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The induced path number of the cartesian product of some graphs

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Abstract

The induced path number $\rho(G)$ of a graph G is defined as the minimum number of subsets into which the vertex set $V(G)$ of G can be partitioned such that each subset induces a path. In this paper we determine $\rho(G)$ for $G = K_m \times K_n$. In addition we show that if $G = C_m \times C_n$, then $\rho(G) \leq 3$.

1 Introduction

We generally use the notation and terminology of [7].

Let $S \subseteq V(G)$. The subgraph of G *induced by* S, denoted $\langle S \rangle$, is the graph having vertex set S and edge set those edges of G having both endpoints in S. For a graph G, the *induced path number* $\rho(G)$ is defined by Chartrand et al. in [6] as the minimum number of subsets into which the vertex set $V(G)$ of G can be partitioned such that each subset induces a path. They investigated the induced path number for bipartite graphs and presented formulas for the induced path number of complete bipartite graphs and complete binary trees. They also determined the induced path number of all trees and considered the induced path numbers of meshes, hypercubes

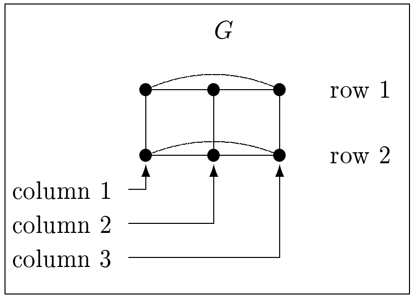

Figure 1: Rows and columns of G

and butterflies. Broere, Jonck and Voigt in [5] and Broere and Jonck in [4] further studied the induced path number of graphs.

In [1] an encompassing theory of partitions of the vertex set $V(G)$ of a graph G is discussed. The contents of this paper does not fit into the framework given in [1] since the property "to be an induced path" is not hereditary. Nevertheless, the topic studied in this paper has given rise to interesting results on notions that are typical in [1], viz. uniquely partitionable graphs in [4] and critical graphs in [3].

The following results for paths, cycles, empty graphs and complete graphs are immediate and take little or no explanation.

Observation 1 *For the path on n vertices, $\rho(P_n) = 1$.*

Observation 2 *For the cycle on n vertices, $\rho(C_n) = 2$.*

Observation 3 *For the complete graph on n vertices, $\rho(K_n) = \lceil \frac{n}{2} \rceil$. In other words, for any positive integer k, $\rho(K_{2k}) = k$ and $\rho(K_{2k+1}) = k + 1$.*

The *cartesian product* of two graphs G_1 and G_2, denoted $G_1 \times G_2$, has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_1, u_2)(v_1, v_2) \mid u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)\}$.

In Figure 1 we indicate, with an example, what we mean by the rows and columns of a graph of the form $G = G_1 \times G_2$ in the sequel (with $G_1 = K_2$ and $G_2 = K_3$).

Note that in this case there is a complete graph K_3 in every row and a complete graph K_2 in every column.

For convenience sake, a vertex in row i and column j is written as (i, j).

The following result is known for the cartesian product of paths $P_{d_1} \times P_{d_2}$ for positive integers d_1 and d_2:

Theorem 1 (Chartrand et al. [6]) *The induced path number of the cartesian product $P_{d_1} \times P_{d_2}$ is two for $d_1, d_2 \geq 2$.*

This result leads us to investigate the induced path number of some products of some other graphs.

The results of this investigation are contained in the next two sections.

2 The induced path number of $K_m \times K_n$

Theorem 2 *Suppose $n \geq m$. Then*

$$\rho(K_m \times K_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ even and } n > m \\ \frac{n}{2} + \lceil \frac{m}{4} \rceil & \text{if } n \text{ is even and } n = m \\ \frac{n-1}{2} + \lceil \frac{m}{2} \rceil & \text{if } n \text{ is odd.} \end{cases}$$

Proof: For the proof of the first case we reason as follows:

Each row and each column of $K_m \times K_n$ induces a complete graph. Therefore, if we want to partition the mn vertices of the graph $K_m \times K_n$ in a number of induced paths, we can choose at most two vertices per row and at most two vertices per column. This means that we need at least $\frac{mn}{2m} = \frac{n}{2}$ partition classes in any partition of $K_m \times K_n$ into induced paths. Thus we have

$$\rho(K_m \times K_n) \geq \frac{n}{2}.$$

A partition of the mn vertices of $K_m \times K_n$ in $\frac{n}{2}$ induced paths, each of order $2m$, for $n > m$, m odd; and for $n > m$, m even, is:

