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Any tree is a large subgraph of a magic tree

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Abstract

A graph $G = (V, E)$ is said to be *magic* if there exists an integer bijection $f : V \cup E \longrightarrow [1, |V \cup E|]$ such that $f(x) + f(y) + f(xy)$ is constant for all edges $xy \in E$. It has been conjectured by Lladó and Ringel that all trees are magic.

If G is a magic graph such that $f(V) = [1, |V|]$, then the graph is said to be *supermagic*. Here we prove that any given tree of order n is contained in a supermagic tree of order at most $2n - 2$.

1 Introduction

We will denote by $G = (V, E)$ a finite simple graph, and we denote by $[m, n]$ the set of all consecutive integers from m to n.

An integer bijection $f : V \cup E \to [1, |V| + |E|]$ is a *magic* total labeling of G if for any edge $xy \in E$, $f(x) + f(y) + f(xy)$ is constant. A graph G is said to be *magic* if it admits a magic labeling.

In 1970 Kotzig and Rosa, [8], introduced the notion of magic graphs and proved that cycles and complete bipartite graphs are magic. Based on a paper of Kotzig [6], these authors also proved, [9], that the complete graphs K_n are magic only for $n = 2, 3, 5, 6$. Several years later the interest on magic graphs was awaken by a paper of Lladó and Ringel [10], where it was conjectured that all trees are magic. Since then many papers have appeared in the literature dealing with magic graphs. See the dynamic survey of Gallian [5] and the references therein.

The problem we consider here is motivated by the conjecture formulated in [10], that all trees are magic. As it happens with other well–known conjectures on trees, like

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the graceful conjecture (all trees are graceful) or the harmonious conjecture (all trees are harmonious), general results on the magic conjecture for trees are still missing. All these conjectures are connected, see [4], so that, it is not surprising that all of them are likely hard.

In the case of the graceful conjecture, it was shown by Kotzig [7] that the subdivision of an edge of a given tree with a large enough number of vertices gives rise to a graceful tree, but there is no estimation on the number of vertices to be added. As far as we know, there is no similar result for harmonious trees, and the families of trees known to be magic include those which are harmonious; see for instance [5].

In this paper we address the problem of embedding a given tree in a magic tree with the addition of some pendant vertices.

For general graphs Enomoto, Masuda and Nakamigawa [3] give the following result. Given a graph H with n vertices and m edges, there is a connected magic graph G with

$$|V(G)| \leq 2m + 2n^2 + o(n^2),$$

which contains H as an induced subgraph.

In this paper we show that, when H and G are restricted to the family of trees, much better bounds can be obtained.

A total labeling f of a graph G is said to be *supermagic* if f is magic and $f(V) = [1, |V|\,]$. In this case we say that the graph is *supermagic*, see [2]. This is equivalent to the notion of strongly indexable graphs introduced by Acharya and Hedge in [1]. It is known [5] that caterpillars are supermagic.

Here we prove the following result.

Theorem 1 *Let T be a tree of order n. There exists a supermagic tree T' of order $N \leq 2n - 2$ which contains T as a subgraph.* □

2 Magic trees containing a given tree

We shall prove a slightly more precise statement than Theorem 1: any tree T is a subgraph of a tree T' which admits a supermagic labeling and is obtained from T with the addition of some pendant vertices.

First we give a simple criteria to obtain a supermagic labeling of a graph from a vertex labeling on it.

Lemma 1 *Let $G = (V, E)$ be a graph of order n and size m. Let $f : V \to [1, n]$ be a bijective function such that $\{f(x)) + f(y),\ xy \in E\}$ is a consecutive set of pairwise different integers. Then, f can be extended to be a supermagic labeling of G.*

Proof. Denote by $S = \{f(x)) + f(y) :\ xy \in E\}$ the consecutive edge sum set of f and let $\mu = \min S$. Then, $S = [\mu, \mu + m - 1]$. Extend f to E by $f(xy) =$

$\mu + n + m - (f(x) + f(y))$. Therefore $f(E) = [n+1, n+m]$ and f is clearly a supermagic labeling with constant $\mu + m + n$. □

Recall that the *base* tree T_B of a tree T is obtained by deleting all the pendant vertices of T.

Theorem 2 *Any given tree T of order n is contained in a supermagic tree T' of order at most $2n-2$. Moreover, T' can be chosen such that T contains the base tree T'_B except perhaps for one leaf.*

Proof. Since caterpillars are supermagic, we may assume that $n \geq 7$ and the maximum degree of T is $\Delta(T) \geq 3$.

