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Embeddings of λ-fold kite systems, $\lambda \geq 2$ *

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Abstract

Necessary and sufficient conditions are given to embed a λ-fold kite system of order n into a λ-fold kite system of order m.

1 Introduction

Let $\mathcal{G}$ be a set of graphs. A *λ-fold $\mathcal{G}$-design of order n* is a pair (X, B) where B is a collection of subgraphs (*blocks*), each isomorphic to a graph of $\mathcal{G}$, which partitions the edge set of λ copies of the complete undirected graph K_n with vertex set X. If we drop the quantification "partitions" we have the definition of a λ-fold *partial* $\mathcal{G}$-design. When $\lambda = 1$ we simply say *$\mathcal{G}$-design.* When $\mathcal{G}$ contains a single graph G, the design is a *G-design.*

Let G be a graph. The λ-fold G-design (X_1, B_1) is said to be *embedded* in the λ-fold G-design (X_2, B_2) provided $X_1 \subseteq X_2$ and $B_1 \subseteq B_2$; we also say that (X_1, B_1) is a *subdesign* (or *subsystem)* of (X_2, B_2), or that (X_2, B_2) contains (X_1, B_1) as a subdesign. Let $N_\lambda(G)$ denote the set of integers n such that there exists a λ-fold G-design of order n. A question which naturally arises is the following: given $n, m \in N_\lambda(G)$, with $m > n$, and a λ-fold G-design (X, B) of order n, does there exist a λ-fold G-design of order m containing (X, B) as subdesign? Doyen and Wilson were the first to pose this problem for $G = K_3$ and $\lambda = 1$ (Steiner triple systems) and in 1973 they showed that given $n, m \in N_1(K_3) = \{v : v \equiv 1, 3 \pmod 6\}$, *a Steiner triple system of order n can be embedded in a Steiner triple system of order m if and only if $m \geq 2n + 1$ or $m = n$* (see [4]). Over the years, any such problem has come to be called a "Doyen-Wilson problem" and any solution a "Doyen-Wilson type theorem". The work along these lines is extensive ([5], [8], [2]) and the interested reader is referred to [3] for a history of this problem.

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In [7] the following theorem is proved: *Given $n, m \in N_1(G)$, where G is a kite, any kite system of order n can be embedded in a kite system of order m if and only if $m \geq \frac{5}{3}n+1$ or $m = n$.* The aim of this paper is to prove a similar result for λ-fold kite systems with any value of $\lambda \geq 2$.

2 Preliminaries and basic lemmas

A *kite* is a triangle with a tail consisting of a single edge. A *λ-fold kite system of order n* (briefly, KS(n, λ)) is a λ-fold G-design, where the graph G is a kite. It is well-known that the spectrum for λ-fold kite systems is the set of all integers n such that

(i) $n \equiv 0, 1 \pmod 8$, $n \geq 8$ for $\lambda \equiv 1 \pmod 2$,

(ii) $n \equiv 0, 1 \pmod 4$, $n \geq 4$ for $\lambda \equiv 2 \pmod 4$, and

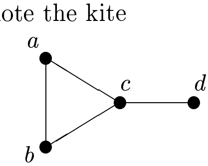
(iii) $n \geq 4$ for $\lambda \equiv 0 \pmod 4$.

It is also evident that if (X, B) is a KS(n, λ), then $|B| = \lambda\frac{n(n-1)}{8}$. From now on, we will assume throughout the paper that the integers n and m belong to the spectrum for λ-fold kite systems, and that $m > n$.