$P_1 : (1, n), (1, 1), (2, 1), (2, 2), (3, 2), (3, 3), \ldots, (m, m-1), (m, m)$

$P_2 : (1, 2), (1, 3), (2, 3), (2, 4), \ldots, (m, [m+1] \bmod n), (m, [m+2] \bmod n)$

$\ldots$

$P_{\frac{n}{2}} : (1, n-3), (1, n-2), (2, n-1), (2, n), (3, n), (3, 1), \ldots, (m, m-3), (m, m-2)$

Note that P_1 ends in the vertex in row m and column m. Also, every P_i contains two vertices from every row.

Therefore we also have that

$$\rho(K_m \times K_n) \leq \frac{n}{2}.$$

For n even and $n > m$, we conclude that

$$\rho(K_m \times K_n) = \frac{n}{2}.$$

The proof of the second case:

Consider an induced path partition of the vertices of $K_m \times K_n$ into $V_1, V_2, \ldots, V_k$. Suppose the set of vertices in a given row is denoted by R. We then consider the non-zero numbers among $|V_1 \cap R|$, $|V_2 \cap R|$, $\ldots$, $|V_k \cap R|$ and we arrange them in non-increasing order — the resulting sequence of positive numbers is called the *form* of R. Note that the form of every row will be $2, 2, \ldots, 2, 1, 1, \ldots 1$ with an even number of 1's.

Among the forms of all the rows, let a be the minimum number of 1's.

If $a = 0$, then there are at least $\frac{n}{2}$ paths as seen in the row with 0 1's. Suppose these $\frac{n}{2}$ paths are longest paths (of order $2n-1$). Suppose further that there are k rows in which these $\frac{n}{2}$ paths end (not begin). Let e_i be the number of endpoints of the $\frac{n}{2}$ paths in row i; then $\frac{n}{2} = \sum_{i=1}^{k} e_i \quad (k \leq \frac{n}{2})$. In row i there are at least $\left\lceil \frac{n-e_i}{2} \right\rceil - (e_1 + \cdots + e_{i-1} + e_{i+1} + \cdots + e_k) = \frac{n}{2} - \left\lfloor \frac{e_i}{2} \right\rfloor - \frac{n}{2} + e_i = \left\lceil \frac{e_i}{2} \right\rceil$ other paths. Thus there are at least $\sum_{i=1}^{k} \left\lceil \frac{e_i}{2} \right\rceil \geq \left\lceil \sum_{i=1}^{k} \frac{e_i}{2} \right\rceil = \left\lceil \frac{n}{4} \right\rceil = \left\lceil \frac{m}{4} \right\rceil$ other paths in total. Thus,

$$\rho(G) \geq \frac{n}{2} + \left\lceil \frac{m}{4} \right\rceil .$$

If $a \geq 2$, then there are at least ma 1's and thus at least $\frac{ma}{2} = \frac{na}{2} > \frac{n}{2}$ paths of which the vertices causing the 1's are the beginning and end vertices. As in the proof of $a = 0$, we can proceed to prove that there are at least $\left\lceil \frac{m}{4} \right\rceil$ other paths. Thus,

$$\rho(G) \geq \frac{n}{2} + \left\lceil \frac{m}{4} \right\rceil .$$

A partition of the mn vertices of $K_m \times K_n$ in $\frac{n}{2} + \left\lceil \frac{m}{4} \right\rceil$ induced paths if m is a multiple of 4 and if m is not a multiple of 4 is:

$P_1 : (m,1), (m-1,1), (m-1,2), (m-3,2), \ldots, (2,n-2), (2,n-1), (1,n-1), (1,n)$

$P_2 : (m-2,1), (m-3,1), (m-3,2), (m-4,2), \ldots, (1,n-2), (m,n-2), (m,n-1), (m-1,n-1), (m-1,n)$

...