Let r be any vertex of T of degree $\Delta(T)$, which will represent its root. Partition the vertices V of T into levels $V_i = \{x \in V : d(r,x) = i\}$, where $d(r,x)$ denotes the distance in T between x and r.

We define a labeling $f : V \to [1,n]$ recursively on the levels of T. Set $f(r) = 1$. Suppose that f has been defined in all vertices of level V_i for some $i \geq 0$. Take the vertex with smallest label in V_i whose neighbors in V_{i+1} have not been yet labeled and label them with the smallest labels not yet used in any order. In this way we define an injective map and the labels in a given level are consecutive. Notice that this labeling is not unique. See Figure 1 as an example.

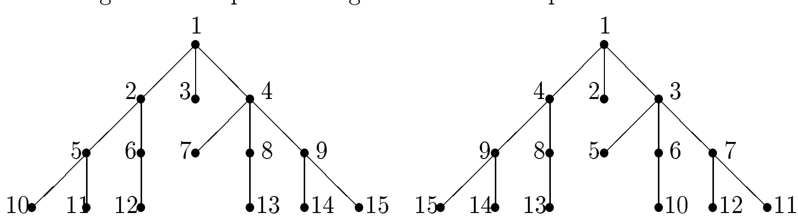


Figure 1: Two vertex labelings on the same rooted tree.

Denote by $S = \{f(x) + f(y) : xy \in E\}$ the sumset of f on the set of edges E of T.

By the definition of f the sums of S are pairwise different, and $|S| = |E| = n - 1$. If the elements of S are consecutive then, by Lemma 1, T is already a super magic tree and we are done.

Suppose that the elements of S are not consecutive numbers and let

$$\bar{S} = [\min S, \max S] \setminus S.$$

We have $\min(S) = 3$ and $n + 1 \leq \max S \leq 2n - 1$, so that

$$|\bar{S}| = \max(S) - \min(S) + 1 - |S| \leq n - 2. \quad (1)$$

In what follows we proceed to extend the tree T in order to fill the gaps in $[3, \max(S)]$. Let

$$\bar{S}_1 = \bar{S} \cap [3, n] \quad \text{and} \quad \bar{S}_2 = \bar{S} \cap [n+1, 2n-1].$$

Denote the elements in these two sets as

$$\bar{S}_1 = \{s_1 < s_2 < \cdots < s_k\} \quad \text{and} \quad \bar{S}_2 = \{s'_1 < s'_2 < \cdots < s'_{k'}\}.$$

We consider two cases.

Case 1. Suppose first that $n+1 \in S$.

For $k > 0$ and $k' > 0$ let

$$V^- = \{0, -1, \ldots, -k+1\} \quad \text{and} \quad V^+ = \{n+1, \ldots, n+k'\}.$$

If either $k = 0$ or $k' = 0$ we define the corresponding set as the empty set.

Notice that, for each $0 \leq i \leq k-1$,

$$1 < s_{k-i} + i \leq s_k \leq n,$$

and for every $1 \leq i \leq k'$, as $n+1 \in S$, we have

$$n > s'_i - (n+i) \geq s'_1 + (i-1) - (n+i) \geq 1.$$

Denote by T' be the graph obtained from T by adding the vertices of $V^- \cup V^+$ and the set of edges $E^- \cup E^+$, where

$$E^- = \{\{-i,\ f^{-1}(i+s_{k-i})\} :\ 0 \leq i < k\},\ E^+ = \{\{n+i,\ f^{-1}(s'_i-(n+i))\} :\ 1 \leq i \leq k'\}.$$

Therefore, the edges in $E^- \cup E^+$ are well defined and the resulting graph is a tree in which $V^- \cup V^+$ are pendant vertices.

So that, the sets of vertices and edges of T' are respectively,

$$V' = V \cup V^- \cup V^+ \quad \text{and} \quad E' = E \cup E^- \cup E^+.$$

Hence, the tree T is contained in T' and contains the base tree T'_B.

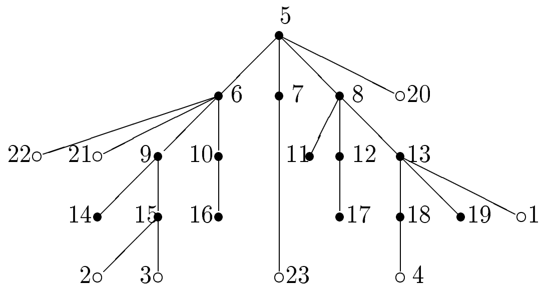
Define a labeling f' on V' as follows

$$f'(x) = \begin{cases} f(x) + k, & x \in V \\ x + k, & x \in V^- \cup V^+ \end{cases}$$

Then, $f'(V') = [1, n + |\bar{S}|]$ and the edge sums of f' form a consecutive set. Therefore, by Lemma 1, T' is a supermagic tree of order $N = n + |\bar{S}|$ and from (1) we have

$$N \leq 2n - 2.$$

See an illustration in Figure 2.



