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A sufficient condition for spanning trees with bounded maximum degree in a graph

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Abstract

We obtain a sufficient condition on the degree sequence of a graph which is an improvement to the condition of E. Flandrin, H.A. Jung and H. Li (*Discrete Math.* 90 (1991), 41–52) for traceability. As a generalization of this condition, a sufficient condition for the existence of a spanning tree with bounded maximum degree in a graph is presented.

1 Introduction

We represent a graph G by an ordered pair $(V(G), E(G))$, where $V(G)$, its vertex set, is nonempty, and $E(G)$, its edge set, consists of unordered pairs of distinct vertices. We write $|G|$ for $|V(G)|$. To show that H is a subgraph of G, we write $H \subset G$. For any $v \in V(G)$, we denote by $d_G(v)$ the degree of v in G; $\Delta(G)$ is maximum degree of vertices of G. A vertex of degree one is called an *end vertex*. For a nonnegative integer i, we put $V_i(G) := \{v \in V(G) : d_G(v) = i\}$. For any nonempty subset U of $V(G)$, we put $N_G(U) := \{v \in V(G) : uv \in E(G) \text{ for some } u \in U\}$, $d_G(U) := \sum_{u \in U} d_G(u)$, $G[U] := (U, \{uv \in E(G) : u, v \in U\})$, and $G - U := G[V(G) - U]$. If $U = \{u\}$ then we write $G - u$ for $G - U$ and $N_G(u)$ for $N_G(U)$. For a nonnegative integer i and any nonempty subset U of $V(G)$, we put $N_i(U) := \{v \in V(G) : |N_G(V) \cap U| = i\}$. If H is a proper subgraph of G, we write $G - H$ for $G - V(H)$. If H and K are subgraphs of G, then $H \cup K := (V(H) \cup V(K), E(H) \cup E(K))$. If $V(K) \subset V(H)$ then we write $H + E(K)$ for $H \cup K$; if $E(K) = \{uv\}$ then we write $H + uv$ for $H + E(K)$. If $S \subset E(H)$ then $H - S := (V(H), E(H) - S)$; if $S = \{uv\}$ then $H - uv := H - S$. A *k-tree* of a connected graph is a spanning tree with maximum degree at most k.

If T is a tree and u, v vertices of T, then the path in T connecting u and v is unique and denoted by $P_T[u, v]$. We assume $P_T[u, v]$ to be oriented from u towards v. A *branch* of T at a vertex r is a component of $T - r$.

We call a nonempty set S of independent vertices of G a *frame* of G, if $G - S'$ is connected for any $S' \subset S$. Obviously, any nonempty subset of a frame is also a frame. If $|S| = k$ then S is called a *k-frame*.

For further explanation of terminology and notation, we refer to Bondy and Murty [2].

In this paper, G denotes a connected graph and $k(\geq 2)$ is an integer. We put $n := |G|$.

In [3], E. Flandrin, H.A. Jung and H. Li gave the following theorem for a graph to have a hamiltonian path.

Theorem A *If $d_G(\{u, v, w\}) - |N_3(\{u, v, w\})| \geq n - 1$ for every independent set $\{u, v, w\}$, then G has a hamiltonian path.*

In this paper, we improve the previous result in the following way.

Theorem B *Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $d_G(v_i) = d_i$ for $1 \leq i \leq n$. Suppose, for any 3-frame $S = \{v_{i_1}, v_{i_2}, v_{i_3}\} \subset V(G)$, $i_1 < i_2 < i_3$ such that $d_{i_j} \leq i_j - j + 1$, $1 \leq j \leq 3$, it holds that $d_G(S) - |N_3(S)| \geq n - 1$. Then G has a hamiltonian path.*

Generalizing Theorem B, we obtain the following result.

Theorem C *Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $d_G(v_i) = d_i$ for $1 \leq i \leq n$. Suppose, for any $k+1$-frame $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{k+1}}\} \subset V(G)$, $i_1 < i_2 < \ldots < i_{k+1}$ such that, $d_{i_j} \leq i_j - j + 1, 1 \leq j \leq k+1$, it holds that $d_G(S) + \sum_{i=2}^{k+1} (k - i)\, |N_i(S)| \geq n - 1$. Then G has a k-tree.*

In the following section, we prove auxiliary results. We prove Theorems B and C in the last section.