$P_{\frac{m}{2}} : (2,1), (1,1), (1,2), (m,2), (m,3), (m-1,3), \ldots, (4,n-2), (4,n-1), (3,n-1), (3,n)$

$P_{\frac{m}{2}+1} : (m,n), (m-2,n)$

$P_{\frac{m}{2}+2} : (m-4,n), (m-6,n)$

...

$P_{\frac{n}{2}+\left\lceil \frac{m}{4} \right\rceil - 1} : (8,n), (6,n)$ if m is a multiple of 4, or $(6,n), (4,n)$ if m is not a multiple of 4

$P_{\frac{n}{2}+\lceil\frac{m}{2}\rceil}$: $(4, n), (2, n)$ if m is a multiple of 4, or $(2, n)$ if m is not a multiple of 4.

Both the paths P_ℓ with $\ell \geq \frac{m}{2} + 1$ are indeed short, most having only two vertices.

Therefore we also have that

$$\rho(K_m \times K_n) \leq \frac{n}{2} + \left\lceil \frac{m}{4} \right\rceil.$$

So, for n even and $n = m$, we conclude that

$$\rho(K_m \times K_n) = \frac{n}{2} + \left\lceil \frac{m}{4} \right\rceil.$$

The proof of the third case:

The proof of this case is similar to the proof of the second case, except that the form of every row will be $2, 2, \ldots, 2, 1, 1, \ldots, 1$ with an odd number of 1's.

Hence there are at least ma 1's in total and thus at least $\frac{ma}{2}$ paths of which the vertices causing the 1's are the beginning and endvertices. But there are also $\frac{n-a}{2}$ other paths as seen in a row containing a 1's. Thus

$$\begin{aligned} \rho(G) &\geq \frac{n-a}{2} + \frac{ma}{2} \\ &= \frac{n-1-a+1}{2} + \frac{m+m(a-1)}{2} \\ &= \frac{n-1}{2} + \frac{m}{2} + \frac{ma-m-a+1}{2} \\ &\geq \frac{n-1}{2} + \frac{m}{2}. \end{aligned}$$

Since $\rho(G)$ is a positive integer, we have that

$$\rho(G) \geq \frac{n-1}{2} + \left\lceil \frac{m}{2} \right\rceil.$$

A partition of the mn vertices of $K_m \times K_n$ into $\frac{n-1}{2} + \lceil\frac{m}{2}\rceil$ induced paths if m is odd and m is even is:

$P_1 : (1, 1), (2, 1), (2, 2), (3, 2), \ldots, (m, m-1), (m, m)$

$P_2 : (1, 2), (1, 3), (2, 3), (2, 4), \ldots, (m, n+1), (m, m+2)$ if $m+1,\ m+2 \leq n$

or $(1, 2), (1, 3), (2, 3), (2, 4), \ldots, (m-2, n), (m-1, n)$ if $m+1 > n$ and m is odd

or $(1, 2), (1, 3), (2, 3), (2, 4), \ldots, (m-1, n), (m, n)$ if $m+1 > n$ and m is even

...

$P_{\frac{n-1}{2}}: (1,n-3)\,, (1,n-2)\,, (2,n-2)\,, (2,n-1)\,, (3,n-1)\,, (3,n)\,, (4,n)$

$P_{\frac{n-1}{2}+1}: (1,n-1)\,, (n,n)\,, (2,n)$

$P_{\frac{n-1}{2}+2}: (3,1)\,, (4,1)\,, (4,2)\,, (5,2)\,, \ldots, (m,m-3)\,, (m,m-2)$

...

$P_{\frac{n-1}{2}+\lceil\frac{m}{2}\rceil-1}: (m-2,1)\,, (m-1,1)\,, (m-1,2)\,, (m,2)\,, (m,3)$ if m is odd

or $(m-3,1)\,, (m-2,1)\,, (m-2,2)\,, (m-1,2)\,, (m-1,3)\,, (m,3)\,, (m,4)$ if m is even

$P_{\frac{n-1}{2}+\lceil\frac{m}{2}\rceil}: (m,1)$ if m is odd

or $(m-1,1)\,, (m,1)\,, (m,2)$ if m is even

Note that in the above argument for the lower bound for $\rho(G)$ we took $a = 1$. That it can be realized is seen by the partition above.