Lemma 2.1 (see [7]) *If a KS(n, λ) is embedded in a KS(m, λ), then $m \geq \frac{5}{3}n + 1$.*

Proof. Suppose (X, B) embedded in (X', B'), with $|X'| = m$. Let $u = m - n$ and c_i be the number of kites each containing exactly i edges in $X' \setminus X$. Then $c_1 + 2c_2 + 3c_3 + 4c_4 = \lambda\binom{u}{2}$ and $3c_1 + 2c_2 + c_3 = \lambda u \cdot n$ from which it follows $c_2 + 2c_3 + 3c_4 = \lambda\frac{u(3u-2n-3)}{8}$ that gives $u \geq \frac{2}{3}n + 1$ and so $m \geq \frac{5}{3}n + 1$. □

In what follows we will denote the kite



by (a, b, c)-d or (b, a, c)-d. Let (Z_n, B) be a partial KS(n, λ). For any kite $k = (a, b, c)$-$d \in B$ and any $x \in Z_n$, let $k + x = (a + x, b + x, c + x)$-$(d + x)$, where the addition is performed modulo n. (Z_n, B) is called *cyclic* if $k + x \in B$ for every $k \in B$ and every $x \in Z_n$. The set $(k) = \{k + x : x \in Z_n\}$ is called the *orbit generated by k*, and k is called a *base block* of (k).

Let S be a set. We define λS to be a multiset in which each element of S appears exactly λ times.

To solve the Doyen-Wilson problem for λ-fold kite systems we use the *difference method* (see [9], [6]). Let D_u denote the following set with elements from Z_u:

$$D_u = \begin{cases} d \,:\, 1 \leq d \leq \frac{u}{2} & \text{if } u \text{ is even;} \\ d \,:\, 1 \leq d \leq \frac{u-1}{2} & \text{if } u \text{ is odd.} \end{cases}$$

The elements of D_u are called *differences* of Z_u. For any $d \in D_u$, the set $\{\{i, i+d\} : i \in Z_u\}$, known as the *orbit* of the pair $(0, d)$ is a single 2-factor. When u is even, the orbit of $(0, \frac{u}{2})$ is the multiset containing the pairs of the 1-factor $\{\{i, i+\frac{u}{2}\} : 0 \leq i \leq \frac{u}{2} - 1\}$ repeated twice. It is also worth remarking that 2-factors obtained from distinct differences are disjoint.

Let $R = \{\infty_1, \infty_2, \dots, \infty_r\}$, $R \cap Z_u = \emptyset$. Denote by $\langle Z_u \cup R, \{d_1, d_2, \dots, d_t\}\rangle$ the graph Γ with vertex set $V(\Gamma) = Z_u \cup R$ and edge set $E(\Gamma) = \{\{x, y\} : x, y \in Z_u,\ x - y \equiv \pm d_i,\ i \in \{1, 2, \dots, t\}\} \cup \{\{\infty_j, k\} : k \in Z_u,\ 1 \leq j \leq r\}$.

Now we introduce some useful lemmas.

Lemma 2.2 (sse [7]) *Let $d \in D_u \setminus \{\frac{u}{2}\}$. The graph $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{d\}\rangle$ can be decomposed into kites.*

Proof. The subgraph $\langle Z_u, \{d\}\rangle$ is regular of degree 2 and so can be decomposed into r-cycles. Let $(x_0, x_1, \dots, x_{r-1})$ a such cycle. Consider the following kites, where the addition is performed modulo r.
If r is odd:

$$(\infty_1, x_{2i}, x_{2i+1})\text{-}\infty_3,\ 0 \leq i \leq \tfrac{r-3}{2};$$
$$(\infty_2, x_{2i+1}, x_{2i+2})\text{-}\infty_3,\ 0 \leq i \leq \tfrac{r-5}{2};$$
$$(x_{r-2}, x_{r-1}, \infty_2)\text{-}x_0;\ (\infty_3, x_0, x_{r-1})\text{-}\infty_1.$$

If r is even:

$$(\infty_1, x_{2i}, x_{2i+1})\text{-}\infty_3,\ (\infty_2, x_{2i+1}, x_{2i+2})\text{-}\infty_3,\ 0 \leq i \leq \tfrac{r-2}{2}.\ \square$$

