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Degree sum conditions and vertex-disjoint cycles in a graph

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Abstract

We consider degree sum conditions and the existence of vertex-disjoint cycles in a graph. In this paper, we prove the following: Suppose that G is a graph of order at least $3k+2$ and $\sigma_3(G) \geq 6k-2$, where $k \geq 2$. Then G contains k vertex-disjoint cycles. The degree and order conditions are sharp.

1 Introduction

We will generally follow notation and terminology of [1]. Let G be a simple graph. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and

$d_G(x) = |N_G(x)|$ is the degree of x in G. For a subgraph H of G and a vertex $x \in V(G) - V(H)$, we also denote $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. For a subgraph H and a subset S of $V(G)$, $d_H(S) = \sum_{x \in S} d_H(x)$, the subgraph induced by S is denoted by $\langle S \rangle$, and $G - S = \langle V(G) - S \rangle$. We often identify an induced subgraph with its vertex set. For a graph G, $|G| = |V(G)|$ is the order of G, $\omega(G)$ is the number of components of G, $\delta(G)$ is the minimum degree of G, $\alpha(G)$ is the independence number of G and

$$\sigma_k(G) = \min \left\{ \sum_{x \in S} d_G(x) : S \text{ is an independent set of } G \text{ with } |S| = k. \right\}$$

(When $\alpha(G) < k$, we define $\sigma_k(G) = \infty$.)

For $X, Y \subseteq V(G)$, $E(X, Y)$ denote the set of edges of G joining a vertex in X and a vertex in Y. If $X = \{x\}$, we denote $E(x, Y)$ instead of $E(\{x\}, Y)$.

K_n denotes a complete graph of order n and P_n denotes a path of order n. For graphs G and H, $G \cup H$ denotes the union of G and H, and $G + H$ denotes the join of G and H. For a graph G, mG denotes the union of m copies of G. If a graph G is isomorphic to a graph H, we denote $G \simeq H$.

A forest is a graph each of whose components is a tree. A leaf is a vertex of a forest whose degree is at most 1.

In this paper, we consider degree sum conditions and the existence of vertex-disjoint cycles. The classical result of this problem was proved by Corrádi and Hajnal.

Theorem 1 (Corrádi and Hajnal [2]) *Suppose that $|G| \geq 3k$ and $\delta(G) \geq 2k$. Then G contains k vertex-disjoint cycles.*

Justesen improved Theorem 1 as follows.

Theorem 2 (Justesen [4]) *Suppose that $|G| \geq 3k$ and $\sigma_2(G) \geq 4k$. Then G contains k vertex-disjoint cycles.*

The degree condition in Theorem 2 is not sharp. Later, Enomoto and Wang independently improved Theorem 2 and got a sharp degree bound.

Theorem 3 (Enomoto [3], Wang [5]) *Suppose that $|G| \geq 3k$ and $\sigma_2(G) \geq 4k-1$. Then G contains k vertex-disjoint cycles.*

Since $G_0 = K_{2k-1} + mK_1$ does not contain k vertex-disjoint cycles, and $\delta(G_0) = 2k - 1$ and $\sigma_2(G_0) = 4k - 2$, the degree conditions in Theorems 1 and 3 are weakest possible.

In this paper, we prove the following theorem.

Theorem 4 *Suppose that $k \geq 2$, $|G| \geq 3k+2$ and $\sigma_3(G) \geq 6k-2$. Then G contains k vertex-disjoint cycles.*

The sharpness of the degree condition is also shown by the graph G_0 since $\sigma_3(G_0) = 6k - 3$.

$K_{3k-1} \cup K_i$ $(i = 1, 2)$ satisfies the degree condition of Theorem 4 since the independence number of this graph is 2, but does not contain k vertex-disjoint cycles. Hence $|G| \geq 3k + 2$ is also weakest possible.

Suppose that $n \geq 5$. Then $\sigma_3(P_n) = 4 = 6 \times 1 - 2$ but P_n does not contain a cycle. Hence $k \geq 2$ is necessary.