Therefore we also have that

$$\rho(K_m \times K_n) \leq \frac{n-1}{2} + \left\lceil \frac{m}{2} \right\rceil.$$

Thus, for n odd, we conclude that

$$\rho(K_m \times K_n) = \frac{n-1}{2} + \left\lceil \frac{m}{2} \right\rceil.$$

□

3 The induced path number of $C_m \times C_n$

The following two results by I. Broere and M.J. Dorfling [2] are based on Grinberg's ideas (in dual form). Consider 2-partitions of the vertex set of a graph, that is, partitions into two subsets. Such subsets will be denoted by V_1 and V_2, and the number of edges between these sets will be denoted by $e(V_1, V_2)$. For such a partition of a graph G, the number of vertices of V_j which have degree i in the graph G will be denoted by $f(i, j)$.

Theorem 3 *If V_1, V_2 is a 2-partition of G and there is a constant k such that $e\langle V_j\rangle = |V_j| + k$ for $j = 1, 2$, then*

$$\sum_i (i-2)(f(i,1) - f(i,2)) = 0.$$

Proof: We have that

$$\begin{aligned}\sum_i (i-2)f(i,1) &= \sum_{v\in V_1}(deg_G v - 2)\\ &= 2e\langle V_1\rangle + e(V_1,V_2) - 2|V_1|\\ &= 2|V_1| + 2k + e(V_1,V_2) - 2|V_1|\\ &= e(V_1,V_2) + 2k\\ \text{Similarly, } \sum_i (i-2)f(i,2) &= e(V_1,V_2) + 2k.\end{aligned}$$

□

We now use this theorem to prove

Corollary 1 *If the graph G is regular of even degree $2n \geq 4$ and of odd order, then G does not have a 2-partition in subsets inducing acyclic subgraphs with the same number of components.*

Proof: If G has such a partition with c components in each of the induced (acyclic) subgraphs, then the equality $e\langle V_i\rangle = |V_i| - c$ holds for $i = 1, 2$; therefore the theorem is applicable. But then $(2n-2)(f(2n,1) - f(2n,2)) = 0$, implying that $f(2n,1) = f(2n,2)$ which is impossible since G is of odd order. □

We are now ready to consider the induced path number of $C_m \times C_n$. Clearly $m \geq 3$ and $n \geq 3$.

Theorem 4 *Suppose m and n are odd natural numbers. Then*

$$\rho(C_m \times C_n) = 3.$$

Proof: Note that the graph $C_m \times C_n$ is 4-regular and has odd order. By Corollary 1 we have that

$$\rho(C_m \times C_n) \geq 3. \qquad (A)$$

A partition of the mn vertices of $C_m \times C_n$ in three induced paths if $n = m$ and if $n > m$ is:

P_1 : $(1,1), \ldots, (1,n-1), (2,n-1), (2,n), (3,n), (3,1), \ldots, (3,n-3), (4,n-3),$
$(4,n-2),\ (5,n-2), (5,n-1), (5,n), (5,1), \ldots,$
$[(m-3,5), (m-2,5), \ldots, (m-2,n), (m-2,1), (m-2,2), (m-1,2), (m-1,3)]$
if $n = m$

or

$[(m-3, n-m+5), (m-2, n-m+5), \ldots, (m-2, n-m+2),$
$(m-1, n-m+2), (m-1, n-m+3)]$ if $n > m$

$P_2 : (2,1), \ldots, (2, n-2), (3, n-2), (3, n-1), (4, n-1), (4, n), (4, 1), \ldots,$
$(4, n-4), (5, n-4), (5, n-3), (6, n-3), \ldots, (6, n), (6, 1), \ldots,$
$[(m-3, 3), (m-2, 3), (m-2, 4), (m-1, 4), \ldots, (m-1, n), (m-1, 1), (m, 1)]$
if $n = m$

or

$[(m-3, n-m+3), (m-2, n-m+4), (m-1, n-m+4), \ldots, (m-1, n),$
$(m-1, 1), (m, 1), \ldots, (m, n-m)]$ if $n > m$

$P_3 : (1, n), (m, n), \ldots, [(m, 2)]$ if $n = m$

or $[(1, n), (m, n), \ldots, (m, n-m+1), (m-1, n-m+1), \ldots, (m-1, 2)]$ if $n > m$

Thus

$$\rho(C_m \times C_n) \leq 3. \qquad (B)$$

By (A) and (B) we have that $\rho(C_m \times C_n) = 3$. □

Theorem 5 *Suppose $a, k \in N$, $n = 4a$ and $m = 2a(2k-1)+1$. Then*

$$\rho(C_m \times C_n) = 2.$$

Proof: Clearly $\rho(C_m \times C_n) \geq 2$.

