AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **36** (2006), Pages 197–212

New formulas for the pentomino exclusion problem*

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Abstract

Existence of constant time algorithms for the Δ-dislocation problem on fascia- and rotagraphs is proved. Application to the Pentomino exclusion problem yields formulas for $6 \times n$ and $7 \times n$ chessboards.

1 Introduction

In [8], the Δ-*dislocation problem* is defined as a generalization of the pentomino exclusion problem. For a graph $G = (V, E)$ and a positive integer Δ, a subset S of vertices of G is a Δ-*dislocation set* of G if and only if all the connected components of $G - S$ have at most Δ vertices. As $S = V(G)$ is always a Δ-dislocation set for any Δ, the optimization problem of interest is to find a set S of minimum cardinality. As observed in [8], if $\Delta = 1$, a Δ-dislocation set is a transversal of G, i.e. the complement of S is an independent set of G. The corresponding decision problem

PROBLEM: Δ-dislocation
Instance: a graph and an integer K
Question: is there a Δ-dislocation set S in G of cardinality $\leq K$?

is known [8] to be NP-complete.

In a special case when G is a $k \times n$ grid and $\Delta = 4$, the Δ-dislocation problem is known as the pentomino exclusion problem. A *polyomino* is a pattern formed by connection of a specified number of equal-sized squares along common edges. A *pentomino* is a polyomino composed of 5 squares. Among interesting problems related to pentominos is the *Pentomino Exclusion Problem* $\text{PEP}_{k\times n}$, due to Golomb

* This work was partially supported by the Ministry of Education, Science and Sport of Slovenia.

[5]. The task is to find the minimum number of unit squares to be placed on a $k \times n$ chessboard so as to exclude all pentominos.

A related decision problem is

PROBLEM: Pentomino Exclusion Problem, PEP(k, n)
Instance: positive integers k, n, and K
Question: is there a set S of positions on the $k \times n$ chessboard so that all pentominos are excluded and the cardinality of S is $\leq K$?

We will later refer to a formally slightly different problem, namely

PROBLEM: Pentomino Exclusion Problem, PEP$_k(n)$
Instance: positive integers n and K
Question: is there a set S of positions on the $k \times n$ chessboard so that all pentominos are excluded and the cardinality of S is $\leq K$?

We will give a polynomial, even constant, time algorithm for PEP$_k(n)$. On the other hand, the complexity of PEP(k, n) seems to be an open problem.

Clearly, taking $G = P_k \Box P_n$ (i.e. G is the Cartesian product of paths) and $\Delta + 1 = 5$, the problem PEP(k, n) is a special case of the Δ-dislocation problem.

The minimal number of positions on the $k \times n$ chessboard will be denoted by $\kappa_{k,n}$. More general, we will use $\kappa_\Delta(G)$ or $\kappa^{(\Delta+1)}(G)$ for the size of the smallest Δ-dislocation set on graph G. In particular, for pentominoes, $\kappa(G) = \kappa^{(5)}(G) = \kappa_4(G)$, and hence $\kappa_{k,n} = \kappa(P_k \Box P_n)$.

Bosch [3] proposed an integer programming formulation and solved the problem $\kappa_{n,n}$ for $n \leq 12$. Gravier, Moncel, and Payan [9, 8] have established the formulas for $\kappa_{k,n}$ for $k \leq 5$.

Theorem 1

$$\kappa_{k,n} = \begin{cases} \lfloor \frac{n}{5} \rfloor & \text{if } k = 1 \\ 2\lfloor \frac{n}{3} \rfloor & \text{if } k = 2 \\ n & \text{if } k = 3 \text{ and } n \geq 2 \\ \lceil \frac{3n}{2} \rceil - 1 & \text{if } k = 4 \text{ and } n \geq 4 \\ 2n - 2 & \text{if } k = 5 \text{ and } n \geq 5 \end{cases},$$

In this paper, we will give formulas for $\kappa_{k,n}$ for $k = 6$ and $k = 7$.

The formulas are results of an algorithm for solving the Δ-dislocation problem on fascia- and rotagraphs with time complexity $O(\Delta^{2k})$ where k is the size of the monograph. The complexity of the algorithm is thus independent on the number of monographs.

The rest of the paper is organized as follows. In the next section a concept of a polygraph is introduced and two special subclasses of graphs, the fasciagraphs and the rotagraphs are defined. In Section 3, the concept of a path algebra is introduced and an algorithm is recalled from [13] which can be used to solve various problems on fasciagraphs and rotagraphs. In Section 4 we give an instance of the algorithm which solves the Δ-dislocation on fasciagraphs and rotagraphs. We then prove that the powers of the matrices which correspond to the solution have a special structure, which implies existence of a constant time algorithm for computing any power. Finally, the algorithm is applied to special cases including the pentomino exclusion problem on $6 \times n$ and $7 \times n$ chessboards.