Note that the degree condition of Theorem 4 is weaker than those of Theorems 1 and 3:

If $\delta(G) \geq 2k$, then it is easy to see that $\sigma_2(G) \geq 4k - 1$ and $\sigma_3(G) \geq 6k - 2$. Suppose that $\sigma_2(G) \geq 4k-1$. If we take three independent vertices x_1, x_2 and x_3 of G, then $d_G(x_1)+d_G(x_2) \geq 4k-1$, $d_G(x_2)+d_G(x_3) \geq 4k-1$ and $d_G(x_3)+d_G(x_1) \geq 4k-1$. Hence we have $2(d_G(x_1)+d_G(x_2)+d_G(x_3)) \geq 12k-3$, and $d_G(x_1)+d_G(x_2)+d_G(x_3) \geq 6k - 3/2$. This implies that $\sigma_3(G) \geq 6k - 2$.

Before proving Theorem 4, we will give some definitions.

Suppose that $C_1, \ldots, C_r$ are r vertex-disjoint cycles of a graph G. If $C'_1, \ldots, C'_r$ are r vertex-disjoint cycles of G and $|\bigcup_{i=1}^r V(C'_i)| < |\bigcup_{i=1}^r V(C_i)|$, then we call $C'_1, \ldots, C'_r$ are shorter cycles than $C_1, \ldots, C_r$. We call $\{C_1, \ldots, C_r\}$ is minimal if G does not contain r vertex-disjoint cycles $C'_1, \ldots, C'_r$ such that $|\bigcup_{i=1}^r V(C'_i)| < |\bigcup_{i=1}^r V(C_i)|$. We call a cycle of order 3 a *triangle*.

We will use $C[u, v]$ to denote the segment of the cycle C from u to v (including u and v) under some orientation of C, and $C[u, v) = C[u, v] - \{v\}$ and $C(u, v) = C[u, v] - \{u, v\}$. Given a cycle C with an orientation, we let v^+ (resp. v^-) denote the successor (resp. the predecessor) of v along C according to this orientation. Analogously, $v^{2+} = (v^+)^+, v^{3+}, v^{2-} = (v^-)^-, v^{3-}, \ldots$ are defined.

2 Proof of Theorem 4

The following lemmas will be used several times in this section.

Lemma 1 *Let r be a positive integer and $C_1, \ldots, C_r$ be r minimal vertex-disjoint cycles of a graph G. Then $d_{C_i}(x) \leq 3$ for any $x \in V(G) - \bigcup_{j=1}^r V(C_j)$ and for any i, $1 \leq i \leq r$. Furthermore, $d_{C_i}(x) = 3$ implies $|C_i| = 3$ and $d_{C_i}(x) = 2$ implies $|C_i| \leq 4$.*

Proof. This is easily seen by the minimality of $\{C_1, \ldots, C_r\}$ □

Lemma 2 *Suppose that F is a forest with at least two components and C is a triangle. Let x_1, x_2 and x_3 be leaves of F from at least two components. If $d_C(\{x_1, x_2, x_3\}) \geq 7$, then there are two vertex-disjoint cycles in $\langle F \cup C \rangle$ or there exists a triangle C' in $\langle F \cup C \rangle$ such that $\omega(\langle F \cup C \rangle - C') < \omega(F)$.*

Proof. Let $C = v_1v_2v_3v_1$ and F_1, F_2 and F_3 be components of F. Suppose that $x_1, x_2 \in V(F_1)$ and $x_3 \in V(F_2)$. If $d_C(x_1) = 3$, then $d_C(\{x_2, x_3\}) \geq 4$ and $N_C(x_2) \cap N_C(x_3) \neq \emptyset$. Hence we may assume that $v_3 \in N_C(x_2) \cap N_C(x_3)$. Then $C' = x_1v_1v_2x_1$ is a triangle such that $\omega(\langle F \cup C\rangle - C') < \omega(F)$. If $d_C(x_3) = 3$, then $d_C(\{x_1, x_2\}) \geq 4$ and $N_C(x_1) \cap N_C(x_2) \neq \emptyset$. Hence we may assume that $v_3 \in N_C(x_1) \cap N_C(x_2)$. Then $x_3v_1v_2x_3$ and $v_3P_{F_1}[x_1, x_2]v_3$ are two vertex-disjoint cycles, where $P_{F_1}[x_1, x_2]$ is the unique path in F_1 connecting x_1 and x_2.

