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The graphs satisfying conditions of Ore's type

Shengjia Li* Ruijuan Li†

School of Mathematical Science
Shanxi University
030006 Taiyuan, P. R. China

Jinfeng Feng‡

Lehrstuhl C für Mathematatik
RWTH Aachen University
52056 Aachen
Germany

Abstract

Let $G = (V, E)$ be a connected simple graph and let $d_G(u, v)$ denote the distance between two vertices u, v in G. Sohel Rahman and Kaykobad (*Inform. Process. Lett.* 94 (2005), 37–41) proved that if $d_G(u) + d_G(v) \geq |V(G)| - d_G(u, v) + 1$ for each pair of nonadjacent vertices u and v, then G has a hamiltonian path. In this paper, we determine the structure of such graphs and prove that for every longest cycle C of G, the subgraph $G - V(C)$ is complete or empty.

1 Introduction and main result

All graphs considered in this paper are connected simple graphs. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. For two vertices $u, v \in V(G)$, the distance $d(u, v)$ between u and v is the length of a shortest path between u and v in G.

For a vertex u of G, the set $N_G(u) = \{v \mid uv \in E(G)\}$ is called the neighborhood of u in G. The degree of u in G is $|N_G(u)|$, denoted by $d_G(u)$ or $d(u)$.

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† Corresponding author. *E-mail address:* ruijuanli@sxu.edu.cn
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A subgraph induced by a subset $X \subseteq V(G)$ is denoted by $G[X]$. In addition, $G - X = G[V(G) - X]$.

A path (a cycle, respectively) of G is called a hamiltonian path (hamiltonian cycle, respectively), if it contains all vertices of G. The graph G is said to be hamiltonian, if it has a hamiltonian cycle.

It is well-known that the hamiltonian cycle (as well as the hamiltonian path) problem is **NP**-complete, and many sufficient conditions, respect to various parameters, have been found (see [1]), e.g.

Theorem 1.1 (Ore [3]) *Let G be a graph with n vertices. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices u and v, then G is hamiltonian.*

We say that a condition, described in the form $d_G(u) + d_G(v) \geq f(|V(G)|, d_G(u, v))$ for every pair of nonadjacent vertices u and v, is of Ore's type.

The next theorem states that the graphs satisfying a condition of Ore's type contain a hamiltonian path.

Theorem 1.2 (Rahman, Kaykobad [4]) *Let G be a connected graph with $n \geq 3$ vertices. If $d(u)+d(v) \geq n - d(u, v) + 1$ for every pair of nonadjacent vertices u and v, then G has a hamiltonian path.*

It seems to be interesting to determine the structure of such graphs. In this paper, we prove the following:

Theorem 1.3 *Let G be a connected graph with $n \geq 5$ vertices. If $d(u) + d(v) \geq n - d(u, v) + 1$ for every pair of nonadjacent vertices u and v, then G contains a cycle and for every longest cycle C in G, the subgraph $G - V(C)$ is complete or empty.*

It is clear that Theorem 1.2 follows from our result.

2 Proof of the main result

The proof of the following lemma can be found in [2].

Lemma 2.1 *Let $P = v_1 v_2 \ldots v_s$ ($s \geq 2$) and $Q = w_1 w_2 \ldots w_t$ ($t \geq 1$) be two disjoint paths in a graph G. If $d_P(w_1) + d_P(w_t) \geq |V(P)| + 2$, then Q can be inserted into P (i.e., $v_1 \ldots v_k Q v_{k+1} \ldots v_s$ is a path in G for some $1 \leq k < s$).*

Note that for $t = 1$, Lemma 2.1 states the following: If $d_P(w_1) \geq \left\lceil \frac{|V(P)|}{2} \right\rceil + 1$, then w_1 can be inserted into P.

Proof of Theorem 1.3: Let G be a graph satisfying the conditions of Theorem 1.3. Under the assumption that G is nonhamiltonian, we determine the structure of G by the following claims.

Claim 1. G contains a cycle.

