AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **36** (2006), Pages 39–68

Universal obstructions for embedding extension problems

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Abstract

Let K be an induced non-separating subgraph of a graph G, and let B be the bridge of K in G. Obstructions for extending a given 2-cell embedding of K to an embedding of G in the same surface are considered. It is shown that it is possible to find a nice obstruction which means that it has bounded branch size up to a bounded number of "almost disjoint" millipedes. Moreover, B contains a nice subgraph $\tilde{B}$ with the following properties. If K is 2-cell embedded in some surface and F is a face of K, then $\tilde{B}$ admits exactly the same types of embeddings in F as B. A linear time algorithm to construct such a universal obstruction $\tilde{B}$ is presented. At the same time, for every type of embeddings of $\tilde{B}$, an embedding of B of the same type is determined.

1 Introduction

Let K be a subgraph of a graph G. A *K-bridge* (or a *bridge* of K) in G is a subgraph of G which is either an edge $e \in E(G)\backslash E(K)$ (together with its endvertices) which has both endvertices in K, or a connected component of $G - V(K)$ together with all edges (and their endvertices) between this component and K. Each edge of a K-bridge B having an endvertex in K is a *foot* of B. Vertices of $B \cap K$ are the *vertices of attachment* of B. A vertex of K of degree different from 2 is a *main vertex* of K. For convenience, if a connected component of K is a cycle, then we choose an arbitrary vertex of it and declare it to be a main vertex of K as well. A *branch* of K is any path in K (possibly closed) whose endvertices are main vertices but no internal vertex on this path is a main vertex. If a K-bridge is attached to a

[*] Supported in part by the Ministry of Science and Technology of Slovenia, Research Program P1-0297.

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single branch of K, it is said to be *local*. The number of branches of K is called the *branch size* of K.

Let $K \subseteq G$, and suppose that we are given a 2–cell embedding of K in some surface. The *embedding extension problem* asks whether it is possible to extend the given embedding of K to an embedding of G in the same surface. In one of the strategies how to solve this problem [2, 6, 7, 19, 16], it is important to decide in which faces of K one can embed each of the K–bridges. A particular case of the embedding extension problem is solved by Juvan and Mohar [15]. They obtained a linear time algorithm that solves the embedding extension problem in case when every K–bridge is restricted (or allowed) to have only two essentially different embeddings in the faces of K. In particular, this algorithm solves the embedding extension problem in case when no face of K is singular. Clearly, if an embedding extension exists, each K–bridge B must be embeddable in at least one of the faces of K. If B cannot be embedded in any face, then we want a simple certificate for this. We provide such a certificate in the form of an *obstruction* — a subgraph Ω of B that has embeddings exactly in those faces of K in which B can be embedded. It is convenient if the branch size of Ω is bounded. Then it is easy to check that Ω has no embedding in faces for which it is claimed that B cannot be embedded in. Our main result, Theorem 5.1, characterizes obstructions within a single bridge. It is stated in a more general and abstract form than just in terms of embedding extensions and shows that also such more general obstructions are nice. Moreover, we devise a linear time algorithm that for all K–bridges simultaneously determines all faces of K in which each of them is embeddable. The algorithm also determines all such embeddings, whenever they exist, and for every K–bridge B provides an obstruction $\Omega = \Omega(B)$ contained in B that has exactly the same types of embeddings as B. Moreover, any embedding of Ω in a face of K can be extended to an embedding of B of the same type. The time used by the algorithm is bounded by cn where $n = |E(G)|$, and c is a number that depends only on the branch size of K. Moreover, Ω is either bounded (i.e., the branch size of $K \cup \Omega$ is bounded by a constant depending only on the branch size of K), or it has a special structure. In the latter case, Ω can be written as the union of a graph Ω_0 of bounded branch size and a bounded number of (almost) disjoint subgraphs $M_1, \ldots, M_t$ (called millipedes). Each of the millipedes M_i is attached only to a distinct segment of a branch of K. It is possible to replace some of the branches of K by other branches joining the same endvertices so that our obstruction becomes a subgraph of bounded branch size in the complement of the new graph.

The case when K has only non-singular faces is relatively easy to deal with. All the main ideas for this case are presented in [16]. The problem becomes more complicated when we have doubly or multiply singular faces. Our results are used in a linear time algorithm for testing (and constructing) embeddability of graphs in any fixed surface [19].

Embeddings in surfaces can be described combinatorially [9] by specifying a *rotation system* (for each vertex v of the graph G we have the cyclic permutation π_v of its neighbors, representing their circular order around v on the surface) together with a *signature* $\lambda : E(G) \to \{-1, 1\}$ having the property that a cycle of G has an odd number of edges e with $\lambda(e) = -1$ if and only if the cycle is one-sided on the surface (i.e., it has a regular neighborhood which is homeomorphic to the Möbius band). In order to make a clear presentation of our algorithm, we use this description only implicitly. Whenever we say that we have an embedding (either given, obtained by some other algorithm, or produced inductively by our algorithm) we mean that we have such a combinatorial description. Whenever used, it is easy to see how one can combine embeddings of some subgraphs of the graph into an embedding of larger subgraphs.

Concerning the time complexity of our algorithms, we assume a random-access machine (RAM) model with unit cost for basic operations. This model of computation was introduced by Cook and Reckhow [5]. It is known as the *unit-cost* RAM where operations on integers, whose value is $O(n)$, need only constant time (n is the order of the given graph).

There are linear time algorithms which for a given graph determine whether the graph can be embedded in the 2-sphere. The first such algorithm was obtained by Hopcroft and Tarjan [11] in 1974. There are several other linear time planarity algorithms (e.g., [3, 8, 25]). Extensions of original algorithms produce also an embedding (rotation system) whenever the given graph is found to be planar [4], or find an obstruction — a forbidden *Kuratowski subgraph* homeomorphic to K_5 or $K_{3,3}$ — if the graph is found to be non-planar [25, 26].

It is known [23] that the general problem of determining the genus, or the non-orientable genus of graphs is NP-hard. However, for every fixed surface there is a polynomial time algorithm which checks if a given graph can be embedded in the surface. Such algorithms were found first by Filotti et al. [7]. For a fixed orientable surface Σ of genus g they discovered an algorithm with time complexity $O(n^{\alpha g+\beta})$ (α, β are constants) which tests if a given graph of order n can be embedded in Σ. Djidjev and Reif [6] replaced the exponent depending on g by a constant. For every fixed surface, there is an $O(n^3)$ algorithm using graph minors (Robertson and Seymour [20, 21, 22]). A constructive version is described by Archdeacon in [1] with the running time proportional to n^{10}. We met the embedding extension problem, whose special case is treated in this paper (i.e., the embedding extension for a single K-bridge), when looking for more efficient algorithms for embedding graphs on surfaces. A considerable progress has been made by the discovery of linear time algorithms for the case of the projective plane [16] and the torus [13]. The latter one and its extension to an algorithm for embedding graphs in general surfaces [19] use the results of this paper to replace possibly large bridges with their subgraphs of bounded branch size and to determine the faces and the types of embeddings in which each particular bridge can be embedded.

2 Types of embeddings

Let G be a connected graph, and K a subgraph of G with given 2-cell embedding in a surface Σ. Let F be a face of K. A branch e of K is *singular* in F if it appears twice on the facial walk ∂F of F. If x is a main vertex of K on ∂F, let $\mathrm{sing}(x, F)$

denote the number of appearances of x on ∂F decreased by one. Then x is *singular* if $\mathrm{sing}(x, F) \geq 1$, and the number $\mathrm{sing}(x, F)$ is called the *degree of singularity* of x in F. The *degree of singularity* $\mathrm{sing}(F)$ of F is equal to the sum of $\mathrm{sing}(x, F)$, where x is an arbitrary main vertex of K on ∂F, plus the number of singular branches on ∂F. The face F is *s–singular* if $\mathrm{sing}(F) = s$.

Let B be a K–bridge. Two embeddings of B in a face F of K are *essentially different* if there is a point on ∂F such that B is attached to it under one of the embeddings, and is not attached to it under the other embedding. We will only distinguish embeddings that are essentially different. If no two embeddings of B in F are essentially different, then B is said to have *essentially unique* embedding in F. This is true, in particular, when $\mathrm{sing}(F) = 0$.

Let e be a branch of K with at least one interior vertex. Then e with its end-vertices removed is called an *open branch*. Main vertices and open branches of K are called *basic pieces* of K. We will also use *oriented basic pieces*. They correspond to basic pieces with the distinction that open branches bear an orientation. Thus, every open branch gives rise to two oppositely oriented basic pieces. If one of them is σ, the other one is denoted by σ^-. Clearly, $(\sigma^-)^- = \sigma$.

Suppose now that B is embedded in a face F of K in Σ. Let $\sigma_1, \sigma_2, \ldots, \sigma_t$ be those appearances of oriented basic pieces on ∂F that B is attached to, enumerated and oriented in the same order as they appear on ∂F. (Note that some appearances of basic pieces on ∂F may not participate in this sequence.) The embedding of B determines the sequence $\sigma_1, \ldots, \sigma_t$ up to cyclic shifts and up to its reflection. Denote by $\lceil \sigma_1, \ldots, \sigma_t \rfloor$ the set of all sequences $\sigma_i, \sigma_{i+1}, \ldots, \sigma_t, \sigma_1, \ldots, \sigma_{i-1}$ and their inverses, $\sigma_{i-1}^-, \sigma_{i-2}^-, \ldots, \sigma_1^-, \sigma_t^-, \ldots, \sigma_i^-$, $1 \leq i \leq t$. Then we say that the given embedding of B in F is of *type* $\Delta = \lceil \sigma_1, \ldots, \sigma_t \rfloor$. Types of embeddings of B can also be considered in cases when B has no appropriate embedding of that type. Types of embeddings of B are partially ordered: We say that (an embedding of) type Δ' is *simpler* than (an embedding of) type Δ, $\Delta' \preceq \Delta$, if Δ' contains a sequence that is a subsequence of an element of Δ. If $\Delta' \preceq \Delta$ and if B has an embedding of type Δ', then this embedding will also be considered as an embedding of type Δ.

The above and all the succeeding definitions also hold if Δ is a type that does not necessarily correspond to a face of some 2-cell embedding of K. It is required that every basic piece of K that B is attached to appears in Δ at least once and that no other basic pieces of K appear in Δ. Moreover, every open branch appears in Δ at most twice (with the same or opposite orientations). Given $\Delta = \lceil \sigma_1, \ldots, \sigma_t \rfloor$, we define a (hypothetical) *face* F corresponding to Δ as follows. Take basic pieces $\sigma_1, \ldots, \sigma_t$ of Δ and join them into a cycle in the obvious way (respecting orientations of oriented open branches). We will denote this cycle by ∂F. Paste a disk on ∂F to get the face F. An embedding of B in F (necessarily an embedding of type Δ) is any embedding of B in F with the obvious identification of ∂F with the vertices of attachment of B. Let σ be a basic piece that B is attached to. The *degree of singularity* of σ in Δ is defined as the number of occurrences of σ and σ^- in any of the sequences in Δ minus 1. Open branches have degree of singularity in Δ at most 1. Δ is *admissible* if the degree of singularity of any main vertex x of K is at most $\deg_K(x) - 1$. Every type of embeddings of B with respect to a face F of a 2-cell embedding of K is admissible but the converse is not necessarily true. Define the *degree of singularity* of the type Δ, $\text{sing}(\Delta)$, as the sum of degrees of singularities of basic pieces in Δ.