Next, suppose that $x_1 \in V(F_1)$, $x_2 \in V(F_2)$ and $x_3 \in V(F_3)$. We may assume that $d_C(x_1) = 3$ and $v_3 \in N_C(x_2) \cap N_C(x_3)$. Then $C' = x_1v_1v_2x_1$ is a triangle such that $\omega(\langle F \cup C\rangle - C') < \omega(F)$. □

Lemma 3 *Let C be a cycle and X be a set of three independent vertices. Suppose that $\langle C \cup X\rangle$ does not contain a cycle C' such that $|C'| < |C|$. If $|E(C, X)| \geq 7$, then $|C| = 3$, and $\langle C \cup X\rangle$ can be partitioned into a vertex-disjoint triangle and a path of order 3 connecting two vertices of X.*

Proof. Since $|E(C, X)| \geq 7$, $d_C(x) \geq 3$ for some $x \in X$. This implies that $|C| = 3$ by Lemma 1. Let $C = v_1v_2v_3v_1$ and $X = \{x_1, x_2, x_3\}$. We may assume that $d_C(x_1) = 3$. Since $d_C(\{x_2, x_3\}) \geq 4$, $N_C(x_2) \cap N_C(x_3) \neq \emptyset$. Without loss of generality, we may assume that $v_1 \in N_C(x_2) \cap N_C(x_3)$. Then $\langle C \cup X\rangle$ is partitioned into a triangle $x_1v_2v_3x_1$ and a path of order 3 $x_2v_1x_3$. □

Lemma 4 *Let C be a cycle and T be a tree with three leaves x_1, x_2 and x_3. If $d_C(\{x_1, x_2, x_3\}) \geq 7$, then there exists a cycle C' in $\langle C \cup T\rangle$ such that $|V(C')| < |V(C)|$, or $\langle C \cup T\rangle$ contains two vertex-disjoint cycles.*

Proof. This is immediate by Lemma 3. □

Lemma 5 *Let G be a graph satisfying the assumption of Theorem 4 and $C_1, \ldots, C_{k-1}$ be $k-1$ minimal vertex-disjoint cycles of G. Suppose that there exists a tree T with at least three leaves, which is a component of $G - \bigcup_{i=1}^{k-1} V(C_i)$. Then G contains k vertex-disjoint cycles.*

Proof. Let $L = \bigcup_{i=1}^{k-1} V(C_i)$ and $X = \{x_1, x_2, x_3\}$ be a set of leaves of T. Since X is independent and $d_T(x) = 1$ for all $x \in X$, $d_L(X) \geq 6k - 2 - 3 = 6k - 5 > 6(k-1)$. Hence $d_{C_i}(X) \geq 7$ for some i, $1 \leq i \leq k-1$. By Lemma 4, there exist two vertex-disjoint cycles in $\langle X \cup C_i\rangle$ since $\{C_1, \ldots, C_{k-1}\}$ is minimal. Hence we have k vertex-disjoint cycles of G. □

Lemma 6 *Let G be a graph satisfying the assumption of Theorem 4 and let $C_1, \ldots, C_{k-1}$ be $k-1$ minimal vertex-disjoint cycles of G. Suppose that $|G - \bigcup_{i=1}^{k-1} V(C_i)| = 4$ and $G - \bigcup_{i=1}^{k-1} V(C_i)$ is not connected and is not isomorphic to $2K_2$. Then there exist $k-1$ minimal vertex-disjoint cycles $C'_1, \ldots, C'_{k-1}$ such that $G - \bigcup_{i=1}^{k-1} V(C'_i)$ is connected.*

Proof. Let $L = \bigcup_{i=1}^{k-1} V(C_i)$, $H = G - L$ and $V(H) = \{x_1, x_2, x_3, x_4\}$. We have to consider the following three cases;

(i) $H \simeq P_3 \cup K_1$,

(ii) $H \simeq K_2 \cup 2K_1$, and

(iii) $H \simeq 4K_1$.