Proof. Suppose to the contrary that G contains no cycle. Then G is a tree. Let $P = v_1v_2\dots v_s$ be a longest path in G. It is clear that $3 \leq s \leq n$. If $s = n$, then the inequality $d(v_1) + d(v_3) = 3 < n - 1 = n - d(v_1, v_3) + 1$ yields a contradiction. In the other case when $s < n$, we have from $d(v_1, v_s) = s - 1$ that $d(v_1) + d(v_s) = 2 < n - s + 2 = n - d(v_1, v_s) + 1$, a contradiction. □

By Claim 1, G has at least one cycle. Let $C = u_1u_2\dots u_mu_1$ be a longest cycle in G.

Claim 2. The subgraph $G - V(C)$ is connected.

Proof. We prove this claim indirectly. Let H and H' be two components of $G - V(C)$ with $|V(H)| = \ell$ and $|V(H')| = \ell'$. Since G is connected, there is at least one edge between C and every component of $G - V(C)$. Let $Q = ux_1x_2\dots x_kv$ be a shortest path from H to H' in $G - \{xy \,|\, x, y \in V(C) \text{ and } xy \notin E(C)\}$. It is clear that $V(Q) \cap V(H) = \{u\}$, $V(Q) \cap V(H') = \{v\}$ and $x_1x_2\dots x_k$ is a segment of C. Assume without loss of generality that $x_i = u_i$ for $i = 1, 2, \dots, k$. Note that $d(u, v) \leq k + 1$ and $m + \ell + \ell' \leq n$.

Since neither u nor v can be inserted into C, it is easy to check

$$d_C(u) \leq \frac{m}{2} \quad \text{and} \quad d_C(v) \leq \frac{m}{2}. \tag{1}$$

In the following, we show that $d(u) + d(v) < n - d(u, v) + 1$ for all $k \geq 1$, which contradicts to the assumption of the lemma.

Firstly, suppose that $k = 1$. Then, we see that $d(u, v) = 2$ and

$$\begin{aligned} d(u) + d(v) &= d_C(u) + d_C(v) + d_H(u) + d_{H'}(v) \\ &\leq \frac{m}{2} + \frac{m}{2} + (\ell - 1) + (\ell' - 1) \leq n - 2 \\ &< n - d(u, v) + 1. \end{aligned}$$

Next, suppose that $k = 2$. Then, it is easy to see that $d(u, v) = 3$. We now consider the case when m is even and $d_C(u) = d_C(v) = \frac{m}{2}$. From the choice of Q, it is easy to check that $N_C(u) = \{u_1, u_3, \dots, u_{m-1}\}$ and $N_C(v) = \{u_2, u_4, \dots, u_m\}$, hence, $uu_3u_2vu_4u_5\dots u_mu_1u$ is a cycle longer than C. This contradiction to the choice of C, together with (1), yields $d_C(u) + d_C(v) < m$. It follows that

$$\begin{aligned} d(u) + d(v) &= d_C(u) + d_C(v) + d_H(u) + d_{H'}(v) \\ &< m + (\ell - 1) + (\ell' - 1) \ \leq \ n - 2 \\ &= n - d(u, v) + 1. \end{aligned}$$

Finally, we consider the case when $k \geq 3$.

If $d_C(u) = 1$, then we see from the choice of Q that v is not adjacent with any vertex of the segment $u_{m-(k-3)} \dots u_m u_1 u_2 \dots u_{k-1}$ of C. Note that $m \geq k+(k-2) = 2k-2$. Furthermore, since v cannot be inserted into the segment $u_k u_{k+1} \dots u_{m-(k-2)}$ of C, we deduce from Lemma 2.1 that $d_C(v) \leq \lceil \frac{[m-(k-2)]-k+1}{2} \rceil \leq \frac{m-2k+4}{2}$. It follows that

$$\begin{aligned} d(u)+d(v) &= d_C(u)+d_C(v)+d_H(u)+d_{H'}(v) \\ &\leq 1+\frac{m-2k+4}{2}+(\ell-1)+(\ell'-1) \\ &\leq 1+n-\frac{m}{2}-k \\ &< n-(k+1)+1 \\ &\leq n-d(u,v)+1. \end{aligned}$$

Thus, we have $d_C(u) \geq 2$. By the same argument as above, we have $d_C(v) \geq 2$, too. Define

$$\alpha = \min\{\, i \mid i \geq 2 \text{ and } uu_i \in E(G)\} \quad \text{and} \quad \beta = \max\{\, j \mid j \leq m \text{ and } vu_j \in E(G)\}.$$