The treatment of types of embeddings of B has several advantages over considering actual 2-cell embeddings of K and its faces, although our main applications concern only the latter case. The approach by using types is more combinatorial and also more general. It will also become clear that most of our results will not really depend on K. Only the way how B is attached to main vertices and branches of K is important.

Lemma 2.1 *Suppose that K has no isolated vertices. Then the degree of singularity of every admissible type Δ for embeddings of B satisfies:*

$$0 \leq \text{sing}(\Delta) \leq 3\epsilon_0 - \nu_0 \,, \tag{1}$$

where ϵ_0 and ν_0 denote the number of (closed) branches and main vertices of K, respectively, that are intersecting B. The number of admissible types for embeddings of B is at most

$$4^{\epsilon_0}(4\epsilon_0)! \,. \tag{2}$$

Proof. The bound on $\text{sing}(\Delta)$ is easy since the sum of degrees of all the main vertices of K is equal to twice the number of branches of K. Let k be the number of basic pieces that B is attached to, and let m be the number of (open) branches among them. The number of types with degree of singularity s is at most $4^m(k+s-1)!$. Thus the total number of types is bounded above by:

$$\sum_{s=0}^{3\epsilon_0-\nu_0} 4^m(k+s-1)! \leq 4^{\epsilon_0}(3\epsilon_0-\nu_0+1)(4\epsilon_0-1)!$$

which yields (2). □

Lemma 2.2 *Suppose that B is attached to $k \geq 2$ basic pieces of K, m of which are (open) branches. Then the number of admissible types with the degree of singularity bounded by d is at most $4^m(k+d)!$.*

Proof. The bound is easily established by using the same method as above. □

3 Obstructions

Throughout this section, we let Δ be a type of embeddings of B and F a hypothetical face corresponding to Δ. We will also assume that B is attached to all vertices on the boundary of F. Several objects (numbers, sets, or graphs) will be referred to as *bounded*. This means that they are bounded above (for sets and graphs their cardinality and the branch size is bounded, respectively) by a certain constant C°

that depends only on the degree of singularity of Δ. We are allowed to change our mind (a few times) about the choice of C°. Thus, any bounded number of arithmetic operations performed on bounded numbers gives rise to a bounded result.

An *obstruction* in B is a subgraph Ω of B. If Ω has no embedding with certain properties, then Ω is said to *obstruct* embeddings of B with these properties. Usually, Ω will have no embeddings of type Δ, in which case Ω obstructs embeddings of this type. An obstruction Ω is *bounded* if $\Omega \cup \partial F$ has bounded branch size. Since we are assuming that $\mathrm{sing}(\Delta)$ is bounded, this is equivalent to the requirement that Ω has bounded branch size and bounded number of feet.

To measure the size of $\Omega \subseteq B$ we will use the number $b(\Omega)$ which is equal to the number of branches of $K \cup \Omega$ that are contained in Ω. If K is connected, or if K has no vertices of degree 1, then Ω is bounded if and only if $b(\Omega)$ is bounded.

Let Φ_0 be the set of feet of B. If B is embedded in F, then the feet of Φ_0 are cyclically ordered according to their attachment on ∂F. Their order is well determined also when several feet attach to the same appearance of a vertex on ∂F. According to this motivation, we will consider *cyclic sequences* of feet of B. Let Ω be a subgraph of B with feet $\Phi \subseteq \Phi_0$, and let Π be a cyclic sequence of Φ. If Ω has an embedding in F whose cyclic order of feet agrees with Π, then Ω has an embedding in F for all placements of feet on ∂F whose cyclic order is Π or Π^{-1} (the cyclic inverse). If Ω has no embedding in F with the cyclic order of its feet equal to Π, then Ω is said to be an *obstruction for the cyclic sequence* Π in F since it obstructs embeddings of B in F inducing the cyclic order Π on Φ. Let us say that Π is Δ–*admissible* if the induced cyclic order of vertices of attachment of Ω corresponding to Π is a subsequence of vertices on ∂F (in one or the other direction). Clearly, cyclic sequences that are not Δ–admissible are obstructed in F. On the other hand, if Ω obstructs a Δ–admissible cyclic sequence Π in F, then Ω obstructs Π in any other (hypothetical) face. Then we say that Ω *obstructs* Π, or that Π is *obstructed* by Ω. If every Δ–admissible cyclic sequence of Φ is obstructed by Ω, then, clearly, Ω has no embedding of type Δ (and vice versa). Therefore, any obstruction for embeddings of type Δ can also be viewed as an obstruction for Δ–admissible cyclic sequences of feet. It will be convenient to view Ω also as an obstruction for some cyclic sequences of feet distinct from Φ. Ω is said to *obstruct* a cyclic sequence Π' of feet $\Phi' \supseteq \Phi$ if Π' induces an obstructed cyclic sequence of Φ. Also, Ω *obstructs* a cyclic sequence Π'' of $\Phi'' \subseteq \Phi$ if it obstructs every cyclic sequence Π of Φ which induces Π''. The obstruction Ω is a *total obstruction* if it obstructs every cyclic sequence of its (and thus also of any other set of) feet.

In Figure 1, two examples of obstructions are presented. The first one obstructs α and β from being attached to different occurrences of the vertex x on ∂F, i.e., it obstructs cyclic sequences $\lceil \alpha, \gamma, \beta, \delta \rfloor$. The second one obstructs α, γ for being embeddable on the same side (and similarly β and δ). In other words, the cyclic sequences in $\lceil \alpha, \gamma, \beta, \delta \rfloor$ and $\lceil \alpha, \gamma, \delta, \beta \rfloor$ are obstructed.

Let us point out that a bounded obstruction $\Omega \subseteq B$ is a simple verifier that certain kinds of embeddings of B (in F) do not exist. The requirement that $b(\Omega)$ is bounded guarantees that there is only a bounded number of possibilities to check for embeddability of Ω in F.

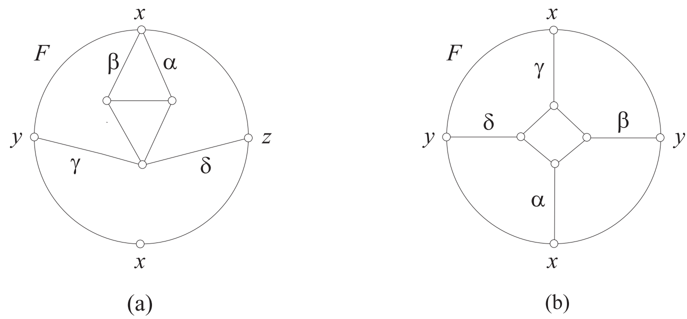

Figure 1: Obstructions

The presence of local bridges of K in G can yield arbitrarily large minimal obstructions also in cases where this would not occur in the absence of local bridges. Therefore the following lemma from [16] proves to be useful.

Lemma 3.1 *([12]) Let G be a 3-connected graph, $K \subseteq G$, and e a branch of K. Let H be a subgraph of G consisting of e and all local K–bridges that are local on e. There is a linear time procedure that either replaces e by another branch $e' \subseteq H$ joining the same pair of main vertices as e and such that there are no local bridges of $K - e + e'$ attached to e', or finds a Kuratowski subgraph contained in H.*

We will also need the following lemma whose proof can be found in [17]:

Lemma 3.2 *([17]) Let H be a graph, C a cycle in H, and let D be a disk. There is a linear time algorithm that either finds an embedding of H in D with C on ∂D, or returns an obstruction Ω for such embeddings such that $b(\Omega) \leq 12$. Moreover, Ω is either a pair of disjoint crossing paths, a tripod, or a Kuratowski subgraph contained in a 3-connected component of H.*

Now we continue with our special case when there is only one bridge of K.

Lemma 3.3 *Let Ω be an obstruction in B. There is a linear time procedure that either determines a non-obstructed cyclic sequence of feet of Ω, or finds a total obstruction $\Omega_0 \subseteq \Omega$ contained in Ω such that $b(\Omega_0) \leq 11$.*

Proof. Let Q be the graph obtained from Ω by identifying all its vertices of attachment to K into a single vertex w. Test planarity of Q. If Q is planar, the local rotation at w determines a cyclic sequence of feet of Ω that is not obstructed. Otherwise, a Kuratowski subgraph of Q determines a subgraph Ω_0 of Ω. It is easy to see that Ω_0 is a total obstruction and that $b(\Omega_0) \leq 11$. $\square$

Given $\Omega \subseteq B$ and a cyclic sequence Π of feet of Ω, it is easy to check (in time proportional to $b(\Omega)$) if Ω is an obstruction for Π. We first construct the *split auxiliary* *graph* Aux(Ω, Π) of Ω and Π. We split each vertex of attachment of Ω into as many new vertices as is the number of feet of Ω at this vertex. Then join all the (new) endvertices of feet of Ω into the cycle (called the *auxiliary cycle*) as determined by Π. Finally, add an additional vertex (called the *auxiliary vertex*) and join it with all vertices on the auxiliary cycle. See an example on Figure 2. It is easy to see that Ω is an obstruction for Π if and only if Aux(Ω, Π) is not a planar graph. This property gives rise to an efficient checking procedure to test if given graphs are obstructions or not.

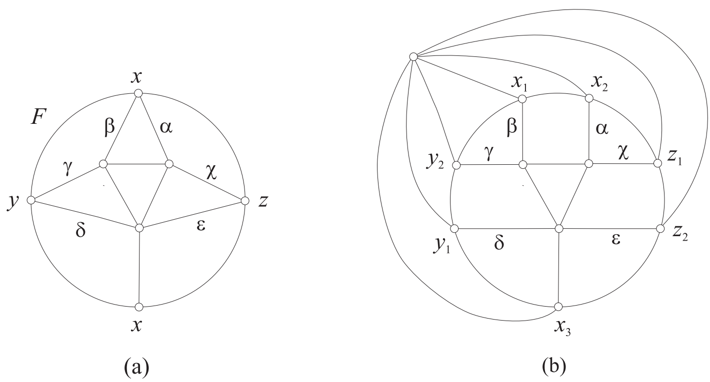

Figure 2: Ω, Π, and Aux(Ω, Π)

We will also use obstructions for embedding extension problems in case of more than just one K–bridge. Definitions follow those given above for the case of a single bridge of K.

Let us recall that a graph H is *nodally 3-connected* if the graph obtained from H by replacing each branch by an edge between the corresponding main vertices is 3-connected. An obstruction $\Omega \subseteq G - E(K)$ is *well connected* if Ω is a single K–bridge in $K \cup \Omega$. The following property of split auxiliary graphs of obstructions will be useful.

Lemma 3.4 *Let $\Omega \subseteq B$ be a well connected obstruction with at least three feet. Then Ω either contains a total obstruction Ω' which has $b(\Omega') \leq 11$, or contains a well connected subgraph Ω' with the following properties:*

(a) Ω' has the same feet as Ω and $b(\Omega') \leq b(\Omega)$,

(b) Ω' obstructs the same cyclic sequences as Ω, and

(c) Aux(Ω', Π') is nodally 3–connected for every cyclic sequence Π' of feet of Ω'.