Without loss of generality, we may assume that $x_1x_2, x_2x_3 \in E(G)$ for (i), and $x_1x_2 \in E(G)$ for (ii). In each of three cases, $X = \{x_1, x_3, x_4\}$ is independent and $d_H(X) \leq 2$. Hence $d_L(X) \geq 6k - 2 - 2 = 6k - 4 > 6(k-1)$ and this implies that $d_{C_i}(X) \geq 7$ for some i, $1 \leq i \leq k-1$. Then by Lemma 3, we can take minimal vertex-disjoint cycles $C'_1, \ldots, C'_{k-1}$ such that $\omega(G - \bigcup_{i=1}^{k-1} V(C'_i)) < \omega(H)$. Moreover, $G - \bigcup_{i=1}^{k-1} V(C'_i)$ contains a path of order 3 connecting two vertices of X. Hence $G - \bigcup_{i=1}^{k-1} V(C'_i) \not\simeq 2K_2$. By repeating this argument, we can get a conclusion. □

Proof of Theorem 4. Let G be an edge-maximal counterexample. Since a complete graph of order at least $3k+2$ contains k vertex-disjoint cycles, G is not complete. Let x and y be non-adjacent vertices of G. Then $G' = G + xy$, the graph obtained from G by adding the edge xy, is not a counterexample by the maximality of G. Hence G' contains k vertex-disjoint cycles $C_1, \ldots, C_k$ and without loss of generality, we may assume that $xy \in E(C_k)$. This means that G contains $k-1$ vertex-disjoint cycles $C_1, \ldots, C_{k-1}$ such that $\sum_{i=1}^{k-1} |V(C_i)| \leq n - 3$. Let $L = \langle \bigcup_{i=1}^{k-1} V(C_i) \rangle$ and $H = G - L$. Take $k-1$ minimal vertex-disjoint cycles $C_1, \ldots, C_{k-1}$ so that

$$\omega(H) \text{ is as small as possible.} \tag{1}$$

Claim 1 *Each component of H is a path.*

Proof. This is immediate by Lemma 5. □

Claim 2 *H is connected, or $|H| = 4$ and $H \simeq 2K_2$*

Proof. Suppose that H is not connected.

If $|H| \geq 5$ and $\omega(H) \geq 3$, then we can take three leaves x_1, x_2 and x_3 from three different components. If $|H| \geq 5$ and $\omega(H) = 2$, then there exists a component H' of H such that $|H'| \geq 3$. Since H' is a path by Claim 1, we can take two leaves x_1, x_2 from H', and take a leaf x_3 from another component. In each case, $X = \{x_1, x_2, x_3\}$ is independent and $d_H(X) \leq 3$. Hence $d_L(X) \geq 6k - 2 - 3 = 6k - 5 > 6(k-1)$ and this means that $d_{C_i}(X) \geq 7$ for some i, $1 \leq i \leq k-1$. Then $d_{C_i}(x) \geq 3$ for some $x \in X$ and $|C_i| = 3$ by Lemma 1. By Lemma 2, we have $k-1$ minimal vertex-disjoint cycles $C'_1, \ldots, C'_{k-1}$ such that $\omega(G - \bigcup_{j=1}^{k-1} V(C'_j)) < \omega(H)$ because G does not contain k vertex-disjoint cycles. But this contradicts the choice of cycles (1).

If $|H| = 4$ and $H \not\simeq 2K_2$, then we can get the conclusion by Lemma 6.

Hence we may assume that $|H| = 3$. Let x and y be non-adjacent vertices of G. Then $G + xy$ contains k vertex-disjoint cycles $D_1, \ldots, D_k$. Without loss of generality, we may assume that $xy \in E(D_k)$. If $|D_k| \geq 4$, then $|\bigcup_{i=1}^{k-1} V(D_i)| < |L|$, but this contradicts the minimality of L. Hence $|D_k| = 3$. If $G - \bigcup_{i=1}^{k} V(D_i) \neq \emptyset$, then $|\bigcup_{i=1}^{k-1} V(D_i)| < |L|$ since $|G - \bigcup_{i=1}^{k-1} V(D_i)| \geq 4$. Therefore, $V(G) = \bigcup_{i=1}^{k} V(D_i)$,

$\{D_1, \ldots, D_{k-1}\}$ is minimal and $G - \bigcup_{i=1}^{k-1} V(D_i)$ is connected. By the choice of cycles (1), H is connected. □

We distinguish two cases according to the value of $|H|$.

CASE 1 $|H| \geq 5$

By Claims 1 and 2, H is a path. Let $x_1 x_2 \cdots x_l$, where $l = |H|$, and let $X = \{x_1, x_3, x_l\}$. Then X is independent.