A partition of the mn vertices of $C_m \times C_n$, in this case in two induced paths, is:

$P_1 : (1,1), \ldots, \left(1, \frac{n}{2}\right), \left(2, \frac{n}{2}\right), \ldots, (2, n-1), (3, n-1), (3, n), (3, 1), (3, 2), (4, 2),$
$\ldots, \left(4, \frac{n}{2}+1\right), \left(5, \frac{n}{2}+1\right), \ldots, (5, n), (6, n), \ldots, (m-2, 1), (m-2, 2), (m-1, 2),$
$\ldots, \left(m-1, \frac{n}{2}+1\right), \left(m, \frac{n}{2}+1\right), \ldots, (m, n)$

$P_2 : \left(1, \frac{n}{2}+1\right), \ldots, (1, n), (2, n), (2, 1), \ldots, \left(2, \frac{n}{2}-1\right), \left(3, \frac{n}{2}-1\right), \ldots, (3, n-2),$
$(4, n-2), (4, n-1), (4, n), (4, 1), (5, 1), \ldots, \left(5, \frac{n}{2}\right), \left(6, \frac{n}{2}\right), \ldots, (m-2, 3), \ldots,$
$(m-2, n-2), \ldots, (m-1, n-2), (m-1, n-1), (m-1, n), (m-1, 1), (m, 1),$
$\ldots, \left(m, \frac{n}{2}\right)$

This pattern can be stopped after the completion of the $2a+1^{th}$ row or the $6a+1^{th}$ row etc. to form a 2-partition of $C_{2a+1} \times C_{4a}$, $C_{6a+1} \times C_{4a}$ etc. respectively.

Thus $\rho(C_m \times C_n) \leq 2$. □

Theorem 6 *Suppose $m, n \in N$, m is odd, n is even and $n > m$. Then*

$$\rho(C_m \times C_n) \leq 3.$$

Proof: We can partition the mn vertices of $C_m \times C_n$ in three induced paths similar to the partition in the proof of Theorem 4 for the case $n > m$. □

Conjecture 1 *Suppose m, $n \in N$, m is odd, n is even and $n > m$. Suppose also that $C_m \times C_n$ is not one of the cases considered in Theorem 5. Then*

$$\rho(C_m \times C_n) = 3.$$

Theorem 7 *Suppose $m, n \in N$, m is even and $n \geq m$. Then*

$$\rho(C_m \times C_n) \leq 3.$$

Proof: A partition of the mn vertices of $C_m \times C_n$ in three induced paths is:

P_1 : $(1,1), \ldots, (1,n-1), (2,n-1), (2,n), (3,n), (3,1), \ldots, (3,n-3), (4,n-3),$ $(4,n-2), (5,n-2), (5,n-1), (5,n), (5,1), \ldots, (m-3,n-m+3),$ $(m-2,n-m+3), (m-2,n-m+4), (m-1,n-m+4), \ldots, (m-1,n),$ $(m-1,1), \ldots, (m-1,n-m+1)$

P_2 : $(2,1), \ldots, (2,n-2), (3,n-2), (3,n-1), (4,n-1), (4,n), (4,1),$ $\ldots, (4,n-4), (5,n-4),$ $(5,n-3), (6,n-3), \ldots, (6,n), (6,1), \ldots, (m-3,n-m+4), (m-3,n-m+5),$ $(m-2,n-m+5), \ldots, (m-2,n), (m-2,1), \ldots, (m-1,n-m+2),$ $(m-1,n-m+3), (m,n-m+3), \ldots, (m,n-1)$

P_3 : $(1,n), (m,n), (m,1), (m,2), \ldots, (m,n-m+2)$ □

We conclude with the following.

Conjecture 2 *Suppose m, $n \in N$, m is even and $n \geq m$. Then*

$$\rho(C_m \times C_n) = 3.$$

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