Moreover, given Ω, the obstruction Ω' can be found in linear time.

Proof. By Lemma 3.3, we either find a total obstruction (in which case we stop), or a non-obstructed cyclic sequence Π. By an algorithm of Hopcroft and Tarjan [10], we can get in linear time the 3-connected components of $\mathrm{Aux}(\Omega, \Pi)$. Let Q be the 3-connected component containing the auxiliary vertex and the auxiliary cycle. We can view Q as a subgraph of $\mathrm{Aux}(\Omega, \Pi)$ by replacing virtual edges by appropriate paths in Ω. Since Ω is well connected and has at least three feet, Q contains all feet of Ω. Let Ω' be the subgraph of Ω corresponding to Q. It is easy to see that Ω' is well connected and that it satisfies (a) and (b). It also satisfies (c) for $\Pi' = \Pi$ since $\mathrm{Aux}(\Omega', \Pi) = Q$. It follows easily that (c) holds also for other cyclic sequences of feet. □

Remark. Lemma 3.4 can be performed also in case when Ω contains at most two feet. In that case we cannot insist on $\mathrm{Aux}(\Omega', \Pi')$ being nodally 3-connected since the auxiliary cycle contains a loop or parallel edges. But then Ω' is a total obstruction or just a path. If the connectivity condition of Lemma 3.4 is not fulfilled, we can use the above procedure "componentwise". The final result is again nodally 3-connected with the only difference that some "components" that gave rise to paths in Ω', may yield parallel edges in $\mathrm{Aux}(\Omega', \Pi')$ for some of the cyclic sequences Π'.

Lemma 3.5 *Let $\Omega \subseteq B$ be a well connected obstruction with $f \geq 3$ feet. Then Ω either contains a total obstruction Ω' such that $b(\Omega') \leq 11$, or it contains a well connected subgraph Ω' with the following properties:*

(a) Ω' has the same feet as Ω,

(b) Ω' obstructs the same cyclic sequences of feet as Ω,

(c) $\mathrm{Aux}(\Omega', \Pi')$ is nodally 3–connected for every cyclic sequence Π' of feet of Ω', and

(d) $b(\Omega') \leq 5f - 9$.

Moreover, given Ω, the obstruction Ω' can be found in linear time.

Proof. By Lemma 3.4, we may assume that $\mathrm{Aux}(\Omega, \Pi')$ is nodally 3-connected for every cyclic sequence Π' of its feet (or we find a total obstruction Ω' with $b(\Omega') \leq 11$). Moreover, we may assume that $\mathrm{Aux}(\Omega, \Pi)$ is planar for some Π. Denote by $F_1, \ldots, F_f$ the faces of $\mathrm{Aux}(\Omega, \Pi)$ containing consecutive edges on the auxiliary cycle but not containing the auxiliary vertex. Denote by Ω' the subgraph of Ω consisting of all vertices and edges of Ω that are contained in boundaries of $F_1, \ldots, F_f$. It is clear that Ω' is a well connected subgraph of Ω and that it satisfies (a).

We claim that $\mathrm{Aux}(\Omega', \Pi)$ is nodally 3-connected. It is easy to see that it is 2-connected since $\mathrm{Aux}(\Omega, \Pi)$ is nodally 3-connected and thus any two faces F_i, F_j are either disjoint or their intersection is a vertex or a branch of $\mathrm{Aux}(\Omega, \Pi)$. Suppose that there is a (non-trivial) 2-separation $\{x, y\}$. Let Q be the part separated by x, y that does not contain the auxiliary vertex. Since Q is not just a branch, it contains a cycle C. Denote by D the disk bounded by C under the plane embedding of $\mathrm{Aux}(\Omega, \Pi)$.

Since Ω is well connected, Q contains none of the auxiliary edges. Therefore, D does not contain any of the faces F_i (assuming that the auxiliary vertex is on the boundary of the infinite face). It follows easily that $Q = C$ and that the two segments from x to y on C are on the boundaries of two faces F_i, F_j, say. Then $\partial F_i \cap \partial F_j$ is disconnected and this contradicts the nodal 3-connectivity of $\mathrm{Aux}(\Omega, \Pi)$.

Since $\mathrm{Aux}(\Omega', \Pi)$ is nodally 3-connected and Ω' is well connected, we have (c). Also, boundaries of faces of $\mathrm{Aux}(\Omega', \Pi)$ are characterized as induced non-separating cycles [24]. If R is such a cycle that is disjoint from the auxiliary cycle, it is an induced non-separating cycle (and hence facial) also in $\mathrm{Aux}(\Omega', \Pi')$ for any Π'. Consequently, having a plane embedding of $\mathrm{Aux}(\Omega', \Pi')$, it can be extended to a plane embedding of $\mathrm{Aux}(\Omega, \Pi')$ by using the embedding of $\mathrm{Aux}(\Omega, \Pi)$. This proves (b).

The bound in (d) will be proved by induction on f. If $f = 3$, it is easy to see that $b(\Omega') \leq 6 = 5f - 9$. Similarly, if $f > 3$ and if any two non-consecutive faces F_i, F_j are disjoint, then $\mathrm{Aux}(\Omega', \Pi)$ is just the f–prism (together with the auxiliary vertex), and $b(\Omega') = 2f < 5f - 9$. Suppose now that non-consecutive faces F_i, F_j $(i < j)$ intersect. Because of (c), they share a vertex or a common branch. By splitting the vertex (removing the branch, respectively) we get a face across the disk bounded by the auxiliary cycle. Denote by x, y the two vertices obtained after splitting (the ends of the removed branch, respectively). We can split the auxiliary cycle into two cycles, each containing one of the split parts between F_i and F_j. By adding an edge from x to the auxiliary cycle in the first part and adding an edge from y to the cycle in second part, we obtain graphs Ω_1, Ω_2 with $j - i + 1$ and $f - j + i + 1$ feet, respectively. They have the same property as $\mathrm{Aux}(\Omega', \Pi)$: every edge is on the boundary of one of the faces containing the auxiliary edges. By induction, $b(\Omega') \leq b(\Omega_1) + b(\Omega_2) - 1 \leq 5(j - i + 1) - 9 + 5(f - j + i + 1) - 9 - 1 = 5f - 9$. □

Corollary 3.6 *Let Ω_1, Ω_2 be obstructions in B such that $\Omega_1 \cup \Omega_2$ contains at least 3 feet. Then B either contains a total obstruction Ω' such that $b(\Omega') \leq 11$, or it contains a well connected subgraph Ω' with the following properties:*

(a) Ω' has the same feet as $\Omega_1 \cup \Omega_2$,

(b) Ω' obstructs all the cyclic sequences of feet that are obstructed by Ω_1 or by Ω_2,

(c) $\mathrm{Aux}(\Omega', \Pi')$ is nodally 3–connected for every cyclic sequence Π' of feet of Ω', and

(d) $b(\Omega') \leq 5f - 9$ where f is the number of feet of $\Omega_1 \cup \Omega_2$.

Moreover, given B, Ω_1 and Ω_2, the obstruction Ω' can be found in linear time.

Proof. By adding some paths of $B - K$ to $\Omega_1 \cup \Omega_2$, one can get (in linear time) a well connected obstruction $\Omega \supseteq \Omega_1 \cup \Omega_2$ with the same feet as $\Omega_1 \cup \Omega_2$. It remains to apply Lemma 3.5. □

The obstruction Ω' constructed by applying Corollary 3.6 is said to be obtained by *combining* Ω_1 and Ω_2. Let us note that linearity is considered with respect to the size of B. Corollary 3.6 can also be used for combining more than two obstructions.

We will use some special subgraphs of bridges in G. Let K' be a subgraph of G and let B' be a bridge of K'. For each open branch e of K' that B' is attached to, let e_1 and e_2 be feet of B' attached as close as possible to one and the other end of e, respectively. If there is just one vertex of attachment on e, we select $e_1 = e_2$. In addition to the above, select one foot of B' at every main vertex of K' that B' is attached to. Let $H \subseteq B'$ be a minimal tree that contains all chosen and no other feet of B'. The obtained graph H is said to be an H–*graph* of B'. Suppose, for example, that B' is attached only to two open branches of K'. Then H contains at most 4 feet and its branch size is at most 5. If there are three or just two distinct feet in H, then H is essentially unique (up to homeomorphisms). But in case of four distinct feet, there are four essentially different cases for H (see Figure 3). H–graphs can be constructed in linear time by standard graph search algorithms. The following simple fact justifies introduction of H–graphs: Let G be a graph and K its subgraph that is 2-cell embedded in some surface. Let B and B' be non-local K-bridges in G that can be embedded in the same non-singular face F of K. Then B and B' overlap in F if and only if their H–graphs overlap in F. (We say that B and B' *overlap* in F if they cannot be simultaneously embedded in F.)

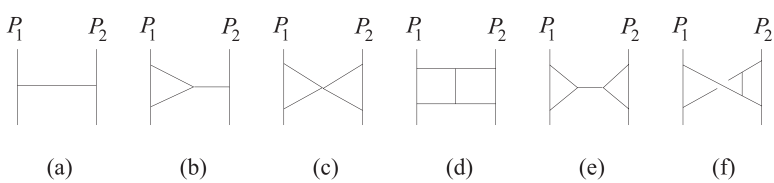

Figure 3: H-graphs

Corollary 3.7 *Let B be a non-local bridge of K in G, and let $\Delta_1, \ldots, \Delta_k$ where $k \geq 1$ is bounded, be non-singular embedding types for B. Then B contains well connected obstructions $\overline{B}$ and $\tilde{B}$ with the following properties:*

(a) $\overline{B}$ and $\tilde{B}$ admit embeddings of exactly those types among $\Delta_1, \ldots, \Delta_k$ that B does.

(b) $b(\overline{B}) \leq 20k - 8$ and $b(\tilde{B}) \leq 20k + 4t - 9$ where t is the number of basic pieces that B is attached to.

(c) Suppose that B' is another non-local bridge of K in G and $\tilde{B}'$ is the corresponding obstruction with respect to non-singular embedding types $\Delta'_1, \ldots, \Delta'_\ell$. If F is a non-singular face, then B and B' have simultaneous embedding in F of types Δ_i ($1 \leq i \leq k$) and Δ'_j ($1 \leq j \leq \ell$), respectively, if and only if $\tilde{B}$ and $\tilde{B}'$ admit such an embedding in F.

Moreover, obstructions $\overline{B}$ and $\tilde{B}$ can be obtained in linear time.