Claim 3 $d_{C_i}(X) \leq 6$ *for any* i, $1 \leq i \leq k-1$.

Proof. Suppose that $d_{C_i}(X) \geq 7$ for some i, $1 \leq i \leq k-1$. Since $d_{C_i}(x) \geq 3$ for some $x \in X$, $|C_i| = 3$ by Lemma 1. Let $C_i = v_1 v_2 v_3 v_1$.

Suppose that $d_{C_i}(x_1) = 3$. Since $d_{C_i}(\{x_3, x_l\}) \geq 4$, $N_{C_i}(x_3) \cap N_{C_i}(x_l) \neq \emptyset$ and we may assume that $v_3 \in N_{C_i}(x_3) \cap N_{C_i}(x_l)$. Then $x_1 v_1 v_2 x_1$ and $v_3 x_3 x_4 \cdots x_l v_3$ are two vertex-disjoint cycles in $\langle H \cup C_i \rangle$, and we have k vertex-disjoint cycles of G, a contradiction.

Hence $d_{C_i}(x_1) \leq 2$. Similarly, we have $d_{C_i}(x_l) \leq 2$. This means that $d_{C_i}(x_3) = 3$ and $d_{C_i}(x_1) = d_{C_i}(x_l) = 2$.

Suppose that $N_{C_i}(x_1) \neq N_{C_i}(x_l)$. Without loss of generality, we may assume that $v_1 \in N_{C_i}(x_1)$ and $v_2, v_3 \in N_{C_i}(x_l)$. Then $x_1 x_2 x_3 v_1 x_1$ and $x_l v_2 v_3 x_l$ are two vertex-disjoint cycles in $\langle H \cup C_i \rangle$, and we have k vertex-disjoint cycles of G, a contradiction. Hence we have $N_{C_i}(x_1) = N_{C_i}(x_l)$ and we may assume that $\{v_1, v_2\} = N_{C_i}(x_1)$. If we take $C'_i = x_1 v_1 v_2 x_1$ and $C'_j = C_j$ for $j \neq i$, then $\{C'_1, \ldots, C'_{k-1}\}$ is minimal and $G - \bigcup_{j=1}^{k-1} V(C'_j)$ is a tree with three leaves x_2, x_l and v_3 since otherwise we can find two vertex-disjoint cycles in $\langle H \cup C_i \rangle$. By Lemma 5, we have k vertex-disjoint cycles of G, a contradiction. Hence the proof is completed. □

By Claim 3, we have

$$d_L(X) \leq 6(k-1).$$

On the other hand, since $d_H(X) = 4$,

$$d_L(X) \geq 6k - 2 - 4 = 6(k-1).$$

Hence $d_L(X) = 6(k-1)$ and $d_{C_i}(X) = 6$ for all i, $1 \leq i \leq k-1$. By Lemma 1, we have $|C_i| \leq 4$ since $d_{C_i}(x) \geq 2$ for some $x \in X$.

Claim 4 $|C_i| = 3$ *for all* i, $1 \leq i \leq k-1$.

Proof. Suppose that $|C_i| = 4$ and let $C_i = v_1 v_2 v_3 v_4 v_1$. By Lemma 1, $d_{C_i}(x) = 2$ for all $x \in X$.

Suppose that $N_{C_i}(x_1) \neq N_{C_i}(x_3)$. Then we may assume that $N_{C_i}(x_1) = \{v_1, v_3\}$ and $N_{C_i}(x_3) = \{v_2, v_4\}$. Note that there do not exist two vertex-disjoint cycles in $\langle H \cup C_i \rangle$, since otherwise we have k vertex-disjoint cycles of G, a contradiction.

Take $C'_i = x_1v_1v_2v_3x_1$ and $C'_j = C_j$ for $j \neq i$. Then $\{C'_1, \dots, C'_{k-1}\}$ is minimal and $G - \bigcup_{j=1}^{k-1} V(C'_j)$ is a tree with three leaves x_2, x_l and v_4. By Lemma 5, we have k vertex-disjoint cycles of G, but this is a contradiction.