2 Polygraphs

Polygraphs are used as graph theoretical models of polymers in mathematical chemistry, see, for example [1, 14, 10, 11, 7, 6]. P_n and C_n will denote the path on n vertices and the cycle on n vertices, respectively. An edge $\{u, v\}$ of a graph will be denoted uv (hence uv and vu mean exactly the same edge). An arc from u to v in a digraph will be denoted (u, v). We consider finite undirected and directed graphs. A graph will always mean an undirected graph, a digraph will stand for a directed graph.

Let $G_1, G_2, \ldots, G_n$ be arbitrary, mutually disjoint graphs, and let $X_1, X_2, \ldots, X_n$ be a sequence of sets of edges such that an edge of X_i joins a vertex of $V(G_i)$ with a vertex of $V(G_{i+1})$. For convenience we also set $G_0 = G_n$, $G_{n+1} = G_1$ and $X_0 = X_n$. This in particular means that edges in X_n join vertices of G_n with vertices of G_1. A *polygraph*

$$\Omega_n = \Omega_n(G_1,\, G_2,\, \ldots,\, G_n;\, X_1,\, X_2,\, \ldots,\, X_n)$$

over *monographs* $G_1,\, G_2,\, \ldots,\, G_n$ is defined in the following way:

$$V(\Omega_n) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n),$$
$$E(\Omega_n) = E(G_1) \cup X_1 \cup E(G_2) \cup X_2 \cup \cdots \cup E(G_n) \cup X_n.$$

For a polygraph Ω_n and for $i = 1,\, 2,\, \ldots n$ we also define

$$D_i = \{u \in V(G_i) \mid \exists v \in G_{i+1} \,:\, uv \in X_i\},$$
$$R_i = \{u \in V(G_{i+1}) \mid \exists v \in G_i \,:\, uv \in X_i\}.$$

In general $R_i \cap D_{i+1}$ need not be empty.

Assume that for $1 \le i \le n$, G_i is isomorphic to a fixed graph G and that we have identified each G_i with G. Let in addition the sets X_i, $1 \le i \le n$, be equal to a fixed edge set $X \subseteq V(G) \times V(G)$. Then we call the polygraph a *rotagraph* and denote it $\omega_n(G; X)$. A *fasciagraph* $\psi_n(G; X)$ is a rotagraph $\omega_n(G; X)$ without edges between the first and the last copy of a monograph. Formally, in $\psi_n(G; X)$ we have $X_1 = X_2 = \cdots = X_{n-1}$ and $X_n = \emptyset$. Since in a rotagraph all the sets D_i and the sets R_i are equal, we will denote them by D and R, respectively. The same notation will be used for fasciagraphs as well, keeping in mind that R_n and D_0 are empty.

3 Path algebras and the algorithm

In this section we recall a general framework for solving different problems on the class of fasciagraphs and rotagraphs [13]. The essence of the method is a computation of powers of matrices over certain semirings. Similar ideas were implicitly used in [10, 14], and later explicitly applied to distance related invariants [11, 12, 6], graph domination, graph coloring, and others [13, 17, 7]. We follow the approach given in [4], see also [16, 19].

A *semiring* $\mathcal{P} = (P, \oplus, \circ, 0, 1)$ is a set P on which two binary operations, $\oplus$ and $\circ$, are defined such that

(i) $(P, \oplus)$ forms an commutative monoid with 0 as unit,
(ii) $(P, \circ)$ forms a monoid with 1 as unit,
(iii) operation $\circ$ is left- and right-distributive over operation $\oplus$,
(iv) for all $x \in P$, $x \circ 0 = 0 = 0 \circ x$.

An idempotent semiring (for all $x \in P$, $x \oplus x = x$) is called a *path algebra.* It is easy to see that a semiring is a path algebra if and only if $1 \oplus 1 = 1$ holds. Examples of path algebras include (for more examples we refer to [4]):

$$\begin{aligned}
\mathcal{P}_1 &: \ (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)\,, \\
\mathcal{P}_2 &: \ (\mathbb{N}_0 \cup \{-\infty\}, \max, +, -\infty, 0)\,, \\
\mathcal{P}_3 &: \ (\{0,1\}, \max, \min, 0, 1)\,.
\end{aligned}$$

Let $\mathcal{P} = (P, \oplus, \circ, 0, 1)$ be a path algebra and let $\mathcal{M}_\mu(\mathcal{P})$ be the set of all $\mu \times \mu$ matrices over P. Let $A, B \in \mathcal{M}_\mu(\mathcal{P})$ and define operations $A \oplus B$ and $A \circ B$ in the usual way:

$$\begin{aligned}
(A \oplus B)_{ij} &= A_{ij} \oplus B_{ij}\,, \\
(A \circ B)_{ij} &= \bigoplus_{k=1}^{\mu} A_{ik} \circ B_{kj}\,.
\end{aligned}$$

$\mathcal{M}_\mu(\mathcal{P})$ equipped with the above operations is a path algebra itself with the zero and the unit matrix as units of the semiring.