Proof. We may suppose that B has no embeddings of types $\Delta_1, \ldots, \Delta_s$ ($0 \le s \le k$), and that it admits embeddings of other types. Let $\Omega_1, \ldots, \Omega_s$ be obstructions for $\Delta_1, \ldots, \Delta_s$, respectively. They can be obtained by applying Lemma 3.2. (We also add some paths, if necessary, so that $\Omega_1, \ldots, \Omega_s$ are well connected.) If some Ω_i is a tripod or a Kuratowski subgraph in a 3-connected component of $K \cup B$, then it can serve as $\overline{B}$, obstructing all types $\Delta_1, \ldots, \Delta_k$. Moreover, $b(\overline{B}) \le 12 \le 20k - 8$. Otherwise, each Ω_i is a pair of disjoint crossing paths. Applying Corollary 3.6 on $\Omega_1 \cup \cdots \cup \Omega_s$, we get a required obstruction $\overline{B} = \Omega'$. By (d), $b(\overline{B}) \le 5(4s) - 9 \le 20k - 9$.

To get $\tilde{B}$ from $\overline{B}$, we add extreme feet of B on every basic piece that B is attached to and corresponding paths to $\overline{B}$ so that $\tilde{B}$ is well connected. Property (c) is then obvious by (a) since $\tilde{B}$ contains an H–graph of B. □

The above upper bound $20k - 8$ can be further decreased with more careful analysis. We will show how to do it in a very specific example.

Lemma 3.8 *Suppose that B is attached only to two (open) branches of K. Then B contains an obstruction $\tilde{B}$ that has exactly the same non-singular admissible embeddings as B and also satisfies property (c) of Corollary 3.7. If B has no non-singular embeddings then $b(\tilde{B}) \le 12$. Otherwise, $b(\tilde{B}) \le 5$. The obstruction $\tilde{B}$ can be found in linear time.*

Proof. There are two types of non-singular embeddings of B. If B admits at least one of them, we take for $\tilde{B}$ an H–graph of B obtained from the "leftmost" and the "rightmost" path with respect to this embedding. If these two paths meet, B also has the other embedding. If they are disjoint, they obstruct the other embedding. If B has no non-singular embeddings, we apply Lemma 3.2 for both types. Obtaining a tripod or a Kuratowski subgraph in any of the cases, this is a required obstruction.

Suppose now that we have disjoint crossing paths in both cases. Let P_1, P_2 be the paths for one of the embedding types. We may assume that all 3-connected components of G contained in B are planar (or we get a Kuratowski subgraph). Then Lemma 3.1 enables us to change P_1 and P_2 so that the graph $K' = K \cup P_1 \cup P_2$ has no local bridges. Consider an embedding extension of B for the other type Δ. The paths P_1, P_2 split the corresponding face F into three faces and every K'–bridge in G has at most one face where it could be embedded (according to its attachments). Lemma 3.2 (used for all three faces) gives rise to an obstruction $\Omega' \subseteq G - E(K')$, where Ω' is either a tripod or disjoint crossing paths with respect to one of the faces. If necessary, we add path(s) Q from $K - (P_1 \cup P_2)$ to $\Omega' - (P_1 \cup P_2)$. It is easy to see that the obtained subgraph $P_1 \cup P_2 \cup \Omega' \cup Q$ of B contains an obstruction Ω with $b(\Omega) \le 12$. □

4 Millipedes

There are cases when embeddability of B in F cannot be obstructed by a bounded obstruction. We will show that in such a case, B contains an obstruction Ω which is nice in the sense that it has bounded number of embeddings of any admissible type and which can be changed into a bounded obstruction if we allow to change some of the branches of K.

Consider a (hypothetical) face F corresponding to an admissible type Δ of embeddings of B. Let e be an open singular branch of K on ∂F. Denote by Q_1 and Q_2 the two segments of $\partial F \backslash e$ in which the two appearances of e on ∂F split the boundary walk of F. Suppose that $\partial F = Q_1 e Q_2 e^-$. We will refer to the part of ∂F consisting of Q_1 as the *left side*, and to Q_2 as the *right side* of F. For example, we say that a vertex x on e is to the left of the vertex y on e if x is closer to Q_1 than y.

A *1-millipede* in B *based on* e is a subgraph M of B which can be expressed as $M = P \cup B_1^\circ \cup B_2^\circ \cup \cdots \cup B_m^\circ$ ($m \geq 12$) where:

(M1) $P \subseteq B$ is a path embedded in F joining a vertex q_1 of Q_1 with a vertex q_2 in Q_2.

(M2) Denote by F_1 and F_2 the subfaces in which P splits the face F and let $K' = K \cup P$. $B_1^\circ \cup B_2^\circ \cup B_3^\circ$ is uniquely embeddable in $F_1 \cup F_2$. Let F_α be the face containing B_1° under this embedding. Similarly, $B_{m-2}^\circ \cup B_{m-1}^\circ \cup B_m^\circ$ is uniquely embeddable, and let F_β be the face containing B_m°. If m is even, then $\alpha = \beta$. If m is odd, then $\alpha \neq \beta$.

(M3) $B_1^\circ, B_2^\circ, \ldots, B_m^\circ$ are H–graphs of distinct K'–bridges $B_1, B_2, \ldots, B_m$ (respectively). $B_2^\circ, \ldots, B_{m-1}^\circ$ are attached to e and to P and are not attached to other pieces of K'.

(M4) For each $i = 1, 2, \ldots, m-1$, B_i° and B_{i+1}° overlap in F_1 and in F_2.

(M5) For $i > 1$ and $i + 2 \leq j < m$, B_i° and B_j° overlap neither in F_1 nor in F_2. $B_1^\circ \cup B_j^\circ$ can be embedded in F_α ($j = 3, 4, \ldots, m-1$). Similarly, $B_j^\circ \cup B_m^\circ$ can be embedded in F_β ($j = 2, 3, \ldots, m-2$). Additionally, $B_1^\circ \cup B_2^\circ \cup B_3^\circ \cup B_{m-2}^\circ \cup B_{m-1}^\circ \cup B_m^\circ$ can be embedded in $F_1 \cup F_2$.

(M6) For $i = 3, 4, \ldots, m-2$, $B_{i+1}^\circ \cup B_{i+2}^\circ$ has a vertex of attachment on e strictly to the right of the rightmost attachment of $B_{i-2}^\circ \cup B_{i-1}^\circ$ on e, and $B_{i-2}^\circ \cup B_{i-1}^\circ$ has a vertex of attachment on e strictly to the left of the leftmost attachment of $B_{i+1}^\circ \cup B_{i+2}^\circ$ on e.

(M7) Let p and q be the first and the last attachment of $B_2^\circ \cup \cdots \cup B_{m-1}^\circ$ on P, and let P° be the segment of P between p and q. For $i = 1, 2, \ldots, m$, denote by l_i and r_i the leftmost and the rightmost vertex of attachment of B_i° on e, respectively. If W is the closed segment on e between l_4 and r_{m-3}, then every K'–bridge that has an attachment in the interior of P° or in the interior of W has all its attachments in $P^\circ \cup e$. All K'–bridges that are attached only to $P^\circ \cup e$ can be simultaneously embedded in $F_1 \cup F_2$.

It is clear by (M2) and (M4) that a 1-millipede M obstructs those embeddings of B in F where P is embedded in F as chosen in (M1). (M5) implies that no bridge B_i° in M is redundant.

Our notion of 1-millipedes is slightly different from the concept of millipedes introduced in [17]. The millipedes in [17] can be shorter (i.e., $m < 12$ is allowed) and their subgraphs B_i° are only required to be subgraphs of H–graphs in order that millipedes become minimal obstructions. On the other hand, the notion of "extended millipedes" introduced in [17] corresponds precisely to our 1-millipedes. In this paper we take advantage of B_i° being H–graphs and do not care of some superfluous branches in the millipede. The following important result has been proved in [17].

Theorem 4.1 *Suppose that the only singular basic piece on ∂F is an open branch e and that $\partial F = Q_1 e Q_2 e^-$. Suppose that in B there is a path from Q_1 to Q_2. Then B cannot be embedded in F if and only if either*

(a) B contains a bounded obstruction Ω for embeddability of B in F, where $b(\Omega) \leq 77$, or

(b) B contains a 1-millipede $\Omega = P \cup B_1^\circ \cup \cdots \cup B_m^\circ$ based on e such that $b(P \cup B_1^\circ \cup B_2^\circ \cup B_3^\circ \cup B_{m-2}^\circ \cup B_{m-1}^\circ \cup B_m^\circ) \leq 43$.

Moreover, such an obstruction Ω can be found in linear time.

A 1-millipede is *thin* if $B_5^\circ \cup B_6^\circ \cup \cdots \cup B_{m-4}^\circ$ is attached to P at a single vertex. Otherwise, M is *thick*.

One can define millipedes also in case when $\partial F = Q_1 e Q_2 e$, i.e., the open face together with e is homeomorphic to the Möbius band. We distinguish one of the ends of e as the "left" side of e, and the other side is on the "right". Note that in this case, the left and the right side of e cannot be defined using Q_1 and Q_2. In such a case we need only thin millipedes (where $(B_2^\circ \cup \cdots \cup B_{m-1}^\circ) \cap P$ is just one vertex; cf. [14]), but another kind of arbitrarily long obstructions arises. They are called *skew millipedes* and are defined analogously as thin millipedes. The branch e is assumed to appear on ∂F twice in the same direction. The path P contains vertices p and q such that no K'–bridge in G is attached to P strictly between p and q. The K'–bridges $B_2^\circ, \ldots, B_{m-1}^\circ$ satisfy (M1), (M2), (M4), and (M6)–(M7) while (M3) and (M5) are replaced by:

(M3′) $B_2^\circ, \ldots, B_{m-1}^\circ$ are H–graphs of distinct K'-bridges. If i is even $(1 < i < m)$, then B_i° is attached to e and to p (and not elsewhere). If i is odd $(1 < i < m)$, then B_i° is attached to e and to q (and not elsewhere).

(M5′) For $i > 1$ and $i + 2 \leq j < m$, B_i° and B_j° do not overlap in F_α if either $i \not\equiv \alpha \pmod 2$, or $j \equiv \alpha \pmod 2$ (or both). They do not overlap in $F_{3-\alpha}$ if either $i \equiv \alpha \pmod 2$, or $j \not\equiv \alpha \pmod 2$ (or both). For $3 \leq j < m$, $B_1^\circ \cup B_j^\circ$ can be embedded in F_α. For $1 < i \leq m-2$, $B_i^\circ \cup B_m^\circ$ can be embedded in F_β. Additionally, $B_1^\circ \cup B_2^\circ \cup B_3^\circ \cup B_{m-2}^\circ \cup B_{m-1}^\circ \cup B_m^\circ$ can be embedded in $F_1 \cup F_2$.

In referring to a *millipede*, we mean either a 1-millipede, a thin or a skew millipede. If M is a millipede, it is an obstruction for embedding extensions of K to

G. Moreover, it is a "minimal" obstruction in the sense that no bridge in M is redundant.

A result similar to Theorem 4.1 has been proved in [18]. It characterizes obstructions in the Möbius band.

Theorem 4.2 *([18]) Suppose that the only singular basic piece on ∂F is a branch e and that $\partial F = Q_1 e Q_2 e$. Then B cannot be embedded in F if and only if either B contains a bounded obstruction Ω for embeddability of B in F, or B contains a thin or a skew millipede. In the former case we have $b(\Omega) \leq 2000$, while in the latter case $b(B_1^\circ \cup B_2^\circ \cup B_3^\circ \cup B_{m-2}^\circ \cup B_{m-1}^\circ \cup B_m^\circ) \leq 2000$. Moreover, such an obstruction can be found in linear time.*

Let us derive some basic properties of millipedes. We will use the notation from (M1)–(M7).