Lemma 2.3 *Let u and s be integers such that $u > 4s$. Then there exists a cyclic partial KS$(u, 2)$, whose base blocks contain every difference $d \in \{1, 2, \dots, 2s\}$ exactly twice.*

Proof. It is a simple matter to check that the orbits of the following s kites define the kites in a cyclic partial kite system (Z_u, B).

$$(1, 2s, 0) - (u - 2s),$$
$$(2, 2s-1, 0) - (u + 2 - 2s),$$
$$\vdots$$
$$(s, s+1, 0) - (u - 2).\ \square$$

Lemma 2.4 *Let $u \equiv 0 \pmod 4$ and $d \in D_u$ be odd. The graph $2\langle Z_u \cup \{\infty_1, \infty_2\}, \{d, \frac{u}{2}\}\rangle$ can be decomposed into kites.*

Proof. Let $r = \frac{u}{gcd(u,d)}$; consider the kites $(4id+j, (4i+1)d+j, \infty_1)$-$((4i+3)d+j)$, $((4i+1)d+j, (4i+2)d+j, \infty_1)$-$((i+1)4d+j)$, $(\infty_1, (4i+2)d+j, (4i+3)d+j)$-$((i+1)4d+j)$, $(\infty_2, (4i+1)d+j, 4id+j)$-$(\frac{u}{2}+4id+j)$, $(\infty_2, (4i+2)d+j, (4i+1)d+j)$-$(\frac{u}{2}+(4i+1)d+j)$, $(\infty_2, (4i+3)d+j, (4i+2)d+j)$-$(\frac{u}{2}+(4i+2)d+j)$, $(\infty_2, (i+1)4d+j, (4i+3)d+j)$-$(\frac{u}{2}+(4i+3)d+j)$, for $0 \le i \le \frac{r}{4}-1$ and $0 \le j \le \frac{u}{r}-1$. □

Lemma 2.5 *Let $u \equiv 0 \pmod 4$ and $d \in D_u$ be odd. The graph $2\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{d, \frac{u}{2}\}\rangle$ can be decomposed into kites.*

Proof. Let $r = \frac{u}{gcd(u,d)}$; consider the kites $(4id+j, \frac{u}{2}+4id+j, \infty_1)$-$((4i+3)d+j)$, $((4i+1)d+j, \frac{u}{2}+(4i+1)d+j, \infty_1)$-$(\frac{u}{2}+(4i+3)d+j)$, $(\infty_1, \frac{u}{2}+(4i+2)d+j, (4i+2)d+j)$-$((4i+1)d+j)$, $(4id+j, (4i+1)d+j, \infty_2)$-$((4i+3)d+j)$, $((4i+1)d+j, (4i+2)d+j, \infty_2)$-$((i+1)4d+j)$, $(\infty_2, (4i+2)d+j, (4i+3)d+j)$-$(\frac{u}{2}+(4i+3)d+j)$, $(4id+j, (4i+1)d+j, \infty_3)$-$((4i+2)d+j)$, $((4i+3)d+j, (i+1)4d+j, \infty_3)$-$((4i+1)d+j)$, $(\infty_3, (4i+2)d+j, (4i+3)d+j)$-$((i+1)4d+j)$, for $0 \le i \le \frac{r}{4}-1$ and $0 \le j \le \frac{u}{r}-1$. □

Lemma 2.6 *Let $u \equiv 0 \pmod 4$ and $d \in D_u$ be odd. The graph $2\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{d, \frac{u}{2}\}\rangle$ can be decomposed into kites.*