Let $\mathcal{P}$ be a path algebra and let $\mathcal{G}$ be a *labeled* digraph, i.e., a digraph together with a labeling function ℓ which assigns to every arc of $\mathcal{G}$ an element of P. Let $V(\mathcal{G}) = \{v_1, v_2, \ldots, v_\mu\}$. The labeling ℓ of $\mathcal{G}$ is extended to paths as follows. For a path $Q = (x_{i_0}, x_{i_1})(x_{i_1}, x_{i_2}) \cdots (x_{i_{k-1}}, x_{i_k})$ of $\mathcal{G}$ let

$$\ell(Q) \ = \ \ell(x_{i_0}, x_{i_1}) \circ \ell(x_{i_1}, x_{i_2}) \circ \cdots \circ \ell(x_{i_{k-1}}, x_{i_k}).$$

Let S_{ij}^k be the set of all paths of order k from x_i to x_j in $\mathcal{G}$ and let $A(\mathcal{G})$ be the matrix defined by $A(\mathcal{G})_{ij} = \ell(x_i, x_j)$ if (x_i, x_j) is an arc of $\mathcal{G}$ and $A(\mathcal{G})_{ij} = 0$ otherwise. Now we can state the following well-known result (see, for instance, [4, p. 99]):

Theorem 2 $(A(G)^k)_{ij} \ = \ \bigoplus_{Q \in S_{ij}^k} \ell(Q)\,.$

Let finally $\ell : E(\mathcal{G}) \to P$ be a labeling of $\mathcal{G}$ where $\mathcal{P}$ is a path algebra on the set P.

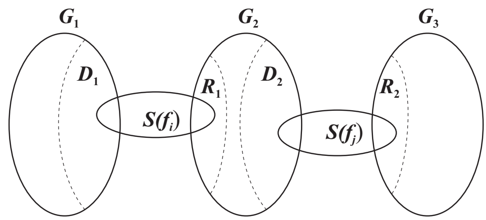

Figure 1: Three consecutive monographs.

The general scheme for computing the invariant is to calculate $A(\mathcal{G})^n$ in appropriate path algebra $\mathcal{M}_\mu(\mathcal{P})$ and select among admissible coefficients of $A(\mathcal{G})^n$ the one which optimizes the corresponding goal function.

A remark on time complexity. It is well known that, in general, computing the n-th power of a matrix can be done with $O(\log n)$ matrix multiplications. In some cases it is possible to compute the powers of the matrix $A(\mathcal{G})^n$ in constant time, i.e. with an algorithm of time complexity which is independent of n. Hence if we assume that the size of G is a given constant (and n is a variable), then the algorithm will run in constant time. Known examples include distance related invariants [12] and the domination number [17]. In the next section we will show the existence of a constant time algorithm for the problem $\mathrm{PEP}_k(n)$.

4 Δ-Dislocation problem on fasciagraphs and rotagraphs

Let $\psi_n(G;X)$ and $\omega_n(G;X)$ be a fasciagraph and a rotagraph, respectively. Set $W_i = D_i \cup R_i$. As all copies of R_i are isomorphic to R and all $D_i = D$ we can write $W_i = W$ and let V be the set of functions $W \to \{0, 1, \ldots, \Delta\}$. Note that $|\mathrm{V}| = (\Delta + 1)^{|W|}$.

Let us define a labeled digraph $\mathcal{G} = \mathcal{G}(G;X)$ as follows. The vertex set of $\mathcal{G}$ is V. The elements of V will be denoted by f_i; in particular we will use f_0 for the constant 0 function, i.e. $f_0 \equiv 0$. For a later reference we define two restrictions $D(f) = f|_D$, and $R(f) = f|_R$. Furthermore, let $S(f)$ be the set of "stones" defined by f, i.e. $S(f) = \{v \mid v \in W, f(v) = 0\}$. Function values can be understood as vertex weights on the polygraph where the weight has the following meaning: weight $f(v) = 0$ means there is a "stone" put on that vertex and positive weight gives the size of the connected component the vertex v is in. An arc joins f_i and f_j if f_i is not in a "conflict" with f_j. Here a "conflict" of f_i with f_j means that using f_i and f_j as a part of a solution in consecutive copies of X would violate the problem constraints. In particular, for the Δ-exclusion problem, the violations occur if the same vertex is assigned different values or if the sum of weights of two adjacent vertices is Δ or more.

Let f_i, $f_j \in V(\mathcal{G}(G;X))$, and consider for a moment $\psi_3(G;X)$. Let $f_i : D_1 \cup R_1 \to \{0, \ldots, \Delta\}$ and $f_j : D_2 \cup R_2 \to \{0, \ldots, \Delta\}$. Recall that all G_i and X_i (for

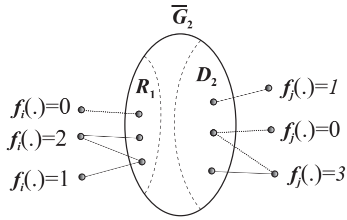

Figure 2: Encoding of boundary conditions - example.