Lemma 4.3 *Let M be a millipede. For each i, $1 \leq i \leq m-2$, and for every t, $2 \leq t \leq m-i$, B_{i+t}° is attached to P and to e to the right of B_i°. If $i+t < m$, then at least one of the attachments (either on P, or on e) is strictly to the right of the corresponding rightmost attachment of B_i° on P, or e.*

Proof. Shown in [17] for 1-millipedes and in [14] for Möbius band thin and skew millipedes. The strict property is clear since otherwise B_{i+t+1}° could not overlap with B_{i+t}° without overlapping with B_i° as well. □

Lemma 4.4 *Let x be the rightmost attachment on e of $B_1^\circ \cup \cdots \cup B_i^\circ$, where $2 \leq i \leq m-5$. Then each of $B_{i+5}^\circ, B_{i+6}^\circ, \ldots, B_m^\circ$ has all its attachments on e strictly to the right of x.*

Proof. By (M6), $B_{i+2}^\circ \cup B_{i+3}^\circ$ has an attachment y strictly to the right of x. By Lemma 4.3, $B_{i+5}^\circ, B_{i+6}^\circ, \ldots$ have all their attachments on e to the right of y, hence our claim. □

Let M be a millipede. Its subgraph

$$\partial M = P \cup (B_1^\circ \cup \cdots \cup B_6^\circ) \cup (B_{m-5}^\circ \cup \cdots \cup B_m^\circ)$$

is called the *boundary part* of M. Since $m \geq 12$, H-graphs B_i° participating in ∂M are pairwise distinct.

We will consider embeddings of M and of ∂M in faces F where we will no longer insist on P being attached to ∂F as assumed in the definition of millipede. A face F can have other singular pieces than just the open branch e. Thus, it may happen that also B_1° or B_m° is embedded differently as allowed in the definition of millipede.

Lemma 4.5 *Under any embedding of ∂M in F, each of the H–graphs B_5°, B_6°, B_{m-5}°, and B_{m-4}° is attached only to one appearance of e on ∂F.*

Proof. Suppose that B_5° is attached to both appearances of e on ∂F. Then F is separated by a path $Q \subseteq B_5^\circ - P$ into the left and the right part. Suppose first that on ∂F the branch e appears with distinct orientations. By (M6), $B_2^\circ \cup B_3^\circ$ has an attachment on e strictly to the left of the leftmost attachment of B_5°. Therefore, B_2° or B_3°, and thus also P is in the left part. Similarly, B_{m-2}° or B_{m-1}° has an attachment to e strictly to the right of Q. Thus we get a contradiction. Note that we used the fact that B_i° are attached to $P - K$.

Suppose now that e appears on ∂F twice in the same direction. Suppose that P is embedded in the left part of F (with respect to Q). Then all bridges B_i° ($2 \le i < m$, $i \ne 5$) are embedded in the left part as well and each of them is attached to only one appearance of e on ∂F. By Lemma 4.3, $B_{m-4}^\circ, \ldots, B_{m-1}^\circ$ are all attached to the same occurrence of e on ∂F. Therefore their attachment intervals on e do not overlap. It follows by (M4) that any two consecutive bridges among $B_{m-4}, \ldots, B_{m-1}^\circ$ contain disjoint paths from e to P but non-consecutive bridges do not contain such disjoint paths. This is easily seen to be a contradiction.

The proof for B_6°, B_{m-5}°, and B_{m-4}° is similar. □

Lemma 4.6 *Let M be a thick 1-millipede. Then every embedding of ∂M in F has P attached to q_1 on the left subwalk Q_1 of ∂F and attached to q_2 on Q_2. Moreover, having such an embedding, e appears on ∂F twice in opposite directions.*

Proof. First we will prove that a face F with $\partial F = Q_1 e Q_2 e$ does not admit embeddings of ∂M. (This also proves that e appears twice on ∂F.) Suppose that ∂M is embedded in such a face F. By Lemma 4.5, each of B_5° and B_6° is attached only to one occurrence of e. If B_5° and B_6° are attached to the same occurrence, then B_6° has an attachment on P that is closer to q_1 than an attachment of B_5° on P (because of (M4)). By Lemma 4.3, B_{m-4}° is attached to the right of B_6°. It follows that B_{m-4}° is attached on ∂F to the other occurrence of e than B_6°. Now it is easy to see that B_2° and B_3° (which are attached to P closer to q_1 than B_6°) have troubles being embedded in F. Consequently, B_5° and B_6° are attached to distinct occurrences of e on ∂F. The same holds for B_{m-5}° and B_{m-4}°. Since M is thick, $B_5^\circ, B_6^\circ, B_{m-5}^\circ, B_{m-4}^\circ$ are attached to at least two vertices on P (by Lemma 4.3). Denote them by x and y, where x is to the left or y. It follows that P is attached to Q_1 and to Q_2. If this were not the case, bridges B_{m-2}° and B_{m-1}° which are attached to the right of y (Lemma 4.3) can not extend the embedding of $P \cup B_5^\circ \cup B_6^\circ$.

By symmetry we may suppose that P is attached to q_1 on the left side (at Q_1) and attached to q_2 on the right and that the orientation of e in the "lower" subface of F (with respect to $K \cup P$) is the same as assumed in (M1)–(M7). Thus, either B_2° or B_3° is embedded in the "upper" subface and either B_{m-2}° or B_{m-1}° is in the "upper" subface. This is possible only if those among these bridges that are in the "upper" subface are attached to P at just one vertex (by Lemma 4.3). But this is not possible (by Lemma 4.3) since M is thick.

Let us now suppose that $\partial F = Q_1 e Q_2 e^-$. Suppose first that ∂M is embedded in F in such a way that P is attached to an appearance of q_1 on Q_2, and to q_2 on

Q_1. Let t be the attachment of $B_5^\circ \cup B_6^\circ$ on P that is closest to q_1. If B_5° and B_6° are embedded in distinct faces F_1', F_2' of $K \cup P$, then it is easy to see that B_2°, B_3° (Lemma 4.3) are attached to P only at t. (M4) implies that intervals of attachment of B_2°, B_3° on e overlap. Hence, they are embedded in distinct faces F_i'. Consequently, $B_2^\circ, B_3^\circ, B_5^\circ, B_6^\circ$ are all attached to P only at t. Then the same holds also for B_4° since it is to the right of B_2° and to the left of B_6° (Lemma 4.3). Since M is thick, there is an attachment y of $B_7^\circ \cup \cdots \cup B_{m-4}^\circ$ on P closer to q_2 than t. Then also $B_{m-1}^\circ \cup B_{m-2}^\circ$ has such an attachment by Lemma 4.3. Thus, $B_{m-1}^\circ \cup B_{m-2}^\circ$ cannot extend the embedding of $P \cup B_5^\circ \cup B_6^\circ$.

Suppose next that B_5° and B_6° are embedded in the same face F_1' of $K \cup P$. Then B_5° and B_6° have attachments x and y, respectively, on P such that $x \neq y$. Let x be closer to q_1 than y. By Lemma 4.3, B_2°, B_3° are both embedded in F_2'. It is easy to see that this blocks the possibility of embedding B_{m-1}°.

Next possibility is that P is attached to Q_2 only. We first consider the case when B_5° and B_6° are attached to P at single vertex t. By Lemma 4.5, they are embedded so that they are attached to e locally. Since they overlap in F_1 and in F_2 and they attach to P at single common vertex, they are attached to distinct occurrences of e. By Lemma 4.3 it follows that for each of B_2° and B_3°, the only possibility is that it is embedded to the left of B_5° and B_6°. Then their only attachment on P is the vertex t. Thus $t = p$ (where p is defined in (M7)). Since M is thick, some of the bridges B_i° ($5 \le i \le m-4$) have an attachment on P strictly closer to q_2 than t. By Lemma 4.3, the same holds for B_{m-2}° and B_{m-1}°. By (M4), they obstruct embeddability of ∂M.

Suppose now that $B_5^\circ \cup B_6^\circ$ has at least two distinct attachments on P. Having B_5° and B_6° attached to the same occurrence of e, it is easy to see that one of them would block B_2°, a contradiction. Thus these bridges are attached to distinct occurrences of e. Then it is easy to see that embeddability of $B_{m-2}^\circ \cup B_{m-1}^\circ$ is obstructed.

The case when P is attached to Q_1 only is symmetric and thus follows from above.

□

Note that the attachments of P on ∂F are not necessarily the same appearances of q_1, q_2 as used in the definition of the millipede M. Note also that the same property does not hold for thin or skew millipedes. Then the path P can be "turned" around or attached to just one of the segments Q_1, Q_2. Moreover, a Möbius band thin or skew millipede (defined for a face F with $\partial F = Q_1 e Q_2 e$) can also admit embeddings in faces with e appearing on the boundary with opposite orientation (or vice versa).

Lemma 4.7 *Let M be a millipede. Under any embedding of M in F and for $i = 5, 6, \ldots, m-5$, the bridges B_i° and B_{i+1}° are attached to distinct occurrences of e on ∂F. If M is thick, the same holds for $i = 2, 3, 4$ and for $i = m-4, m-3, m-2$.*

Proof. Clear by Lemma 4.6 if M is thick. On the other hand, if M is thin or skew, then the property (M4) for B_i° and B_{i+1}° is satisfied because of their overlapping on e. Thus Lemma 4.5 applies. □

Corollary 4.8 *Every millipede has only a bounded number of essentially different admissible embeddings.*

Let M be a millipede. Let f be the rightmost foot of B_5° on e. Subdivide f by inserting a new vertex v_5 of degree 2. Introduce similarly vertices v_6 in B_6°, and v_{m-5}, v_{m-4} in $B_{m-5}^\circ, B_{m-4}^\circ$, respectively (here with respect to leftmost feet). If m is even, then add to ∂M the edges $f_1 = v_5 v_{m-5}$ and $f_2 = v_6 v_{m-4}$. If m is odd, then add edges $f_1 = v_5 v_{m-4}$ and $f_2 = v_6 v_{m-5}$. Denote the obtained graph by $\tilde{M}$ and call it the *squashed millipede*. This way we reduce the size of M, while essentially preserving its embedding extension properties. (This will be proved below.)

Let M be a millipede in B. Using notation from (M1)–(M7), if M is thick, let D be the union of all $(K \cup P)$–bridges that have an attachment on e (strictly) between l_5 and r_{m-4}. If M is thin or skew we define D as follows. Properties (M4) (for $i = 4, 5, 6$) and (M5) or (M5′) (for $i = 4, 5$, $j = i + 2$) imply that there is an edge $f \subset e$ such that B_5° and B_6° each have an attachment to the left of f and an attachment to the right of f (possibly an end of f). Define similarly f' for $B_{m-5}^\circ, B_{m-4}^\circ$. Then let D be the union of all $(K \cup P)$–bridges that have an attachment on e between f and f'. It follows by Lemma 4.7 that every $(K \cup P)$–bridge in D is "blocked" by M. More precisely, under any embedding of M, H–graphs $B_2^\circ, B_3^\circ, B_{m-2}^\circ, B_{m-1}^\circ$ (if M is thick), or $B_5^\circ, B_6^\circ, B_{m-4}^\circ, B_{m-5}^\circ$ (if M is thin or skew), are embedded in such a way that D must be embedded in the strip in between of them. On the other hand, by (M7) and Lemma 4.7 we know that any embedding of M can be extended to an embedding of $M \cup D$.