From the choice of Q, we conclude that $\alpha \geq 2k-1$ and $\beta \leq m-k+2$. Since u (v, respectively) cannot be inserted into the segment $u_\alpha \dots u_m u_1$ with $m+2-\alpha$ vertices (the segment $u_k u_{k+1} \dots u_\beta$ with $\beta-k+1$ vertices, respectively) of C, we have

$$\begin{aligned} d_C(u) &\leq \left\lceil \frac{m+2-\alpha}{2} \right\rceil \leq \left\lceil \frac{m+2-(2k-1)}{2} \right\rceil = \left\lceil \frac{m-2k+3}{2} \right\rceil \\ \text{and} \quad d_C(v) &\leq \left\lceil \frac{\beta-k+1}{2} \right\rceil \leq \left\lceil \frac{(m-k+2)-k+1}{2} \right\rceil = \left\lceil \frac{m-2k+3}{2} \right\rceil. \end{aligned}$$

It follows that

$$\begin{aligned} d(u)+d(v) &= d_C(u)+d_C(v)+d_H(u)+d_{H'}(v) \\ &\leq 2\left\lceil \frac{m-2k+3}{2} \right\rceil+(\ell-1)+(\ell'-1) \\ &\leq [(m-2k+3)+1]+(\ell-1)+(\ell'-1) \leq n-2k+2 \\ &\leq n-(k+1)+(3-k) \\ &< n-(k+1)+1 \\ &\leq n-d(u,v)+1. \end{aligned}$$

The proof of Claim 2 is complete. □

Claim 3. $G-V(C)$ is a complete graph.

Proof. By Claim 2, the subgraph $G-V(C)$ is connected. Denote $H = G-V(C)$ and $\ell = |V(H)|$. Clearly, we only need to consider the case when $\ell \geq 3$.

Firstly, we show the following statements: If uu_i is an edge of G with $u \in V(H)$ and $1 \leq i \leq m$, then we have

1) $d_C(u_{i-1}) \leq m-d_C(u)$, where $u_0 = u_m$ for $i = 1$,

2) $d_H(u) = \ell - 1$.

To prove 1), we assume that there is an integer j with $1 \leq j \leq m$ with $uu_j, u_{i-1}u_{j-1} \in E(G)$. Then, $u_{i-1}u_{j-1}u_{j-2}\dots u_i u u_j \dots u_{i-1}$ is a cycle longer than C. This contradiction to the choice of C implies that $d_C(u_{i-1}) \leq m - d_C(u)$.

The statement 2) can be confirmed indirectly. Suppose thus that $d_H(u) < \ell - 1$. From the choice of C and the fact that H is connected, we see $d_H(u_{i-1}) = 0$ and $d(u, u_{i-1}) = 2$. It follows from 1) that

$$\begin{aligned} d(u) + d(u_{i-1}) &= d_H(u) + d_C(u) + d_H(u_{i-1}) + d_C(u_{i-1}) \\ &\leq d_H(u) + d_C(u) + 0 + (m - d_C(u)) \\ &< (\ell - 1) + m \ = \ n - 1 \\ &= n - d(u, u_{i-1}) + 1, \end{aligned}$$

a contradiction. Therefore, $d_H(u) = \ell - 1$ holds.

Next, we show that H is complete. Suppose to the contrary that H is not complete. Then, H contains a vertex v with $d_H(v) < \ell - 1$. Since G is connected, there exists an edge uu_k with $u \in V(H)$ and $1 \leq k \leq m$. By 2) above, we have $d_H(u) = \ell - 1$ and $d_C(v) = 0$. It follows that $d(u_{k-1}, v) = 3$. Combining with 1) above, we obtain

$$\begin{aligned} d(u_{k-1}) + d(v) &= d_H(u_{k-1}) + d_C(u_{k-1}) + d_H(v) + d_C(v) \\ &< 0 + (m - d_C(u)) + (\ell - 1) + 0 \\ &= n - d_C(u) - 1 \ \leq \ n - 2 \\ &= n - d(v, u_{k-1}) + 1, \end{aligned}$$

a contradiction. ∎

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