Proof. Let $r = \frac{u}{gcd(u,d)}$; consider the kites $(4id+j, \frac{u}{2}+4id+j, \infty_1)$-$((4i+3)d+j)$, $((4i+2)d+j, \frac{u}{2}+(4i+2)d+j, \infty_1)$-$(\frac{u}{2}+(4i+3)d+j)$, $(\infty_1, \frac{u}{2}+(4i+1)d+j, (4i+1)d+j)$-$\infty_4$, $((i+1)4d+j, (4i+3)d+j, \infty_2)$-$((4i+2)d+j)$, $((4i+1)d+j, (4i+2)d+j, \infty_2)$-$((4i+3)d+j)$, $(\infty_2, (4i+1)d+j, 4id+j)$-$\infty_4$, $(4id+j, (4i+1)d+j, \infty_3)$-$((4i+2)d+j)$, $((4i+2)d+j, (4i+3)d+j, \infty_3)$-$((4i+1)d+j)$, $(\infty_3, (4i+3)d+j, (i+1)4d+j)$-$\infty_4$, $((4i+1)d+j, (4i+2)d+j, \infty_4)$-$((4i+3)d+j)$, $(\infty_4, (4i+2)d+j, (4i+3)d+j)$-$(\frac{u}{2}+(4i+3)d+j)$, for $0 \le i \le \frac{r}{4}-1$ and $0 \le j \le \frac{u}{r}-1$. □

Lemma 2.7 *Let $d \in D_u \setminus \{\frac{u}{2}\}$. The graph $2\langle Z_u \cup \{\infty\}, \{d\}\rangle$ can be decomposed into kites.*

Proof. Consider the kites $(\infty, i+d, i)$-$(u-d+i)$, $i \in Z_u$. □

Lemma 2.8 *Let $d \in D_u \setminus \{\frac{u}{2}\}$. The graph $4\langle Z_u \cup \{\infty_1, \infty_2\}, \{d\}\rangle$ can be decomposed into kites.*

Proof. Consider the kites $(\infty_2, i, d+i)$-∞_1 twice and $(\infty_1, d+i, i)$-$(u-d+i)$, $i \in Z_u$. □

Lemma 2.9 *Let u be even and $d \in D_u \setminus \{\frac{u}{2}\}$. The graph $4\langle Z_u \cup \{\infty_1, \infty_2\}, \{d, \frac{u}{2}\}\rangle$ can be decomposed into kites.*

Proof. Consider the kites $(\infty_1, d+i, i)$-∞_2, $(\infty_1, d+i, i)$-$(\frac{u}{2}+i)$, for $i \in Z_u$, and $(\infty_2, u-d+i, i)$-$(d+i)$, $(\infty_2, i, \frac{u}{2}+i)$-$(\frac{u}{2}+d+i)$, $(\infty_2, \frac{u}{2}-d+i, \frac{u}{2}+i)$-$i$, for $0 \le i \le \frac{u}{2}-1$. □

Lemma 2.10 *Let u be even and $d \in D_u \setminus \{\frac{u}{2}\}$. The graph $4\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{d, \frac{u}{2}\}\rangle$ can be decomposed into kites.*

Proof. Consider the kites $(\infty_2, d+i, i)$-∞_3, $(\infty_3, d+i, i)$-∞_1, for $i \in Z_u$, and $(\infty_1, \frac{u}{2}+i, i)$-$(d+i)$, $(\infty_2, i, \frac{u}{2}+i)$-$(\frac{u}{2}+d+i)$, $(\infty_3, \frac{u}{2}+i, i)$-$\infty_2$, $(\infty_1, d+i, i)$-$(\frac{u}{2}+i)$, $(\infty_1, \frac{u}{2}+d+i, \frac{u}{2}+i)$-$\infty_2$, for $0 \le i \le \frac{u}{2}-1$. □

Lemma 2.11 *Let u be even and $d \in D_u \setminus \{\frac{u}{2}\}$. The graph $4\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{d, \frac{u}{2}\}\rangle$ can be decomposed into kites.*