Let us define graphs B' and $\tilde{B}$ as follows. Let $B' = (B \backslash D) \cup (D \cap M)$. In other words, B' is a subgraph of B obtained by replacing the $(K \cup P)$–bridges $B_5, B_6, \ldots, B_{m-4}$ by their H–graphs $B_5^\circ, B_6^\circ, \ldots, B_{m-4}^\circ$, respectively, and deleting the "superfluous" $(K \cup P)$–bridges that are in D. To get $\tilde{B}$, add to B' the edges f_1, f_2 as introduced above, and remove H–graphs $B_7^\circ, \ldots, B_{m-6}^\circ$. Note that $\tilde{M}$ is contained in $\tilde{B}$. Note that B' (and also $\tilde{B}$) is a single K–bridge in $K \cup B'$ (in $K \cup \tilde{B}$, respectively). The operation of replacing B by $\tilde{B}$ and M by $\tilde{M}$ is called *squashing* of the millipede M.

Theorem 4.9 *Let $M \subseteq B$ be a millipede based on e. Graphs $\tilde{B}$ and B' introduced above admit essentially the same types of embeddings as B. More precisely, every embedding of B gives rise to embeddings of $\tilde{B}$ and B' of the same type that coincide on the intersection of B with $\tilde{B}$ or B'. Conversely, having an embedding of $\tilde{B}$ (or B'), the embedding of $B \cap \tilde{B}$ ($B \cap B'$, respectively) can be extended to an embedding of B of the same type.*

Proof. By Lemma 4.7 it follows that for any embedding of M, and for $5 \le i \le m-5$, B_i° and B_{i+1}° are attached to distinct occurrences of e. Hence, B_5° and B_{m-5}° are attached to the same copy of e if m is even. If m is odd, then B_5° and B_{m-4}° are attached to the same copy of e. This implies that every embedding of B in F yields an embedding of $\tilde{B}$ in F which coincides on $B \cap \tilde{B}$ with the embedding of B. It is obvious that the same holds for $B' \subseteq B$.

To show the converse, we need (M7). This property guarantees that D contains only $(K \cup P)$–bridges that are attached to e and P° only. Moreover, the graph $D \cup (M \backslash (B_1^\circ \cup B_m^\circ))$ has an embedding in F. Fix this embedding and denote it by ϕ. By (M7) and Lemmas 4.6 and 4.7, no interference with the remaining part of B is lost if we remove D from B. By Lemma 4.7 and millipede properties, having an embedding of M in F (and this is equivalent to having an embedding of $\tilde{M}$ in F), this embedding combinatorially coincides with the restriction of ϕ to M (up to "up/down" reflection). Thus there is a unique extension of this embedding to an embedding of $M \cup D$ which combinatorially coincides with ϕ. Finally, having an embedding of $M \cup D$, we can extend it to an embedding of the graph in which all the H–graphs B_i° are replaced by B_i $(i = 5, \ldots, m-4)$ since B_i° interferes with the rest the same as B_i does and since (M7) guarantees that there are no obstructions hidden in B_i. □

Let Ω be an obstruction. A subgraph $M = P \cup B_1^\circ \cup \cdots \cup B_m^\circ$ $(m \geq 12)$ of Ω is a *millipede* in Ω if M is a millipede, defined with respect to some 1-singular face F. If Ω is bounded and does not contain millipedes, we say that it is *0-nice*. For $k > 0$, Ω is *k-nice* if Ω contains a millipede M such that $\tilde{\Omega}$ obtained after squashing M is $(k-1)$–nice. In particular, every millipede is a 1-nice obstruction. Finally, Ω is *nice* if it is k-nice for some bounded $k \geq 0$. By Corollary 4.8, nice obstructions have bounded number of embeddings.

Suppose that we have a k-nice obstruction Ω_0. Let $\tilde{G}$ be the graph obtained by successive squashing of k millipedes in Ω_0. Denote by $\tilde{\Omega}_0$ corresponding squashed obstruction Ω_0. Suppose that Ω_1 is an ℓ-nice obstruction in $\tilde{G}$. Then $\tilde{\Omega} = \tilde{\Omega}_0 \cup \Omega_1$ determines an obstruction Ω (after replacing squashed parts by millipedes) which is at most $(k+\ell)$–nice. We will use such a recursive construction of obstructions in our algorithms and simply refer to it as taking a *union* of Ω_0 and Ω_1 (or $\tilde{\Omega}_0$ and Ω_1). This operation will also be called *combining* of obstructions.

5 Universal obstruction

Let G be a connected graph, and $K \subset G$ an induced non-separating subgraph of G, i.e., K has exactly one bridge in G. Denote this bridge by B. We will assume that every non-main vertex of K is a vertex of attachment of B and that the number of branches that intersect B is bounded. The main result of this section is:

Theorem 5.1 *Let G, K, B be as above. Then B contains a nice subgraph $\tilde{B} \subseteq B$ which has embeddings of exactly the same admissible types as B. Moreover, there is a linear time algorithm that finds such a universal obstruction $\tilde{B}$ and for every admissible type of embeddings not obstructed by $\tilde{B}$ constructs an embedding of B of the same type.*

Proof. If B has at most two feet or only one vertex of attachment, the claim of the theorem is easy to verify. Otherwise, we can apply Lemma 3.5 with $\Omega = B$.

We assume henceforth that this has been done and that $\mathrm{Aux}(B, \Pi)$ is nodally 3-connected for every cyclic sequence Π of feet of B. We will also assume (Lemma 3.3) that $\mathrm{Aux}(B, \Pi_0)$ is planar for some Π_0 and that Π_0 is known to us.

Construction of $\tilde{B}$ will be performed consecutively through all admissible types of embeddings of B. For $s = 0, 1, 2, \ldots,$ we will assume that we have a well connected nice obstruction Ω that obstructs all admissible types Δ with $\mathrm{sing}(\Delta) < s$ for which B has no embedding of type Δ. By considering all possible s–singular embeddings of Ω, we will be able to extend Ω to a nice obstruction satisfying the same condition for $s + 1$. This obstruction will be used in the next iteration. At the same time, we will find embeddings of all non-obstructed s–singular types. By Lemma 2.1 and by assumption that only a bounded number of branches intersect B, the number of admissible types is bounded.

When $s = 0$, we take as Ω an H–graph of B. If necessary, we add to Ω another branch so that Ω contains at least 3 feet. Then we repeat for $s = 0, 1, 2, \ldots, D$ (D is the maximal admissible degree of singularity) the following steps. Each of the steps is discussed in more details in the sequel.

(i) Let $R_1, \ldots, R_r$ be H-graphs of $(K \cup \Omega)$–bridges in $B \cup K$ that are attached to two or more basic pieces of K. Add them to Ω, i.e., $\Omega := \Omega \cup R_1 \cup \cdots \cup R_r$.

(ii) Determine representatives of types, $Q_1, \ldots, Q_q$.

(iii) Determine 1-Möbius band obstructions, $\Omega'_1, \ldots, \Omega'_z$.

(iv) Let $\Omega' = \Omega \cup Q_1 \cup \cdots \cup Q_q \cup \Omega'_1 \cup \cdots \cup \Omega'_z$. Remove local bridges.

(v) For every s–singular admissible type Δ of embeddings of B and for every embedding of Ω' of type Δ repeat the following:

   (1) Find an embedding extension of $K \cup \Omega'$ to B or an obstruction for such extensions. (If Δ has been marked in some of the previous steps, an embedding of B of type Δ is already known.)

   (2) If an embedding ϕ of B of type Δ has been found in (1), mark all types Δ' for which $\Delta \preceq \Delta'$ so that ϕ will serve as an embedding of type Δ'.

   (3) If an obstruction containing a millipede has been obtained, perform the corresponding squashing and proceed in the next steps with the squashed bridge B.

(vi) Combine Ω' with all the obstructions that have been found in step (1) into a single nice obstruction that obstructs all the types of embeddings that Ω does, and also obstructs all s–singular B–obstructed types. Denote the obtained obstruction by Ω, again.

Since the number of admissible types is bounded and since every nice obstruction has only bounded number of s–singular embeddings, we only need to convince ourselves that Ω' is nice and that one can perform each of the steps (i)–(vi) and (1)–(3) in linear time.

STEP (i). Consider the bridges of $\Omega \cup K$ in $B \cup K$. Note that each of them is attached to a vertex of $\Omega - K$. Let $R_1, \ldots, R_r$ be H–graphs of those $(K \cup \Omega)$–bridges which are attached to two or more basic pieces of K. Let $\mu_i \geq 2$ $(1 \leq i \leq r)$ be the number of basic pieces of K that R_i is attached to. If K has k basic pieces that B is attached to, then a short calculation shows that if $\sum_{j=1}^{p}(\mu_j - 1) > k + s$ for some $p \leq r$, then $\Omega \cup R_1 \cup \cdots \cup R_p$ obstructs any embedding types with degree of singularity equal to s. Thus, we may assume that $\sum_{j=1}^{r}(\mu_j - 1) \leq k + s$. In particular, r is bounded. After adding $R_1 \cup \cdots \cup R_r$ to Ω, (the new) Ω has the property that under any s-singular embedding of Ω of type Δ considered in step (1), every $(\Omega \cup K)$–bridge that is attached to two or more basic pieces of K can be embedded in at most one of the faces of Ω. This property easily follows from the fact that Ω (even without $R_1, \ldots, R_r$) obstructs all types that are simpler than Ω.

We change Ω, if necessary, so that split auxiliary graphs $\mathrm{Aux}(\Omega, \Pi)$ have properties assured by Corollary 3.6. We also take care so that no $(\Omega \cup K)$–bridge in G is local. This can be achieved by applying Lemma 3.1 for every branch e of $\Omega \cup K$ that is contained in Ω. Lemma 3.1 successfully removes local bridges on e since $\mathrm{Aux}(B, \Pi_0)$ is 3-connected and planar.

STEP (ii). In this step we determine *representatives of types*. Suppose that in step (v), $\Omega \subseteq \Omega'$ is embedded in an s–singular face F. Let B' be a $(K \cup \Omega)$–bridge that can be embedded in two faces. By step (i), we may assume that B' is attached to exactly one basic piece of K on ∂F. Since $\mathrm{Aux}(\Omega, \Pi)$ is nodally 3-connected, the intersection of the two faces is either a main vertex of $\Omega - \partial F$, or a branch of Ω. More precisely, we have the following four mutually exclusive possibilities for the attachment sets of B':

(a) x, v
(b) e, v
(c) e, f
(d) x, f

where x, e are a main vertex of $K \cup \Omega$ and an open branch of $K \cup \Omega$ on ∂F, respectively, and v, f are a main vertex and a (closed) branch of Ω, respectively, that are not contained in ∂F. In case of types (c) or (d) we assume that such a bridge is not attached to f just at one of its ends since in that case the bridge is covered by (b) or (a), respectively. Define the *type* of B' to be the corresponding pair (x, v), (e, v), (e, f), or (x, f). Observe that the number of different types is bounded.