2 Auxiliary Results

We call a tree T in (a connected graph) G a k^+*-tree* of G if there exists a vertex r in T such that $d_T(r) = k + 1$ and $d_T(x) \leq k$ for every $x \in V(T) - \{r\}$. Note that 2^+-tree is a tree with only three end vertices.

We call a tree k^+-tree T in G a k^+*-maximal tree* of G if there does not exist a k^+-tree T' in G such that $V(T) \subset V(T')$, and $|T| < |T'|$.

Lemma 2.1 *Suppose G does not have a k-tree. Let T be a k^+-maximal tree of G. Then, there does not exist a tree T' in G such that $\Delta(T') \leq k$ and $V(T) = V(T')$.*

Proof. Suppose G does not have a k-tree. Let T be a k^+-maximal tree of G, T' a tree of G such that $\Delta(T') \leq k$ and $V(T) = V(T')$. Since G does not have a k-tree, there exists a vertex $v \in V(T')$ such that $N_G(v) \cap V(G - T') \neq \emptyset$. Pick a vertex w of $N_G(v) \cap V(G - T')$, then the tree $T' + uv$ contradicts the maximality of T. □

Lemma 2.2 *Suppose G does not have a k-tree. If T is a k^+-maximal tree of G, then the following hold:*

i. $N_G(V_1(T)) \subset V(T)$.

ii. $G - S$ *is connected for any* $S \subset V_1(T)$.

iii. *If x, y, r are three distinct vertices of T such that $d_T(r) = k+1$, $r \in V(P_T[x, y])$, and $xy \in E(G)$, then $d_T(x) = k$ or $d_T(y) = k$.*

Proof. (i) The result holds by the maximality of T.

(ii) The result is direct consequence of (i).

(iii) Let x, y, r be vertices of T satisfying the conditions in (iii), and, suppose that $\max\{d_T(x), d_T(y)\} < k$. Let r^- be the vertex that precedes r on $P_T[x, r]$. Then, G has the tree $T' = (T + xy) - rr^-$ such that $\Delta(T') \leq k$ and $V(T') = V(T)$, a contradiction by Lemma 2.1. Hence (iii) holds. □

Before presenting the next result, we introduce some additional terminology.

Let T be a k^+-maximal tree of G, r the vertex of T with $d_T(r) = k+1$. A subset U of $V_1(T)$ containing one and only one vertex from each branch of T at r and no others is called a *representative set* of T with respect to r. After fixing a representative set U of T with respect to r, we will always use the following notation:

$B_1, B_2, ..., B_{k+1}$ are vertex sets of branches of T at r. For $1 \leq i \leq k+1$, the only vertex of $U \cup B_i$ is denoted by u_i and the only vertex of $N_T(r) \cap B_i$ by v_i. For each $x \in B_i - \{u_i\}$, the vertex that precedes x on $P_T[u_i, x]$ is denoted by x^-, and we set $N_T^*(x) := N_T(x) - \{x^-\}$. If $|N_T^*(x)| = 1$, then the only vertex of $N_T^*(x)$ is denoted by x^+ . We label $\bigcup_{i=2}^{k+1} N_i(U) - \{r\} = \{y_1, y_2, \ldots, y_m\}$ if it is nonempty.

To prove Theorem B (Theorem C for $k = 2$), we need the following result.

Lemma 2.3 *Suppose G does not have a hamiltonian path. Let T be a 2^+-maximal tree of G, r the vertex of T with $d_T(r) = 3$, and U a representative set of T with respect to r. Then, the following hold:*

i. U *is independent.*

ii. *For $1 \leq i \leq 3$, $1 \leq j \leq 3$, $i \neq j$, if $x \in B_i \cap N_G(u_j)$, then $x \neq v_i$ and $x^+ \notin N_G(U - \{u_j\})$.*

iii. *For* $1 \leq i \leq 3$, $1 \leq j \leq 3$, $|B_i| \geq 1 + \sum_{j=1}^{3} |N_G(u_j) \cap B_i| - |N_3(U) \cap B_i|$.

iv. $|T| \geq 2 + d_G(U) - |N_3(U)|$.