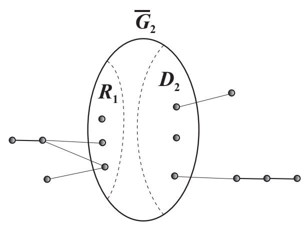

Figure 3: Boundary conditions of Fig. 2 transformed to graph patch.

$i = 1, \ldots, n-1$) are identical: $D_1 = D_2 = D$ and $R_1 = R_2 = R$ (see Fig. 1).

Let $\kappa_{ij}(G; X)$ be the size of a smallest Δ-exclusion set on G_2 such that all polyominos are excluded on G_2 provided f_i are not in conflict f_j. More precisely, $\kappa_{ij}(G; X)$ is the minimal size of a Δ-exclusion set on the graph $\overline{G_2}$ constructed as follows. For each edge $e = uv \in X_1$ such that $f_i(u) > 0$ add a path of length $f_i(u)$ to the graph and connect the first vertex of the path with the vertex $v \in G_1$. Similarly, for each edge $e = uv \in X_2$ such that $f_j(v) > 0$ add a path of length $f_j(v)$ to the graph and connect the first vertex of the path with the vertex $u \in G_\ell$ (See Fig. 2 and Fig. 3).

Then set

$$\ell(f_i, f_j) \quad = \quad |S(f_i) \cap R| + \kappa_{ij}(G; X) + |D \cap S(f_j)| - |S(f_i) \cap S(f_j)| \qquad (1)$$

Summarizing, the definition of $\ell(f_i, f_j)$ is given by the following rules

1. The weights inferred by f_i and f_j must coincide: if for $v \in V(G_2)$ both f_i and f_j are defined, then we must have $f_i(v) = f_j(v)$.
2. The weights may not be too large: if $u \sim v$ are two adjacent vertices of G_2, and the weights of u and v are given by $w(u)$ and $w(v)$, then we must have $w(u) + w(v) \leq \Delta$.
3. if either 1. or 2. is violated, then set $\ell(f_i, f_j) = \infty$, otherwise let $\ell(f_i, f_j)$ be the number of stones used on G_2, given by expression (1).

Algorithm 1

1. Label $\mathcal{G}(G;X)$ as defined by (1).

2. Calculate $A(\mathcal{G})^n$ in the path algebra $\mathcal{M}_N(\mathcal{P}_1)$, where $\mathcal{P}_1 = (\mathbb{N}_0 \cup \{\infty\}$, min, $+$, ∞, 0).

3. Let $\kappa_\Delta(\psi_n(G;X)) = \min_{L(f_i)=0,R(f_j)=0} (A(\mathcal{G})^n)_{ij}$ and $\kappa_\Delta(\omega_n(G;X)) = \min_i (A(\mathcal{G})^n)_{ii}$.

The correctness of the algorithm follows from the first two lemmas. For proofs, see [8] or [18].

Lemma 1 *Algorithm 1 correctly computes $\kappa_\Delta(\psi_n(G;X))$.*

Lemma 2 *Algorithm 1 correctly computes $\kappa_\Delta(\omega_n(G;X))$.*

We now prove a lemma which will imply the existence of a constant time algorithm for computing the powers of $A(\mathcal{G})$ in Step 2 of Algorithm 1.

Let us denote $A_\ell = A(\mathcal{G})^\ell$. The meaning of the value of $(A_\ell)_{ij}$ is the size of the Δ-exclusion set of a certain graph. More precisely, provided that the functions f_i and f_j are not conflicting, $(A_\ell)_{ij}$ is the mimimum size of a Δ-exclusion set of the graph constructed as follows. Start with a graph induced on the vertices

$$V(G_1) \cup V(G_2) \cup \ldots \cup V(G_{\ell-1}) \cup V(G_\ell).$$

For each edge $e = uv \in X_0$ such that $f_i(u) > 0$ add a path of length $f_i(u)$ to the graph and connect the first vertex of the path with the vertex $v \in G_1$. Similarly, for each edge $e = uv \in X_\ell$ such that $f_j(v) > 0$ add a path of length $f_j(v)$ to the graph and connect the first vertex of the path with the vertex $u \in G_\ell$.

It can be shown that for large enough indices ℓ, the matrices A_ℓ have a special structure that enables us to compute them efficiently. The following proposition is a variant of the "cyclicity" theorem for the "tropical" semiring $(\mathbb{N}_0 \cup \infty, \min, +, \infty, 0)$, see, e.g., [2, Theorem 3.112]. By a constant matrix we mean a matrix with all entries equal.