$\bar{S}_1 = \{6, 9, 10, 14\}, \quad \bar{S}_2 = \{17, 19, 20, 22\}$

Figure 2: Vertex labeling of a tree T' obtained from the left labeled tree of Fig. 1.

Case 2. Suppose now that $n + 1 \notin S$.

Since $|S| = n - 1$ and $S \subset [3, 2n - 1]$ we have

$$n + 2 \leq \max(S) \leq 2n - 1.$$

Let

$$s = \begin{cases} \min(\bar{S}_2 \setminus \{n+1\}), & \bar{S}_2 \setminus \{n+1\} \neq \emptyset \\ \max(S) + 1, & \text{otherwise} \end{cases}$$

Then, $s-(n+1) \in [1, n-1]$, so that there is a vertex v of T such that $f(v) = s-(n+1)$.

Let T^+ be the tree obtained from T by adding a new vertex w and the edge $\{v, w\}$. Define on the vertices of T^+ the following labeling,

$$f^+(x) = \begin{cases} f(x), & x \in V \\ n+1, & x = w \end{cases}$$

Note that the tree T^+ has order $n + 1$, the set $S^+ = S \cup \{s\}$ of edge sums of f^+ are pairwise distinct and $n + 2 \in S^+$. Therefore we can apply the procedure of Case 1 to T^+ and obtain a supermagic tree $(T^+)'$ of order $N = n + 1 + |\bar{S}^+|$.

See an example in Figure 3

Let us show that $|\bar{S}^+| \leq n - 3$.

If $\bar{S}_2 \setminus \{n + 1\} \neq \emptyset$, then $s \in \bar{S}$ and therefore $|\bar{S}^+| = |\bar{S}| - 1 \leq n - 3$.

On the other hand, if $\bar{S}_2 \setminus \{n + 1\} = \emptyset$ then $|\bar{S}^+| = |\bar{S}_1 \cup \{n + 1\}| \leq n - 4$. This last inequality follows from the fact that, since the degree of the root is at least 3, we have $\bar{S}_1 \subset [6, n]$.

Hence $(T^+)'$ has order $N \leq 2n - 2$. Moreover T^+ contains the base tree of $(T^+)'$ so that T contains the base tree of $(T^+)'$ except possibly for the added vertex w (see for instance Fig. 3, where $w = 15$). This completes the proof. □

Remark: Consider $S_{p_1,\cdots,p_k}$, a star of $k \geq 3$ paths whose lengths differ in at least two. The procedure described in the above proof gives a supermagic tree of order

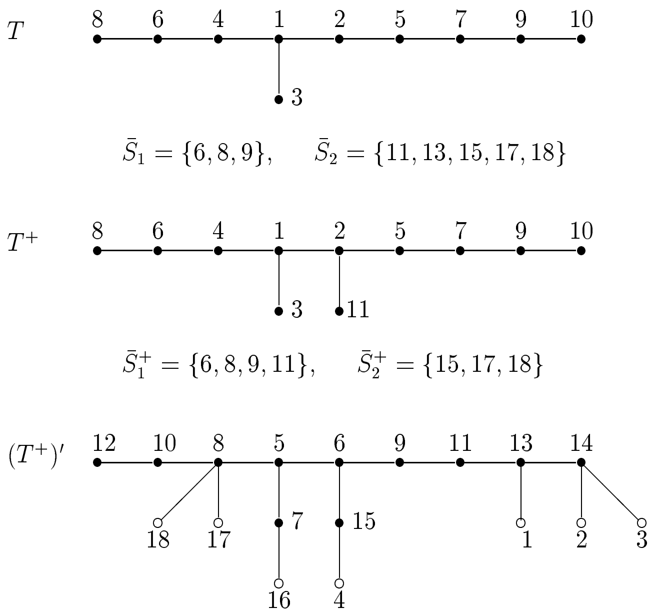


Figure 3: Vertex labellings of the star $S_{1,3,5} = T$, T^+ and $(T^+)'$

exactly $N = 2n - 2$, which is the best general upper bound achievable with this method. See Figure 3.

Since every forest is a subgraph of a tree with the same order, Theorem 2 gives the following Corollary.

Corollary 1 *Any forest of order n is a subgraph of a supermagic tree of order $N \leq 2n - 2$.*

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