Proof. (i) The result follows directly from Lemma 2.2(iii).
(ii) Let x be a vertex in B_i, and suppose, $xu_j \in E(G)$ for some $j \neq i$. If $x = v_i$, then G has the tree $T' = (T + v_iu_j) - rv_i$ such that $\Delta(T') \leq 2$ and $V(T') = V(T)$, a contradiction by Lemma 2.1. Suppose, $x \neq v_i$. Then the tree $T' = (T + xu_j) - xx^+$ is also a 2^+-maximal tree with $V_1(T') = (V(T_1) - \{u_j\}) \cup \{x^+\}$, $d_{T'}(r) = 3$, and $V(T') = V(T)$ so that $x^+ \notin N_G(U - \{u_j\})$ by Lemma 2.2(iii). Hence (ii) holds.
(iii) For $1 \leq i \leq 3$, from (i) and (ii), $\{u_i\}$, $N_G(u_i) \cap B_i$, $(N_G(U - \{u_i\}))^+ \cap B_i$, and $(N_2(U) - N_G(u_i)) \cap B_i$ are pair-wise disjoint subsets of B_i where

$$(N_G(U - \{u_i\}))^+ = \{x^+ : x \in N_G(U - \{u_i\})\}.$$

So

$$\begin{aligned}
|B_i| &\geq 1 + |N_G(u_i) \cap B_i| + |(N_G(U - \{u_i\}))^+ \cap B_i| \\
&\quad + |(N_2(U) - N_G(u_i)) \cap B_i| \\
&= 1 + |N_G(u_i) \cap B_i| + |(N_G(U - \{u_i\})) \cap B_i| \\
&\quad + |(N_2(U) - N_G(u_i)) \cap B_i| \\
&= 1 + \sum_{j=1}^{3} |N_G(u_j) \cap B_i| - |(N_2(U - \{u_i\}) \cap B_i| \\
&\quad + |(N_2(U) - N_G(u_i)) \cap B_i| \\
&= 1 + \sum_{j=1}^{3} |N_G(u_j) \cap B_i| - |(N_3(U) \cap B_i)|.
\end{aligned}$$

(iv) Since $2 \geq \sum_{j=1}^{3} |N_G(u_j) \cap \{r\}| - |N_3(U) \cap \{r\}|$, and from (iii),

$$\begin{aligned}
\sum_{i=1}^{3} |B_i| + 2 &\geq 3 + \sum_{i=1}^{3}\sum_{j=1}^{3} |N_G(u_j) \cap B_i| - \sum_{i=1}^{3} |N_3(U) \cap B_i| \\
&\quad + \sum_{j=1}^{3} |N_G(u_j) \cap \{r\}| - |N_3(U) \cap \{r\}|
\end{aligned}$$

Since $N_G(V_1(T)) \subset V(T)(= \bigcup_{i=1}^{3} B_i \cup \{r\})$ by Lemma 2.2(i),

$$|T| \geq 2 + \sum_{j=1}^{3} |N_G(u_j)| - |N_3(U)|.$$

□

To prove Theorem C for $k \geq 3$, we need the following result.

Lemma 2.4 *Suppose $k \geq 3$ and G does not have a k-tree. Let T be a k^+-maximal tree of G, r the vertex of T with $d_T(r) = k+1$, and U a representative set of T with respect to r. Then the following hold:*

i. *U is independent.*

ii. *For $1 \leq i \leq k+1$, if $x \in B_i \cap N_G(V_1(T) - B_i)$ then $x \neq v_i$ and $N_T^*(x) \cap N_G(u_i) = \emptyset = N_T(x) \cap N_G(V_1(T) - B_i)$.*

iii. *For any two distinct vertices x, y of $V(T) - U$, if $N_T^*(x) \cap N_T^*(y) \neq \emptyset$ then $\{x, y\} \subset N_T(r)$. In particular, $N_T^*(x) \cap (\bigcup_{i=1}^{m} N_T^*(x)) = \emptyset$ if $x \notin \bigcup_{i=2}^{k+1} N_i(U)$.*

iv. *$U, N_G(U), N_T^*(y_1), N_T^*(y_2), \ldots, N_T^*(y_m)$ are pair-wise disjoint subsets of $V(T)$.*