Lemma 3 *Let $\mu = |V(\mathcal{G}(G;X))|$, and $k = |V(G)|$. Then there is an index $q \leq (2k+2)^{\mu^2}$ such that $D_q = D_p + C$ for some index $p < q$ and some constant matrix C. Let $P = q - p$. Then for every $r \geq p$ and every $s \geq 0$ we have*

$$A_{r+sP} = A_r + sC\,.$$

Proof. First we prove the claim: *For any $\ell \geq 1$, the difference between any pair of entries of A_ℓ, both different from ∞, is bounded by $2k$.*

Assume $(A_l)_{ij} \neq \infty$. Then clearly

$$(A_l)_{ij} \geq \kappa(\psi_l(G;X))$$

and, using $|S(D(f_i))| + |S(R(f_j))| \leq 2|V(G)|$, we have

$$(A_l)_{ij} \leq \kappa(\psi_l(G;X)) + 2|V(G)|$$

Hence the claim follows.

For $\ell \geq 1$, define $K_\ell = \min\{(A_\ell)_{ij}\}$ and let $A'_\ell = A_l - (K_\ell)J$, where J is the matrix with all entries equal to 1. Since the difference between any two elements of A_ℓ, different from ∞, cannot be greater than $2k$. (Note that $\infty - x = \infty$ for any x.) The entries of A'_ℓ can therefore have only values $0, 1, \ldots, 2k, \infty$ and hence there are indices $p < q \leq (2k+2)^{\mu^2}$ such that $A'_p = A'_q$. This proves the first part of the proposition.

The equality $A_{r+sP} = A_r + sC$ follows from the fact that for arbitrary matrices D, E and a constant matrix C we have

$$(D \oplus C) \circ E = D \circ E \oplus C.$$

This can easily be seen by computing the values of ij-th entries of both sides of the equality: $(D\oplus C)\circ E)_{ij} = \min_k\{((D)_{ik}+(C)_{ik})+(E)_{kj}\} = \min_k\{(D)_{ik}+(E)_{kj}\}+(C)_{ij}$ and $(D \circ E \oplus C)_{ij} = \min_k\{(D)_{ik} + (E)_{kj}\} + (C)_{ij}$. □

Note that since $\mu = |V(\mathcal{G}(G;X))| = \Delta^{2k}$, and $k = |V(G)|$, the constants p, q and P in Lemma 3 depend only on the size of the monograph and are thus independent of n.

Lemma 4 *Algorithm 1 can be implemented to run in constant time.*

Proof. First note that time for computing each of the $\mu^2 = \Delta^{4k}$ entries of $A(\mathcal{G})$ only depends on the size of the monograph G and the number of edges in X. It is obviously independent on the number of monographs n. Thus Step 1 is of constant time complexity in n, and similarly is Step 3.

Finally, the time complexity of Step 2 is constant in n because of Lemma 3. For any n, only constant number, q, of powers of the matrix has to be computed. Then the formula $A_n = A_{n-jP} + jC$ can be used, where $j = \lfloor \frac{n-p}{P} \rfloor$. □

Combining Lemmas 1, 2 and 4 we have

Theorem 3 *Algorithm 1 correctly computes $\kappa_\Delta(\psi_n(G;X))$ and $\kappa_\Delta(\omega_n(G;X))$ and can be implemented to run in constant time.*

5 Formulas for the pentomino exclusion problem

We know that the Δ-exclusion problem on some polygraphs and thus also the pentomino exclusion problems $\mathrm{PEP}_k(n)$ can be solved in constant time. However, the algorithm is useful for practical purposes only if the number of vertices of the monograph G is relatively small, because the time complexity is exponential in the number of vertices of the monograph G.

We now briefly explain the encoding used in our implementation. The boundary condition is encoded in the matrix index as follows. The last column of the first factor and the first column of the second factor have $k + k = 2k$ fields. Each field carries a number between 0 and Δ, depending on the size of the polyomino it is a part of (in the factor). Therefore there are $(\Delta + 1)^{2k}$ possible indices.

For computing $\kappa_{6,n}$ and $\kappa_{7,n}$, the matrices have sizes $5^{12} \times 5^{12}$ and $5^{14} \times 5^{14}$, which means $5^{24} = 59604644775390625 \approx 5.9 \times 10^{16}$ and $5^{28} = 37252902984619140625 \approx 3.7 \times 10^{19}$. The size of the matrices is too large to implement the method with matrix multiplications. We therefore have computed only the necessary elements of the matrices. As the $\mathrm{PEP}_k(n)$ problem is a fasciagraph instance we only need a part of the n-th power of the matrix. Given the necessary elements of the n-th power of the matrix we have computed only those elements of the $(n + 1)$-st power, which are needed in the next step. More precisely, for the fasciagraph with n monographs, it is sufficient to compute only the rows of the matrix corresponding to the indices with $L(f_i) = 0$. Using a simple trick, we need only one row: Add two columns with all fields covered with stones to the beginning of the chessboard, and compute the κ from the second column on. Here the left index is clearly 0. To obtain the solution for fasciagraph just neglect the columns added and subtract the k stones added.

The matrix multiplication is performed without explicitly storing the elements of A^1. Instead of multiplication, we have performed a loop where the effect of adding any column to the end of any solution (i.e. any boundary condition) of a chessboard with n columns has been investigated and the cummulative results were stored in the new matrix (again with only one row) corresponding to the chessboard with $n + 1$ columns.