Since $\mathrm{Aux}(B, \Pi_0)$ is planar, we may assume that for each type (x, v) there is at most one bridge $B(x, v)$ of that type (and that this bridge is just an edge). We take $B(x, v)$ as the *representative* of type (x, v). Note that there can be more than two ways of embedding $B(x, v)$. However, any bridges of types corresponding to (b)–(d) admit at most two distinct embeddings. Consider such a type (e, v). If all bridges of this type admit simultaneous embedding in a non-singular face of $K \cup \Omega$ (for some embedding of Ω), let $B_1(e, v)$ and $B_2(e, v)$ be the "leftmost" and the "rightmost" bridge, respectively. Otherwise, let $B_1(e, v)$ and $B_2(e, v)$ be an overlapping pair of such bridges. The obtained bridges are said to be *representatives* of the type (e, v). Similarly we define representatives of a type (x, f). Note that simultaneous embeddability or existence of overlapping bridges is independent of the choice of embedding of Ω. In case of a type (e, f), there are two possibilities for appearance of e and f on the boundary of a non-singular face of $K \cup \Omega$. For each of them we get a pair of bridges, say $B_1(e, f)$, $B_2(e, f)$ and $B_3(e, f)$, $B_4(e, f)$, respectively, in the same way as for types of the form (c) or (d). The obtained bridges $B_i(e, f)$, $i = 1, 2, 3, 4$, are *representatives* of the type (e, f).

Let $Q_1, \ldots, Q_q$ be the representatives of all types of bridges. They have the following property which will be used in the sequel. Suppose that we have an embedding of $\Omega \cup Q_1 \cup \cdots \cup Q_q$. If all representatives of some type are in the same face of $K \cup \Omega$, then all other bridges of this type can be embedded in the same face without obstructing any other bridges more than their representatives do.

STEP (iii). Suppose that Δ is an s-singular type of embeddings from step (v). Denote by F the corresponding (hypothetical) face. Suppose that a branch e of K is singular in Δ and that its appearances in Δ are consecutive and with the same orientation. Suppose that we have an embedding of Ω of type Δ so that there is a face F' of $\Omega \cup \partial F$ with a segment of e singular in F' (cf. Figure 4(b)).

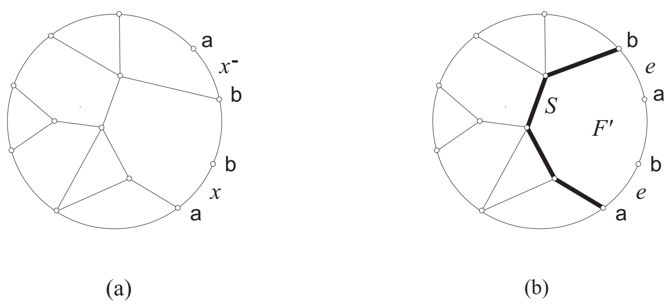

Figure 4: A singular branch on ∂F

Consider the $(\Omega \cup \partial F)$–bridges in $B \cup \partial F$ that can be embedded in F'. They are either attached only to Ω, or they have an attachment in the singular part of e which is not a vertex of Ω. If a bridge is attached to Ω only, then it has essentially different embeddings in F' only if it is attached to a singular vertex of F' (see Figure 5).

Denote by α and β the two extremities of e in F'. Let S be the segment on $\partial F'$ from α to β that is internally disjoint from ∂F (see Figure 4(b)). Since there are no local $(\Omega \cup \partial F)$–bridges in $B \cup \partial F$, any bridge that is a candidate to be embedded in F' cannot be embedded in any other face of Ω in F. It is now easy to see that all bridges that are candidates to be embedded in F' can be checked for all being simultaneously embeddable in F' by applying the 1-Möbius band embedding extension algorithm (Theorem 4.2). In case of positive outcome, the embedded bridges in F' will not comply with the remaining ones. Otherwise, an obstruction Ω_1' for the 1-Möbius band embedding extension is discovered. Note that Ω_1' may contain a (thin or skew) millipede in which case we perform the squashing operation on B and on Ω_1'. In step (v), Ω_1' will be present in Ω'. Thus, for an embedding considered in (1) with a branch e singular as assumed above, bridges embeddable in F' can be simultaneously

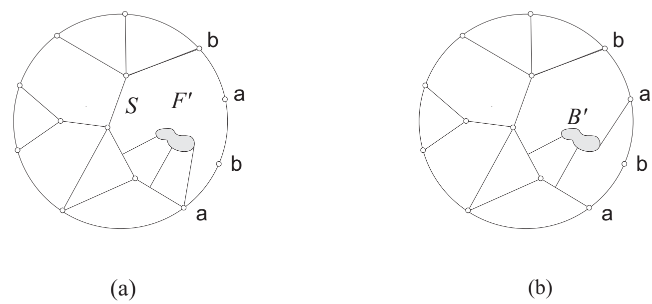

Figure 5: A bridge in F'

embedded in F'.

We repeat the same procedure for all s–singular embeddings of Ω and all corresponding faces F'. Let $\Omega'_1, \ldots, \Omega'_z$ ($z \geq 0$) be the 1-Möbius band obstructions obtained during this process.

Step (iv). We combine Ω with representatives of types $Q_1, \ldots, Q_q$ and the 1-Möbius band obstructions $\Omega'_1, \ldots, \Omega'_z$. Next we make sure that there are no local $(\Omega' \cup K)$–bridges in $B \cup K$. This is achieved by repeated use of Lemma 3.1. Note that the branches of Ω need not to be considered again. Therefore, properties of Q_i and Ω'_j described above still hold after this step.

Step (v). Note that because of step (2) we easily check which admissible types Δ have a strictly simpler type that admits an embedding of B. Therefore we may assume that the obstruction Ω obstructs all simpler types. Since it is bounded, it has only a bounded number of s–singular embeddings. Note that every such embedding is uniquely determined by the corresponding cyclic sequence Π of feet of Ω since $\mathrm{Aux}(\Omega, \Pi)$ is nodally 3-connected.

Steps (1)–(3). Step (1) is described in more details in the sequel while step (2) is obvious. Whenever a millipede is found as a part of an obstruction Ω'', we change B and Ω'' by squashing the millipede. By Theorem 4.9, the new B has exactly the same types of admissible embeddings as the original bridge. The same holds for Ω''. Of course, at the very end we have to replace the squashed obstruction by a subgraph of the original bridge.

Step (vi). Apply Corollary 3.6.

The rest of the proof is devoted to a presentation of step (1). Fix an embedding of Ω' of type Δ. Let Π be the corresponding cyclic sequence of feet of Ω' and F be the corresponding (hypothetical) face. Since $\Omega' \supseteq \Omega$ obstructs all simpler types and because of our initial choice of Ω as an H–graph in B, every basic piece of K on ∂F has a foot of Ω attached to it. Our goal is to find an embedding extension or to extend Ω' to an obstruction that obstructs Π as well.

Consider the induced embedding of Ω. We claim that Ω dissects F into "almost non-singular" 2-cells. Well, suppose that one of the cells, say F' is singular. The singularity in F' appears on ∂F. Let x be a singular basic piece of K in F'. Since every basic piece on ∂F has a foot of Ω attached to it and since Ω is well connected, no other basic piece appears between the two occurrences of x on ∂F. If x is a vertex, then it is clear that the embedding of Ω in F can be changed in such a way that one of the occurrences of x will not have any foot of Ω. This contradicts our choice of Ω. The other possibility is that x is a singular branch. If the two appearances of x on ∂F are in opposite directions, then one of the extreme attachments of B on x must have a foot of Ω (see Figure 4(a)) attached to it, and the only singular point is this vertex (call it β). There are no other basic pieces between the two occurrences of x. Thus the singularity in β is not real: We may assume that all feet of B at β are attached to just one of its occurrences. The remaining case is when the singular branch x appears on ∂F twice in the same direction (Figure 4(b)). In this case we can have an entire segment of x singular. Note that in this case no foot of Ω is attached between the two occurrences of x.

F is dissected by Ω into a number of non-singular faces $F_1, \ldots, F_t$ and some singular faces $F'_1, \ldots, F'_l$. First of all, we determine for each $(\Omega \cup \partial F)$–bridge in $B \cup \partial F$ in which of these faces it is embeddable (according to its attachments). Since Ω' contains $\Omega'_1, \Omega'_2, \ldots, \Omega'_z$ and the embedding of Ω is induced by an embedding of Ω', all $(K \cup \Omega)$–bridges that are candidates to be embedded in $F'_1, \ldots, F'_l$ are embeddable only in these faces. Moreover, they can be embedded so that no two of them obstruct each other. Therefore, we can consider only those embeddings of Ω' for which all these bridges are simultaneously embedded in $F'_1, \ldots, F'_l$. From now on, we do not care about the singular faces $F'_1, \ldots, F'_l$ any more. (Note that we may get embedding extensions of type Δ that differ from our embedding of Ω'. The same will happen also in the sequel.)

It remains to show how to distribute the $(\Omega' \cup \partial F)$–bridges in $F_1, \ldots, F_t$, and how to find their embeddings. As the first step we determine in which of the faces $F_1, \ldots, F_t$ each bridge is embeddable. This can be done for all bridges in linear time as explained in Section 6. If a bridge B' is not embeddable in any face, then we get an obstruction and stop. In step (i) we have added to Ω all bridges that were attached to two or more basic pieces of K. Among the representatives of types $Q_1, \ldots, Q_q$ we have, in particular, all $(K \cup \Omega)$–bridges that are attached just to a main vertex of K on ∂F and to a main vertex of $\Omega - \partial F$. It follows that no $(\Omega \cup \partial F)$–bridge B' is embeddable in 3 or more faces among $F_1, \ldots, F_t$. Thus, every bridge B' has at most two essentially different embeddings in $F_1, \ldots, F_t$. Moreover, if B' can be embedded in two faces, we may assume that B' is attached to exactly one basic piece of K on ∂F and either to a main vertex of $\Omega - \partial F$ or to a branch of Ω. Recall that Ω' contains representatives of types, $Q_1, \ldots, Q_q$.

For a fixed type (e, v) or (e, f), the two faces F_i, F_j of Ω that can accommodate bridges of this type are uniquely determined by e. Similarly, f determines the faces F_i, F_j for bridges of types (x, f) or (e, f).

For every type (α, β) of bridges that can be embedded in two faces of Ω, say in F_1 and in F_2, Ω' contains representatives $B_1(\alpha,\beta), B_2(\alpha,\beta)$ (and $B_3(\alpha,\beta), B_4(\alpha,\beta)$ when applicable) of this type. There are three possibilities:

(a) All representatives are in F_1.

(b) All representatives are in F_2.

(c) At least one representative of type (α, β) is in F_1 and at least one is in F_2.

If we have case (a), representatives do not overlap in F_1. By the choice of representatives, all other bridges of type (α, β) can be embedded without loss of generality in F_1 "between" the representatives. Therefore, they can be without loss of generality eliminated from B (for this embedding of Ω' only). Similarly in case (b). Consequently, whenever we have not fixed embeddings of all bridges of some type, we have (c).