Hence $N_{C_i}(x_1) = N_{C_i}(x_3)$. Similarly, we have $N_{C_i}(x_3) = N_{C_i}(x_l)$. Without loss of generality, we may assume that $N_{C_i}(x) = \{v_1, v_3\}$ for all $x \in X$. Taking $C'_i = x_1x_2x_3v_1x_1$ and $C'_j = C_j$ for $j \neq i$, then $\{C'_1, \dots, C'_{k-1}\}$ is minimal and $G - \bigcup_{j=1}^{k-1} V(C'_j)$ is a tree with three leaves x_4, v_2 and v_4. By Lemma 5, this is also a contradiction. □

Claim 5 *Let $x \in \{x_1, x_l\}$. If $A \subset N_{C_i}(x)$ and $|A| = 2$, then $N_{C_i}(x_3) \setminus A = \emptyset$.*

Proof. Let $C_i = v_1v_2v_3v_1$. Suppose that the claim does not hold, and let $x = x_1$. Without loss of generality, we may assume that $v_1, v_2 \in N_{C_i}(x_1)$ and $v_3 \in N_{C_i}(x_3)$.

Take $C'_i = x_1v_1v_2x_1$ and $C'_j = C_j$ for $j \neq i$. Then $\{C'_1, \dots, C'_{k-1}\}$ is minimal and $G - \bigcup_{j=1}^{k-1} V(C'_j)$ is a tree with three leaves x_2, x_l and v_3 since there do not exist two vertex-disjoint cycles in $\langle H \cup C_i \rangle$. By Lemma 5, we have k vertex-disjoint cycles of G, a contradiction.

For the case $x = x_l$, we can prove similarly. □

Claim 6 *There exist only two type of configurations between H and C_i for all i, $1 \leq i \leq k-1$. (See Figure 1.)*

Proof. Suppose that $d_{C_i}(x_1) = 3$. By Claim 5, we have $d_{C_i}(x_3) = 0$ and $d_{C_i}(x_l) = 3$ since $d_{C_i}(X) = 6$. (This is Type 1.)

Next, suppose that $d_{C_i}(x_1) \leq 1$. Since $d_{C_i}(X) = 6$, $d_{C_i}(\{x_3, x_l\}) \geq 5$. But this contradicts Claim 5.

Finally, suppose that $d_{C_i}(x_1) = 2$. By Claim 5, we have $N_{C_i}(x_1) = N_{C_i}(x_3)$. Since $d_{C_i}(X) = 6$, we have $d_{C_i}(x_l) = 2$ and $N_{C_i}(x_l) = N_{C_i}(x_3)$. (This is Type 2.)

Hence the claim is proved. □

In each configuration, we find that $d_{C_i}(x_2) = d_{C_i}(x_4) = 0$ for any i, $1 \leq i \leq k-1$, since otherwise we can find two vertex-disjoint cycles in $\langle H \cup C_i \rangle$. This means that $d_G(x_2) = d_G(x_4) = 2$.

Let $C_1 = v_1v_2v_3v_1$. Since $\{x_2, x_4, v_3\}$ is independent,

$$6k - 2 \leq d_G(\{x_2, x_4, v_3\}) \leq 2 + 2 + 3(k-2) + 4 = 3k + 2,$$

but this is a contradiction since $k \geq 2$. This completes the proof of CASE 1.

CASE 2 $|H| \leq 4$.

Let $V(H) = \{x_1, \dots, x_{|H|}\}$. By Claims 1 and 2, we may assume that $x_1x_2, x_2x_3 \in E(G)$ if $|H| = 3$ and that $x_1x_2, x_3x_4 \in E(G)$ if $|H| = 4$.

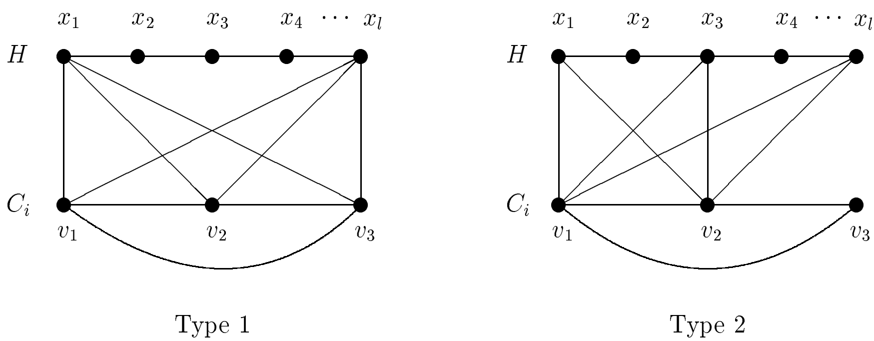

Figure 1: Configurations Type 1 and Type 2

Claim 7 *There exists* i, $1 \leq i \leq k-1$ *such that* $|C_i| \geq 4$ *and* $|E(y, C_j)| \leq 3$ *for any* $y \in V(C_i)$ *and* $j \neq i$.