By Lemma 3, one should compare the last matrices in order to see when the difference is a constant matrix. As we did not store the matrices, the following straightforward lemma was applied [18]. Let $a_{ij}^{(k)}$ stand for the ij-th element of the matrix A^k.

Lemma 5 *Assume that the 0-th row of A^{n+P} and A^n differ for a constant, $a_{0i}^{(n+P)} = a_{0i}^{(n)} + C$. Then $\min_i a_{0i}^{(n+P)} = \min_i a_{0i}^{(n)} + C$.*

Therefore, in our implementation the algorithm stores computed rows and checks the difference of the new row against the previously stored rows until a constant difference, i.e. a period is detected. As the periods of the examples considered are relatively short, the approach was very effective. Note that the bounds given by Lemma 3 are much larger and searching for long periods may again cause implementation problems due to large space requirements.

For completeness, we give solutions for small n (on Fig. 4 and Fig. 6) and a table of κ values for small n not included in Theorem 4 and Theorem 5. The periodic solutions corresponding to rotagraphs (bands) are given in Fig. 5 and Fig. 7. In other words, Figs. 5 and 7 prove $\kappa(P_6 \Box C_8) \leq 19$ and $\kappa(P_7 \Box C_4) \leq 11$. An argument using the formulas for $\kappa_{6,n}$ and $\kappa_{7,n}$ gives $\kappa(P_6 \Box C_8) = 19$ and $\kappa(P_7 \Box C_4) = 11$. For example, $\kappa(P_6 \Box C_8) < 19$ would contradict Theorem 4. We omit the details.

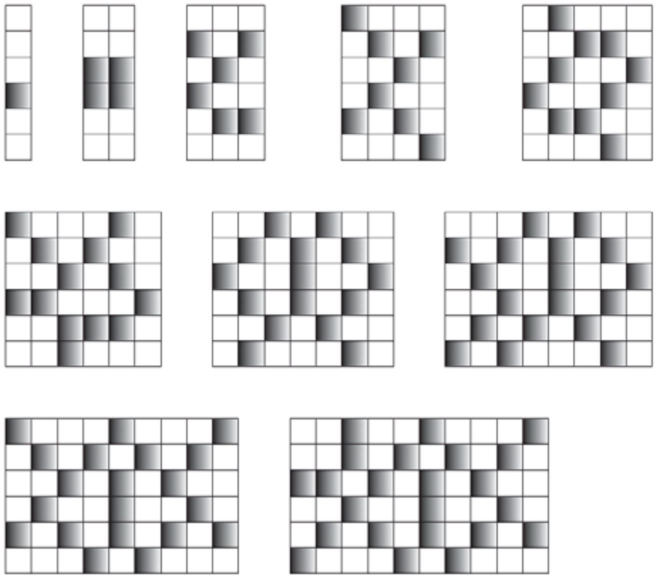

Figure 4: Pentomino exclusion on $6 \times n$ grid for some small n.

Theorem 4 *For $n \geq 6$,*

$$\kappa_{6,n} = \begin{cases} 19\lfloor\frac{(n-6)}{8}\rfloor + 18 & \text{if } n \bmod 8 = 0 \\ 19\lfloor\frac{(n-6)}{8}\rfloor + 20 & \text{if } n \bmod 8 = 1 \\ 19\lfloor\frac{(n-6)}{8}\rfloor + 23 & \text{if } n \bmod 8 = 2 \\ 19\lfloor\frac{(n-6)}{8}\rfloor + 25 & \text{if } n \bmod 8 = 3 \\ 19\lfloor\frac{(n-6)}{8}\rfloor + 27 & \text{if } n \bmod 8 = 4 \\ 19\lfloor\frac{(n-6)}{8}\rfloor + 30 & \text{if } n \bmod 8 = 5 \\ 19\lfloor\frac{(n-6)}{8}\rfloor + 13 & \text{if } n \bmod 8 = 6 \\ 19\lfloor\frac{(n-6)}{8}\rfloor + 15 & \text{if } n \bmod 8 = 7 \end{cases},$$

Theorem 5 *For $n \geq 6$,*

$$\kappa_{7,n} = \begin{cases} 11\lfloor\frac{(n-6)}{4}\rfloor + 21 & \text{if } n \bmod 4 = 0 \\ 11\lfloor\frac{(n-6)}{4}\rfloor + 23 & \text{if } n \bmod 4 = 1 \\ 11\lfloor\frac{(n-6)}{4}\rfloor + 15 & \text{if } n \bmod 4 = 2 \\ 11\lfloor\frac{(n-6)}{4}\rfloor + 17 & \text{if } n \bmod 4 = 3 \end{cases},$$

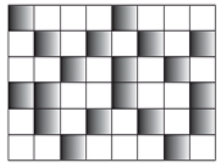

Figure 5: Pentomino exclusion on 6×8 band (rotagraph, or Cartesian product $P_6 \square C_8$).