Proof. Consider the kites $(\infty_1, d+i, i)$-∞_4, $(\infty_2, d+i, i)$-∞_3, $(\infty_3, d+i, i)$-∞_2, for $i \in Z_u$, and $(\infty_1, \frac{u}{2}+i, i)$-$\infty_3$, $(\infty_4, i, \frac{u}{2}+i)$-$\infty_3$, $(\infty_4, d+i, i)$-$(\frac{u}{2}+i)$, $(\infty_4, \frac{u}{2}+d+i, \frac{u}{2}+i)$-$\infty_1$, $(\infty_2, \frac{u}{2}+i, i)$-$\infty_1$, for $0 \le i \le \frac{u}{2}-1$. □

3 The case $\lambda = 2$

Let (R, B) be a KS$(n, 2)$, $m \equiv 0, 1 \pmod 4$, $m > n$, and $u = m - n$; we note that if u is even then $u \equiv 0 \pmod 4$.

For the sake of convenience, we classify the necessary condition in Lemma 2.1 as follows.

1. if $n = 12k$, $k > 0$, then $m = 20k + 4s + \alpha + 1$, with $\alpha \in \{0, 3\}$;
2. if $n = 12k + 1$, $k > 0$, then $m = 20k + 4s + \alpha + 4$, with $\alpha \in \{0, 1\}$;
3. if $n = 12k + 4$, then $m = 20k + 4s + \alpha + 8$, with $\alpha \in \{0, 1\}$;
4. if $n = 12k + 5$, then $m = 20k + 4s + \alpha + 12$, with $\alpha \in \{0, 1\}$;
5. if $n = 12k + 8$, then $m = 20k + 4s + \alpha + 16$, with $\alpha \in \{0, 1\}$;
6. if $n = 12k + 9$, then $m = 20k + 4s + \alpha + 16$, with $\alpha \in \{0, 1\}$.

Step 1: *u even*

Proposition 3.1 *Any KS$(n, 2)$ can be embedded in a KS$(m, 2)$ for every $m \ge \frac{5}{3}n+1$ such that $u = m - n$ is even.*

Proof. Let $R = \{\infty_1, \infty_2, \ldots, \infty_{12k+a}\}$ and $\frac{u}{2} = 4k + 2s + b$, with $(a, b) \in \{(0, 2), (1, 2), (4, 2), (5, 4), (8, 4), (9, 4)\}$. (Note that $\frac{u}{2} \geq 2s + 2$.)
By using the base blocks of Lemma 2.3 we obtain a cyclic partial KS(u, 2), say (Z_u, B_1), which partitions $2\langle Z_u, D\rangle$, where $D = \{1, 2, \ldots, 2s\}$. By Lemmas 2.4, 2.5, or 2.6 arrange the differences $\frac{u}{2}$ and $\frac{u}{2} - 1$ with t different infinite points, to obtain a decomposition of $2\langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_t\}, \{\frac{u}{2} - 1, \frac{u}{2}\}\rangle$ into a collection of kites, say B_2. By Lemmas 2.7 and 2.2 it is possible to decompose the remaining differences of D_u into two sets X of size $x = b - 3 + \frac{b-a+t}{2}$ and Y of size $y = 4k + 1 - \frac{b-a+t}{2}$, with $(a, b, t) \in \{(0, 2, 2), (1, 2, 3), (4, 2, 4), (5, 4, 3), (8, 4, 2), (9, 4, 3)\}$, such that each difference in X is arranged with one infinite point and each difference in Y with three infinite points, respectively, and $x + 3y = 12k + a - t$. The result is a partial KS$(u, 2)$, say (Z_u, B_3). $\cup_{j=1}^{3} B_j$ is a decomposition of $2\langle Z_u \cup R, D_u\rangle$ into kites and so $(R \cup Z_u, B \cup (\cup_{j=1}^{3} B_j))$ is a KS$(m, 2)$ which contains (R, B) as a subsystem. □

Step 2: *u odd*

Proposition 3.2 *Any KS*$(n, 2)$ *can be embedded in a KS*$(m, 2)$ *for every* $m \geq \frac{5}{3}n+1$ *such that* $u = m - n$ *is odd.*