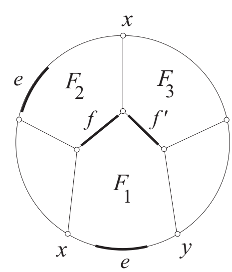

Figure 6: Overlapping of 2-embeddable bridges

Consider distinct types (α, β) and (γ, δ) of bridges. Suppose that we have case (c) for both of these types. Let F_i, F_j be the faces where bridges of type (α, β) can be embedded, and let F_k, F_l be the faces for (γ, δ). If $\{i, j\} \cap \{k, l\} = \emptyset$, then bridges of these two types do not interfere with each other at all. If $i \neq k$ and $j = l$, then it is easy to see that either bridges B', B'' of types (α, β) and (γ, δ), respectively, do not interfere in F_j at all, or B', B'' overlap in F_j independently of their choice. Thus, in such cases the choice of particular representatives of types is not important since for distinct types, *any* bridges of those types in the chosen faces will overlap with each other, or they do not interfere at all. For example, any bridge of type (x, f') (see Figure 6) will overlap in F_1 with any bridge of type (e, f) or (y, f). On the other hand, some other types do not overlap at all (for example, (x, f) and (e, f')). In every face, either one or the other of these two possibilities occurs for any two types that do not have the same pair of faces in which the corresponding bridges embed.

Every embedding of Ω' extends some embedding of Ω. For faces of Ω we use notation and assumptions given above. Now, it suffices to test for simultaneous embeddability for all pairs F_i, F_j of faces that have a type of bridges that is by the chosen embedding of representatives of types embeddable in both of them, and to test for all other faces whether all bridges chosen to be in such a face can be simultaneously embedded in that face. By the above, embeddability problems for two distinct pairs of such faces, say F_i, F_j and F_k, F_l, $\{i, j\} \neq \{k, l\}$, are independent, i.e., no bridge embeddable in F_i and in F_j overlaps with any bridge that is embeddable in F_k and F_l.

The problem of simultaneous embedding of several bridges in one of the faces F_i is easy. Using Lemma 3.2 we either get simultaneous embeddings for all bridges assigned to be in such a face, or we find a bounded obstruction that will be added to Ω in step (vi). The possibility when some types of bridges go into two faces is analyzed in the sequel.

Consider faces F_1, F_2 of Ω in F such that there is a type of bridges that is decided by (c) to be embeddable in both of them. There may be many bridges that are required to be embedded in F_1 (either by the choice (a) or (b), or since they can be embedded in F_1 only). Since F_1 is non-singular, these bridges have essentially unique embeddings in F_1 (or we get an obstruction and stop with this case). The same holds for F_2.

Since $\mathrm{Aux}(\Omega, \Pi)$ is nodally 3-connected, the faces F_1 and F_2 intersect in a main vertex of Ω, or they share a branch. The first case is merely a special case of what we have when F_1 and F_2 intersect in a branch. Thus we will provide details only for the latter possibility.

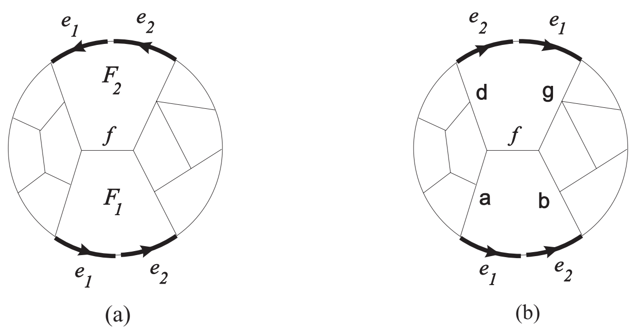

Figure 7: Pairs of singular branches of F

Suppose now that F_1 and F_2 intersect in a branch f of Ω and that F_1, F_2 share a basic piece of K on their boundary. It is easy to see that neither ∂F_1 nor ∂F_2 contains (parts of) three or more basic pieces of K since Ω is well connected and is attached to all appearances of basic pieces on ∂F. Thus F_1, F_2 can share parts of a pair of basic pieces on ∂F. If they have just one basic piece of K in common we can use Theorem 4.1 or 4.2 to either obtaining an obstruction or a simultaneous embedding in $F_1 \cup F_2$ of all applicable bridges. Thus, we may assume that F_1 and F_2 really have two basic pieces e_1, e_2 of K in common and that e_1, e_2 are open branches of K since other possibilities behave like special cases of this one. Since Ω contains an H–graph of B, the branches e_1 and e_2 are oriented as shown in Figure 7(a) or (b). There are no other basic pieces between e_1 and e_2 on ∂F.

Suppose first that e_1, e_2 are as in Figure 7(a). Consider $(\Omega \cup \partial F)$–bridges that are embeddable in F_1 and in F_2. By step (i), no such bridge is attached to e_1 and e_2. Therefore two such bridges overlap in F_1 if and only if they overlap in F_2. If no such bridge attached to e_1 overlaps with such a bridge attached to e_2, we need to solve two independent 1-Möbius band embedding extension problems (Theorem 4.2); first with the bridges attached to e_1 (together with all bridges embeddable in just one of the faces), and then with the bridges attached to e_2 (and bridges embeddable in just one of the faces). Obtaining embedding extensions in both cases, they can be combined to an embedding extension for all of the bridges. On the other hand, an obstruction for any of the two 1-Möbius band embedding extension problems can be taken as the required obstruction for embeddings of B that are extending the given embedding of Ω'.

The other case is when a bridge B_1 attached to e_1 overlaps (in F_1) with a bridge B_2 attached to e_2 (B_1 and B_2 embeddable in F_1 and in F_2). This is easily checked in linear time by comparing attachments of such bridges on f. If this is the case, then B_1 can be embedded in F_1 and B_2 in F_2, and vice versa. In each of the cases, F_1 and F_2 split into two pairs of faces. One pair shares a part of e_1 on their boundaries, the other pair shares a part of e_2. We can then solve two independent 1-Möbius band embedding extension problems and we either get an embedding, or an obstruction (Theorem 4.2). If obstructions are found in both cases, their combination gives an obstruction for extending the embedding of Ω'.

Finally, suppose that we have the situation of Figure 7(b). Let B_1 be an $(\Omega \cup \partial F)$–bridge. For $i = 1, 2$, if B_1 is embeddable only in F_i, we call it (F_i)–bridge. We will assume that e_1 and e_2 are parts of basic pieces of K that are shared by F_1 and F_2. If a part of a corresponding branch of K is only in one of the faces, we assume that it is a part of α, β, γ, or δ (cf. Figure 7; $\alpha, \beta, \gamma, \delta$ denote open segments and they have to be understood as open segments from f to the part of e_1 or e_2 that is shared by F_1 and F_2). Note that every (F_1)–bridge is attached to α or to β. If not, its embedding in F_1 gives rise to an embedding in F_2 as well. Similarly, (F_2)–bridges are attached to γ or to δ. Since there are no local bridges, every bridge can be replaced by its H–graph. (Then any embedding extension to the simplified graph gives rise to an embedding extension of the original graph.)

Try to embed all (F_i)–bridges in F_i at the same time ($i = 1, 2$). If this is not possible, we get a bounded obstruction (Lemma 3.2). Thus we may assume that we have succeeded in embedding all (F_1)–bridges and all (F_2)–bridges. If these bridges remove the "double singularity" (i.e., no pair of new faces shares two basic pieces of K on their boundaries), then we are left with a 1-prism embedding extension problem which can be solved by using Theorem 4.1. Obtaining an obstruction, we add to it also the "outermost" (F_1)–bridges and (F_2)–bridges and stop.

We are left with $(\Omega \cup \partial F)$–bridges that are embeddable in F_1 and in F_2. Such a bridge is called an (e_1)–bridge if it is attached to e_1, and it is an (e_2)–bridge if attached to e_2. Every (e_i)–bridge ($i \in \{1, 2\}$) is attached only to e_i and the (closed) branch f.

Suppose that an (e_1)–bridge B_1 overlaps with another (e_1)–bridge B_2. Such a pair of bridges is then among representatives of the type (e_1, f). Since B_1 and B_2 are in Ω', "doubly singular" pair of faces F_1, F_2 does not occur in embeddings of Ω'. Given an embedding of $\Omega \cup B_1 \cup B_2$ we define subtypes of bridge types (e_1, f) and (e_2, f) in the obvious way. For every embedding of their representatives we get cases (a)–(c) and, as explained above, we either get an embedding in some of the cases, or we get a combined obstruction.

We have the same outcome when an (e_1)–bridge overlaps with an (F_1)–bridge or with an (F_2)–bridge that is attached to e_1. Similarly with e_2.

Suppose now that none of the above cases occurs. Then we may contract e_1 to a vertex x_1 and contract e_2 into a vertex x_2. Any obstruction found in the new graph will give rise to an obstruction of the same branch size in the original and any embedding extension gives rise to an embedding extension for the original graph. After the contractions have been made, the problem transforms into a 1-Möbius band embedding extension problem with the fixed path P_1 containing only the two vertices x_1 and x_2. We are done by Theorem 4.2.

Let us summarize. In case of discovering an embedding extension we are done. If we get a bounded obstruction, we just add it to Ω'. Obtaining a millipede M, we squash B and M, and we proceed as in case of a bounded obstruction. If some previously constructed obstructions for extending particular embeddings of Ω' intersect the part of M that has been changed by the squashing, we simply replace such a part by the squashed millipede. By Theorem 4.9, the squashed bridge B admits exactly the same embedding extensions, and the squashed obstruction obstructs exactly the same embedding types.

It is easy to see that the algorithm described in our proof is linear. □

Let Ξ be a bounded set of admissible types that is $\preceq$–closed, i.e., $\Delta \in \Xi$ and $\Delta' \preceq \Delta$ imply that $\Delta' \in \Xi$. The same proof as above can be used to show that one can find in linear time a nice obstruction $\tilde{B} \subseteq B$ such that for every type $\Delta \in \Xi$ any embedding of $\tilde{B}$ of type Δ can be extended to an embedding of B of the same type. Note that this result can be used also in case when K is not bounded.

6 All the bridges at the same time

Let K be a subgraph of G having bounded branch size. Suppose now that K has more than just one K bridge. By using a modified Depth First Search we can find all K–bridges in linear time. At the same time we mark for each foot, to which bridge it belongs. The next step consists of traversing every main vertex and every open branch of K, and for each K–bridge B, linking the vertices of attachment of B to basic pieces of K into a circular list in the order as they appear on branches of K. It is easy to see that this can be achieved in linear time if we construct circular lists for all bridges simultaneously as their feet are met along branches of K.

For every bridge B of K, the constructed circular list enables us to use the results of previous sections to find universal obstruction $\tilde{B}$ of B in time proportional to $|E(B)|$ (with a bounded constant). Since K is bounded and the edge sets of distinct bridges are disjoint, we have a linear time procedure that for every K–bridge B returns its universal obstruction $\tilde{B}$ and displays embeddings of B of all admissible types that are not obstructed by $\tilde{B}$.

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(Received 12 Jan 2005; revised 9 May 2006)