v. *$|T| \geq 2 + d_G(U) + \sum_{i=2}^{k+1} (k-i)|N_i(U)|$.*

Proof. (i) The result follows directly from Lemma 2.2(iii).
(ii) Let x be a vertex in B_i, and suppose, $xz \in E(G)$ for $z \in (V_1(T) - B_i$. Put $T' := (T + xz) - v_i r$. If $x = v_i$ then $\Delta(T') \leq k$ and $V(T') = V(T)$, a contradiction by Lemma 2.1, which shows $x \neq v_i$. Let x' be a vertex in $N_T(x)$. If $x'z' \in E(G)$ for some $z' \in V_1(T) - B_i$, then G has the tree $T'' = (T' + x'z') - xx'$ such that $\Delta(T'') \leq k$ and $V(T'') = V(T)$, a contradiction by Lemma 2.1. If $x' \neq x^-$ and $x'u_i \in E(G)$, then G has the tree $T'' = (T' + x'u_i) - xx'$ such that $\Delta(T'') \leq k$ and $V(T'') = V(T)$, a contradiction by Lemma 2.1. Hence (ii) holds.
(iii) The result is a direct consequence of the definition of $N_T^*(x)$ and $N_T^*(y)$.
(iv) Using (i), (ii), (iii), and the definition of $N_T^*(y_i)$, one can easily check that (iv) holds.
(v) For $1 \leq i \leq m$, it holds by Lemma 2.2(iii) that $d_T(y_i) = k$, and hence $|N_T^*(y_i)| = k-1$. Then, by (iv),

$$\begin{aligned} |T| &\geq |U| + |N_G(U)| + \sum_{i=1}^{m} |N_T^*(y_i)| \\ &\geq 1 + k + \sum_{i=1}^{k+1} |N_i(U)| + (k-1)m. \end{aligned}$$

By the definition of $m = |\bigcup_{i=2}^{k+1} N_i(U) - \{r\}|$,

$$\begin{aligned} |T| &\geq 1 + k + \sum_{i=1}^{k+1} |N_i(U)| + (k-1)((\sum_{i=2}^{k+1} |N_i(U)|) - 1) \\ &= 2 + \sum_{i=1}^{k+1} i|N_i(U)| + \sum_{i=2}^{k+1} (k-i)|N_i(U)| \\ &= 2 + d_G(U) + \sum_{i=2}^{k+1} (k-i)|N_i(U)|. \end{aligned}$$

□

3 Proof of Theorems B and C

Suppose G does not have a k-tree but satisfies the hypothesis of the Theorem C. Let T be a k^+-maximal tree of G. Assume that T has the least possible number of end vertices among all k^+-maximal trees with vertex set $V(T)$. Let r be the vertex of T with $d_T(r) = k+1$, and U a representative set of T with respect to r. Let $U = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{k+1}}\}$, where $i_1 < i_2 < \ldots < i_{k+1}$. Assume that $b = \sum_{j=1}^{k+1} i_j$ is maximum among all possible choices of T, r, and U (with fixed vertex set $V(T)$). Now we show that $d_{i_j} \leq i_j - j + 1$ for $1 \leq j \leq k+1$.

For each $v_{i_j} \in U$ and each $v_q \in N_G(v_{i_j}) - N_T(v_{i_j})$, consider the graph $T + v_{i_j}v_q$. Recall Lemmas 2.3(i) and 2.4(i) to see that $v_q \notin U$. Let v_qv_s, where $s \neq i_j$, be the edge in the unique cycle of $T + v_q$. Set $T' := (T + v_{i_j}v_q) - v_qv_s$. Clearly, $|V_1(T')| \leq |V_1(T)|$. But, $|V_1(T)| \leq |V_1(T')|$ by the choice of T. Hence, $|V_1(T)| = |V_1(T')|$ and $d_{T'}(v_s) = 1$. Then, by the choice of b, we have $s < i_j$. Therefore, for each j, $1 \leq j \leq k+1$, there are at least $d_{i_j} - 1$ vertices in $T - U$ whose indices are less than i_j. In U, there are $j-1$ vertices which indices less than i_j. Therefore, $i_j > d_{i_j} - 1 + j - 1$ or equivalently, $d_{i_j} \leq i_j - j + 1$.

Moreover, by Lemmas 2.2(ii), 2.3(i) and 2.4(i), U is a $k+1$-frame, and by Lemmas 2.3(iv) and 2.4(v),

$$|T| \geq 2 + d_G(U) + \sum_{i=2}^{k+1} (k-i)|N_i(U)|,$$

$$n - 2 \geq d_G(U) + \sum_{i=2}^{k+1} (k-i)|N_i(U)|,$$

we obtain a contradiction. The proof of Theorem C is complete.