Table 1: $\kappa_{k,n}$ values for small n not included in Theorems 1, 4, and 5.

n	1	2	3	4	5
$k=3$	0	1			
$k=4$	0	2			
$k=5$	1	2	5	7	
$k=6$	1	4	6	8	10
$k=7$	1	4	7	10	12

6 More examples

The algorithm can with obvious modifications be applied to other polyomino exclusion problems. For example, on the $5 \times n$ chessboards the number of positions needed to be covered in order to exclude all trominoes (3-ominoes), tetrominoes (4-ominoes), hexominoes (6-ominoes), and heptominoes (7-ominoes) are given below. Here $\kappa_{i,n}^{(k)}$ is the number of positions which have to be covered on a $i \times n$ grid so that all k-ominoes are excluded.

$$n \geq 4, \kappa_{5,n}^{(3)} = \begin{cases} 7\lfloor\frac{(n-4)}{3}\rfloor + 11 & \text{if } n \bmod 3 = 2 \\ 7\lfloor\frac{(n-4)}{3}\rfloor + 14 & \text{if } n \bmod 3 = 0 \\ 7\lfloor\frac{(n-4)}{3}\rfloor + 9 & \text{if } n \bmod 3 = 1 \end{cases},$$

$$n \geq 4, \kappa_{5,n}^{(4)} = \begin{cases} 13\lfloor\frac{(n-4)}{6}\rfloor + 7 & \text{if } n \bmod 6 = 4 \\ 13\lfloor\frac{(n-4)}{6}\rfloor + 9 & \text{if } n \bmod 6 = 5 \\ 13\lfloor\frac{(n-4)}{6}\rfloor + 12 & \text{if } n \bmod 6 = 0 \\ 13\lfloor\frac{(n-4)}{6}\rfloor + 14 & \text{if } n \bmod 6 = 1 \\ 13\lfloor\frac{(n-4)}{6}\rfloor + 16 & \text{if } n \bmod 6 = 2 \\ 13\lfloor\frac{(n-4)}{6}\rfloor + 18 & \text{if } n \bmod 6 = 3 \end{cases},$$

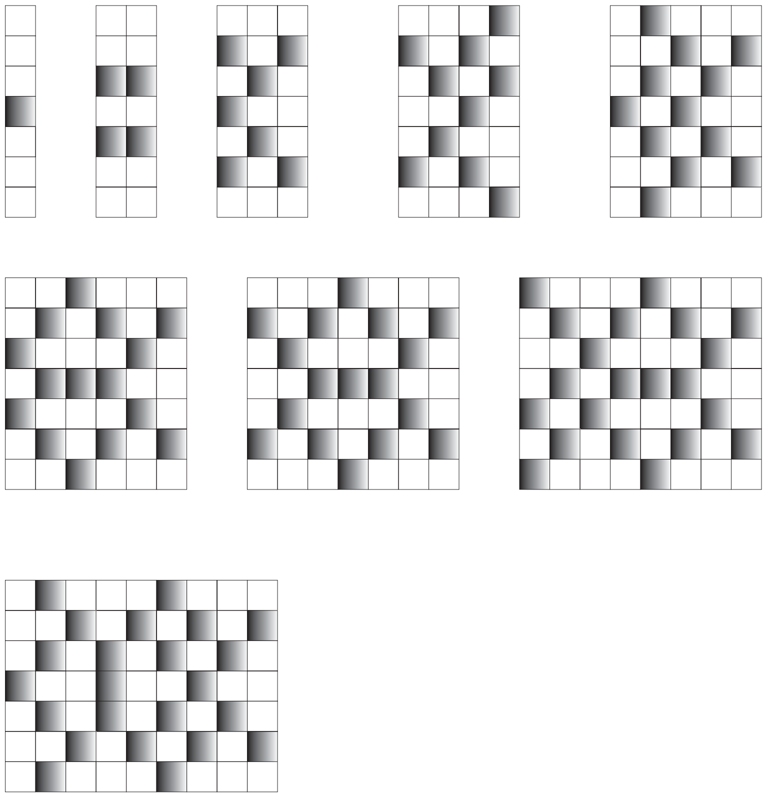

Figure 6: Pentomino exclusion on $7 \times n$ grid for some small n.

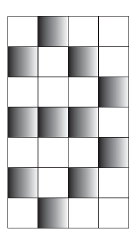

Figure 7: Pentomino exclusion on 7×4 band (rotagraph, or Cartesian product $P_7 \square C_4$).

