2^6, u_5^6, u_6^6, u_3^7, u_4^7, u_5^7 \notin \mathfrak{D}$ and $u_4^1, u_1^5, u_6^5, u_6^7, u_4^8, u_5^8 \in \mathfrak{D}$. This forces $u_5^1, u_6^4, u_n^4, u_7^5, u_7^6, u_3^8, u_6^8, u_7^8, u_4^9, u_5^9, u_6^9, u_3^m$, $u_4^m \notin \mathfrak{D}$ and $u_6^2, u_7^4, u_2^7 \in \mathfrak{D}$. But then $u_7^3, u_8^5, u_1^6 \notin \mathfrak{D}$ and $u_8^4, u_n^5, u_8^6, u_7^7 \in \mathfrak{D}$. This implies $u_8^3, u_n^6 \notin D$ and $u_7^2, u_1^7 \in \mathfrak{D}$. Now, we must have $u_6^1, u_7^1, u_8^2 \notin \mathfrak{D}$ and $u_5^m, u_6^m \in \mathfrak{D}$. This forces $u_7^m \notin \mathfrak{D}$ and $u_8^1 \in \mathfrak{D}$. So far we have proved if $u_1^2, u_2^2, u_3^1 \in \mathfrak{D}$ then $u_1^7, u_2^7, u_3^6, u_6^2, u_7^2, u_8^1 \in \mathfrak{D}$. In a similar fashion we can prove that if $u_1^7, u_2^7, u_3^6 \in \mathfrak{D}$ then $u_1^{12}, u_2^{12}, u_3^{11} \in \mathfrak{D}$. Moreover, if $u_6^2, u_7^2, u_8^1 \in \mathfrak{D}$ then $u_{11}^2, u_{12}^2, u_{13}^1 \in \mathfrak{D}$. A simple induction argument shows that

$$u_1^{5k+2}, u_2^{5k+2}, u_3^{5k+1} \in \mathfrak{D} \quad \text{and} \quad u_{5k+1}^2, u_{5k+2}^2, u_{5k+3}^1 \in \mathfrak{D}$$

for every non-negative integer k. (Note that the superscript $5k + a$ in u_i^{5k+a} is the unique integer $r \in \{1, 2, \dots, m\}$ such that $5k + a \equiv r \pmod{m}$ and the subscript $5k + b$ in u_{5k+b}^j is the unique integer $s \in \{1, 2, \dots, n\}$ such that $5k + b \equiv s \pmod{n}$.) Now since $u_1^{5k+2} \in \mathfrak{D}$ for every non-negative integer k it follows that 5 divides m, otherwise $u_1^i \in \mathfrak{D}$ for every $i = 1, 2, \dots, m$ which is impossible. Similarly, 5 divides n since $u_{5k+1}^2 \in \mathfrak{D}$ for every non-negative integer k.

Case 2 $u_3^1 \notin \mathfrak{D}$. This forces $u_4^1, u_2^m, u_3^m \in \mathfrak{D}$. Now by relabeling the vertices of $C_n \times C_m$ this case can be converted to Case 1. So 5 divides m and n. □

Now we are ready to state the main results of this section.

Theorem 5 *Let $n, m \geq 3$. Then $C_n \times C_m$ has a perfect double dominating set if and only if 5 divides both m and n. If this holds the size of any such set is $2mn/5$.*

Corollary 1 *For $n, m \geq 3$, $C_n \times C_m$ has a perfect 3-tuple dominating set if and only if 5 divides both m and n.*

References

[1] J. Borges and I.J. Dejter, *On perfect dominating sets in hypercubes and their complements*, J. Combin. Math. Combin. Comput. **20** (1996), 161–173.

[2] M. Chellali, A. Khelladi and F. Maffray, *Exact double domination in graphs*, Discuss. Math. Graph Theory **25** (2005), 291–302.

[3] T.Y. Chang, W.E. Clark and E.O. Hare, *Domination numbers of complete grid graphs, I*, Ars Combin. **38** (1994), 97–111.

[4] M.R. Garey and D.S. Johnson, *Computers and intractability: a guide to the theory of NP-completeness*, (1979) W.H. Freeman & Co. New York.

[5] D.R. Guichard, *A lower bound for the domination number of complete grid graphs*, J. Combin. Math. Combin. Comput. **49** (2004), 215–220.

[6] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.

[7] F. Harary and T.W. Haynes, *Double domination in graphs*, Ars Combin. **55** (2000), 201–213.

[8] T.W. Haynes, S.T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, Inc., New York, 1998.

[9] M.S. Jaconson and L.F. Kinch, *On the domination number of products of a graph, I*, Ars Combin. **10** (1983), 33–44.

[10] D.S. Johnson, *The NP-completeness column: an ongoing guide*, J. Algorithms **6** (1958), 434–451.

[11] S. Klavźar and N. Seifter, *Dominating Cartesian products of cycles*, Discrete Appl. Math. **59** (1995), 129–136.

[12] M. Livingston and Q.F. Stout, *Perfect dominating sets*, Proceedings of the Twenty-first Southeastern Conference on Combinatorics. Graph Theory, and Computing (Boca Raton, FL, 1990). Congr. Numer. **79** (1990) 187–203.

[13] D. Van Wieren, M. Livingston and Q.F. Stout, *Perfect dominating sets on cube-connected cycles*, Proceedings of the Twenty-fourth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1993). Congr. Numer. **97** (1993), 51–70.

(Received 3 Oct 2005; revised 5 Aug 2006)