As a simple corollary of the above theorem we get the following condition:

Corollary 1 *Suppose that for any set of* $k+1$ *vertices* $S = \{v_{i_1}, v_{i_2}, \dots, v_{i_{k+1}}\}$ *of* G, $i_1 < i_2 < \cdots < i_k + 1$ *such that* $d_{i_j} \le i_j - j + 1$, $1 \le j \le k+1$, *it holds that* $d_G(S) + \sum_{i=2}^{k+1} (k-i)|N_i(S)| \ge n-1$. *Then* G *has a* k*-tree.*

From Theorem C, we also have the following results of Aung Kyaw [4] and Min Aung and Aung Kyaw [1].

Corollary 2 *Let* G *be a connected graph and* $k(\ge 2)$ *an integer. If*

$$d_G(S) + \sum_{i=2}^{k+1} (k-i)|N_i(S)| \ge n-1$$

for every $k+1$*-frame* S *in* G, *then* G *has a* k*-tree.*

Corollary 3 *Let* $V(G) = \{v_1, v_2, \dots, v_n\}$ *and* $d_G(v_i) = d_i$ *for* $1 \le i \le n$. *Suppose, for any* k*-frame* $S = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\} \subset V(G)$, $i_1 < i_2 < \cdots < i_k$ *such that* $d_{i_j} \le i_j - j + 1$, $1 \le j \le k$, *it holds that* $d_G(S) + \sum_{i=2}^{k-1} (k-i)|N_i(S)| \ge n-1$. *Then* G *has a* k*-tree.*

Example 1. The degree sequence of a graph drawn in Figure 2.1 is $1, 1, 3, 3, 4, 4, 4$. The corresponding values of $(d_i - i - 1)$ are $1, 2, 1, 2, 2, 3, 4$. Then the minimum value of $d_G(S) + \sum_{i=2}^{3} (2-i)|N_i(S)|$ subject to the conditions of Corollary 1 is 6, hence G has a hamiltonian path. But $d_G(\{v_1, v_2, v_3\}) - |N_3(\{v_1, v_2, v_3\}| = 5$; one cannot say by Theorem A that G has a hamiltonian path.

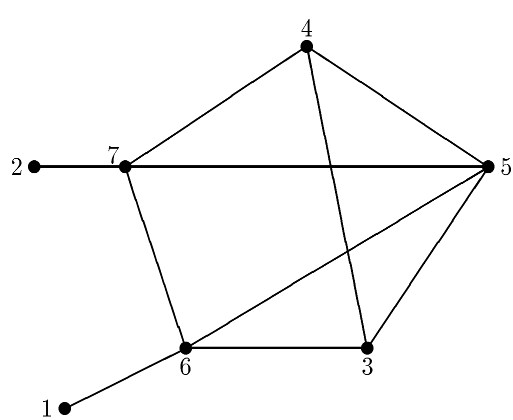

Figure 2.1

Example 2. The degree sequence of a graph drawn in Figure 2.2 is $1, 1, 1, 2, 3, 4, 4, 4$. The corresponding values of $(d_i - i - 1)$ are $1, 2, 3, 3, 3, 3, 4, 5$. Then the minimum value of $d_G(S) + \sum_{i=2}^{4} (3-i)|N_i(S)|$ subject to the conditions of Corollary 1 is 9, hence G has a 3-tree. But $d_G(\{v_1, v_2, v_3, v_4\}) + |N_2(\{v_1, v_2, v_3, v_4\}| - |N_4(\{v_1, v_2, v_3, v_4\}| = 6$; one cannot say by Corollary 2 that G has a 3-tree.

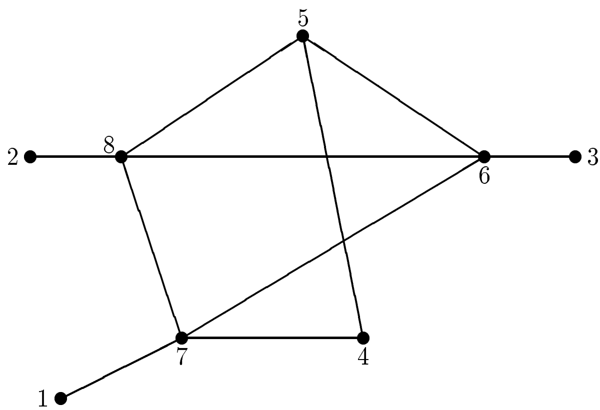

Figure 2.2

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