$$n \geq 4, \kappa_{5,n}^{(6)} = \begin{cases} 7\lfloor\frac{(n-4)}{4}\rfloor + 6 & \text{if } n \bmod 4 = 0 \\ 7\lfloor\frac{(n-4)}{4}\rfloor + 8 & \text{if } n \bmod 4 = 1 \\ 7\lfloor\frac{(n-4)}{4}\rfloor + 9 & \text{if } n \bmod 4 = 2 \\ 7\lfloor\frac{(n-4)}{4}\rfloor + 10 & \text{if } n \bmod 4 = 3 \end{cases},$$

$$n \geq 6, \kappa_{5,n}^{(7)} = \begin{cases} 8\lfloor\frac{(n-6)}{5}\rfloor + 9 & \text{if } n \bmod 5 = 1 \\ 8\lfloor\frac{(n-6)}{5}\rfloor + 10 & \text{if } n \bmod 5 = 2 \\ 8\lfloor\frac{(n-6)}{5}\rfloor + 12 & \text{if } n \bmod 5 = 3 \\ 8\lfloor\frac{(n-6)}{5}\rfloor + 14 & \text{if } n \bmod 5 = 4 \\ 8\lfloor\frac{(n-6)}{5}\rfloor + 16 & \text{if } n \bmod 5 = 0 \end{cases}, \tag{2}$$

See Figs. 8, 9, 10 and 11 for some solutions for small n.

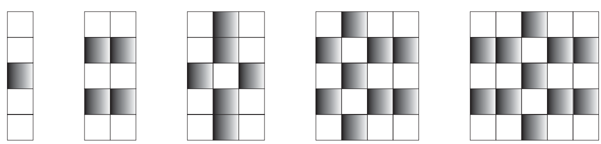

Figure 8: Triomino exclusion on $5 \times n$ grid for some small n.

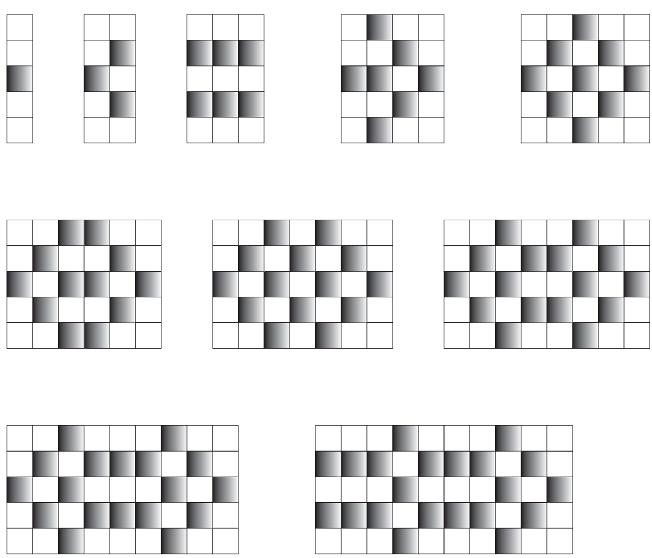

Figure 9: Tetromino exclusion on $5 \times n$ grid for some small n.

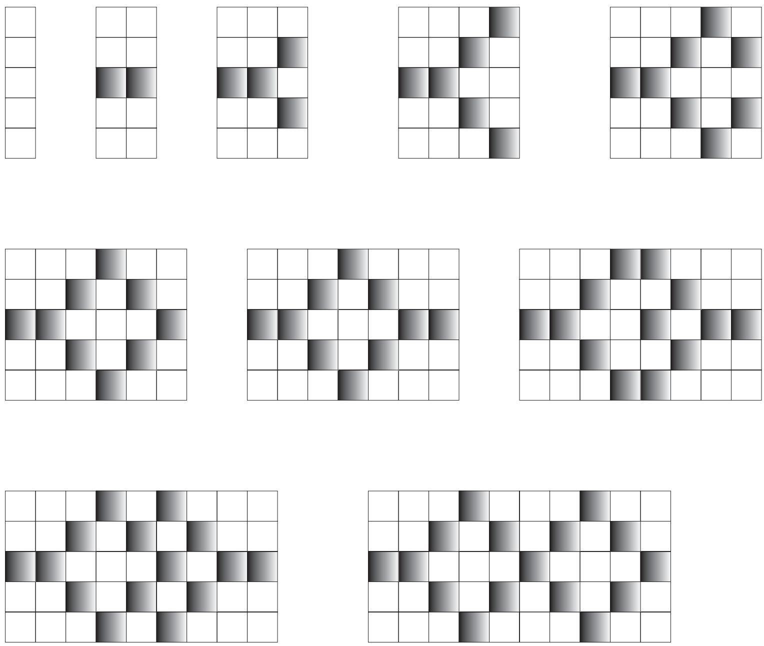

Figure 10: Heksomino exclusion on $5 \times n$ grid for some small n.

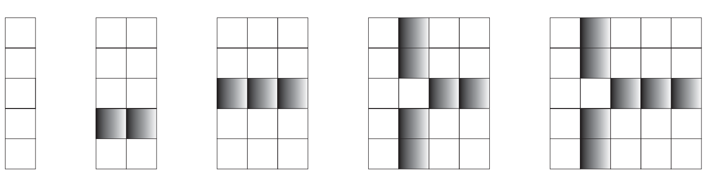

Figure 11: Heptomino exclusion on $5 \times n$ grid for some small n.

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(Received 12 June 2005)