Proof. Let $R = \{\infty_1, \infty_2, \ldots, \infty_{12k+a}\}$ and $\frac{u-1}{2} = 4k + 2s + b$, with $(a, b) \in \{(0, 0), (1, 1), (4, 2), (5, 3), (8, 4), (9, 3)\}$.
By using the base blocks of Lemma 2.3 we obtain a cyclic partial KS$(u, 2)$, say (Z_u, B_1), which partitions $2\langle Z_u, D\rangle$, where $D = \{1, 2, \ldots, 2s\}$. By Lemmas 2.7 and 2.2 it is possible to decompose the remaining differences of D_u into two sets X of size $x = b + \frac{b-a}{2}$ and Y of size $y = 4k - \frac{b-a}{2}$, such that each difference in X is arranged with one infinite point and each difference in Y with three infinite points, respectively, and $x + 3y = 12k + a$. The result is a partial KS$(u, 2)$, say (Z_u, B_2). $B_1 \cup B_2$ is a decomposition of $2\langle Z_u \cup R, D_u\rangle$ into kites and so $(R \cup Z_u, B \cup B_1 \cup B_2)$ is a KS$(m, 2)$ which contains (R, B) as a subsystem. □

Combining Lemma 2.1 and Propositions 3.1 and 3.2 gives the following:

Theorem 3.1 *Any KS*$(n, 2)$ *can be embedded in a KS*$(m, 2)$ *if and only if* $m \geq \frac{5}{3}n+1$ *or* $m = n$.

4 The case $\lambda = 3$

Let (R, B) be a KS$(n, 3)$, $m \equiv 0, 1 \pmod 8$, $m \geq \frac{5}{3}n + 1$, and $u = m - n$. If B_1 is a decomposition of $\langle Z_u \cup R, D_u\rangle$ into kites (see [7]), then $3B_1$ partitions $3\langle Z_u \cup R, D_u\rangle$ and so $(R \cup Z_u, B \cup 2B_1)$ is a KS$(m, 3)$ which contains (R, B) as a subsystem. Therefore, the following result can be proved:

Theorem 4.1 *Any KS*$(n, 3)$ *can be embedded in a KS*$(m, 3)$ *if and only if* $m \geq \frac{5}{3}n+1$ *or* $m = n$.

5 The case $\lambda = 4$

Let (R, B) be a KS$(n, 4)$ and $m \geq \frac{5}{3}n + 1$. Write $n = 3k + a$, where $a = 0, 1, 2$ and $k \geq 1$, and $m = 5k + 2a + 1 + 4s + b$, where $b = 0, 1, 2, 3$. We note that: if $(a, b) \in \{(0,1), (0,3), (1,0), (1,2), (2,1), (2,3)\}$, then $u = m - n$ is even; while if $(a, b) \in \{(0,0), (0,2), (1,1), (1,3), (2,0), (2,2)\}$ $u = m - n$ is odd.

Case 1: *u even*

Proposition 5.1 *Any KS$(n, 4)$ can be embedded in a KS$(m, 4)$ for every $m \geq \frac{5}{3}n+1$ such that $u = m - n$ is even.*

Proof. Let $R = \{\infty_1, \infty_2, \ldots, \infty_{3k+a}\}$ and $\frac{u}{2} = k + 2s + \frac{a+b+1}{2}$, with $(a, b) \in \{(0,1), (0,3), (1,0), (1,2), (2,1), (2,3)\}$. (Note that $\frac{u}{2} \geq 2s + 2$.)
By using twice the base blocks of Lemma 2.3 we obtain a cyclic partial KS$(u, 4)$, say (Z_u, B_1), which partitions $4\langle Z_u, D\rangle$, where $D = \{1, 2, \ldots, 2s\}$. By Lemmas 2.9, 2.10, or 2.11 arrange the differences $\frac{u}{2}$ and d, $d \notin \{1, 2, \ldots, 2s, \frac{u}{2}\}$, with t different infinite points, to obtain a decomposition of $4\langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_t\}, \{d, \frac{u}{2}\}\rangle$ into a collection of kites, say B_2. By Lemmas 2.7 and 2.2 it is possible to decompose the remaining differences of D_u into two sets X of size $x = b - 2 + t - \frac{b-a+1+2t}{4}$ and Y of size $y = k + 1 - \frac{b-a+1+2t}{4}$, $(a, b, t) \in \{(0,1,3), (0,3,2), (1,0,4), (1,2,3), (2,1,4), (2,3,3)\}$, such that each difference in X is arranged with one infinite point and each difference in Y with three infinite points, respectively, and $x + 3y = 3k + a - t$. The result is a partial KS$(u, 4)$, say (Z_u, B_3). $\cup_{j=1}^{3} B_j$ is a decomposition of $4\langle Z_u \cup R, D_u\rangle$ into kites and so $(R \cup Z_u, B \cup (\cup_{j=1}^{3} B_j))$ is a KS$(m, 4)$ which contains (R, B) as a subsystem. □

Case 2: *u odd*

Proposition 5.2 *Any KS$(n, 4)$ can be embedded in a KS$(m, 4)$ for every $m \geq \frac{5}{3}n+1$ such that $u = m - n$ is odd.*

Proof. Let $R = \{\infty_1, \infty_2, \ldots, \infty_{3k+a}\}$ and $\frac{u-1}{2} = k + 2s + \frac{a+b}{2}$, with $(a, b) \in \{(0,0), (0,2), (1,1), (1,3), (2,0), (2,2)\}$.
By using twice the base blocks of Lemma 2.3 we obtain a cyclic partial KS$(u, 4)$, say (Z_u, B_1), which partitions $4\langle Z_u, D\rangle$, where $D = \{1, 2, \ldots, 2s\}$. By Lemmas 2.7, 2.2, and 2.8 it is possible to decompose the remaining differences of D_u into three sets X of size $x = b - \frac{b-a+2z}{4}$, Y of size $y = k - \frac{b-a+2z}{4}$, and Z of size z, with $(a, b, z) \in \{(0,0,0), (0,2,1), (1,1,0), (1,3,1), (2,0,1), (2,2,0)\}$, such that each difference in X is arranged with one infinite point, each difference in Y with three infinite points, each difference in Z with two infinite points, respectively, and $x + 3y + 2z = 3k + a$. The result is a partial KS$(u, 4)$, say (Z_u, B_2). $B_1 \cup B_2$ is a decomposition of $4\langle Z_u \cup R, D_u\rangle$ into kites and so $(R \cup Z_u, B \cup B_1 \cup B_2)$ is a KS$(m, 4)$ which contains (R, B) as a subsystem. □

Combining Lemma 2.1 and Propositions 5.1 and 5.2 gives the following:

Theorem 5.1 *Any KS$(n, 4)$ can be embedded in a KS$(m, 4)$ if and only if $m \geq \frac{5}{3}n+1$ or $m = n$.*

6 Main Theorem

Main Theorem *Any KS(n,λ) can be embedded in a KS(m,λ) if and only if $m \geq \frac{5}{3}n+1$ or $m=n$.*

Proof. The necessary condition follows from Lemma 2.1. Now, let (R,B) be a KS(n,λ), $m \geq \frac{5}{3}n+1$, and $u=m-n$. Write $\lambda = 4h+r$, $r=0,1,2,3$, and let B_1 and B_2 be two collections of kites decomposing $4\langle Z_u \cup R, D_u\rangle$ and $r\langle Z_u \cup R, D_u\rangle$, respectively (see [7] and Sections 3, 4, and 5). Then $(R \cup Z_u, B \cup hB_1 \cup B_2)$ is a KS(m,λ) containing (R,B) as a subsystem. □

A KS(n,λ) with no repeated blocks is called *simple*. An open problem is to find out whether or not it is possible to embed a simple KS(n,λ) into a simple KS(m,λ).

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