Proof. Since $|G| \geq 3k+2$ and $|H| \leq 4$, $|L| \geq 3k-2 > 3(k-1)$. Hence there exists i, $1 \leq i \leq k-1$ such that $|C_i| \geq 4$.

We define a directed graph $\vec{D} = (V(\vec{D}), E(\vec{D}))$ as follows:

$$\begin{aligned} V(\vec{D}) &= \{C_i : |C_i| \geq 4,\ 1 \leq i \leq k-1\} \\ E(\vec{D}) &= \{(C_i, C_j) : |E(y, C_j)| \geq 4 \text{ for some } y \in V(C_i) \text{ and } j \neq i\} \end{aligned}$$

Suppose that $\vec{D}$ contains a directed cycle. Without loss of generality, we may assume that $(C_1, C_2), (C_2, C_3), \ldots, (C_m, C_1) \in E(\vec{D})$, where $m \geq 2$. Take $y_i \in V(C_i)$ so that $|E(y_i, C_{i+1})| \geq 4$. (Hereafter in the proof of this claim, let $C_{m+1} = C_1$.) Then there exist $v_{i+1}, w_{i+1} \in N_{C_{i+1}}(y_i)$ such that $y_{i+1} \notin C_{i+1}[v_{i+1}, w_{i+1}]$ and $C_{i+1}(v_{i+1}, w_{i+1}) \cap N_{C_{i+1}}(y_i) = \emptyset$. For $1 \leq i \leq m$, we define new cycles as

$$C'_i = y_i C_{i+1}[v_{i+1}, w_{i+1}] y_i.$$

Then $|\bigcup_{i=1}^m V(C'_i)| < |\bigcup_{i=1}^m V(C_i)|$, but this contradicts the minimality of L since $V(C'_{i+1})$ misses at least one neighbor of $N_{C_{i+1}}(y_i)$ for each i, $1 \leq i \leq m$. Hence $\vec{D}$ does not contain a directed cycle and an endvertex of a directed path is a desired cycle. □

Without loss of generality, we may assume that C_1 satisfies the property of Claim 7.

Claim 8 $|\{y \in V(C_1) : |E(y, C_j)| = 3\}| \leq 2$ *for any* j, $2 \leq j \leq k-1$.

Proof. Suppose not. Without loss of generality, we may assume that $|E(y, C_j)| = 3$ for any $y \in \{y_1, y_2, y_3\} \subset V(C_1)$. Let $v_1, v_2, v_3 \in N_{C_j}(y_1)$ and suppose that v_1, v_2 and v_3 appear in this order in C_j.

Suppose that $y_2v_1 \notin E(G)$. If $|N_{C_j}(y_2) \cap C_j(v_1, v_3)| \geq 2$, we can find two shorter cycles than C_1 and C_j. Since $d_{C_j}(y_2) = 3$, we have $|N_{C_j}(y_2) \cap C_j[v_3, v_1)| \geq 2$. In this case, we also find two shorter cycles than C_1 and C_j since $v_2 \in N_{C_j}(y_1)$.

Hence $y_2v_1 \in E(G)$. By symmetry, $v_1, v_2, v_3 \in N_{C_j}(y)$ for $y \in \{y_2, y_3\}$. But we can find two shorter cycles than C_1 and C_j since $|C_1| \geq 4$. □

Claim 9 $E(x_2, C_1) \neq \emptyset$.

Proof. Suppose that $E(x_2, C_1) = \emptyset$. Let $Y = \{y_1, y_2, y_3, y_4\} \subset V(C_1)$ and suppose that y_1, y_2, y_3 and y_4 appear in this order in C_1.

Subclaim 9.1 $f = 2|E(x_2, C_i)| + |E(Y, C_i)| \leq 12$ *for any* i, $2 \leq i \leq k-1$.

Proof. Suppose that $f \geq 13$ for some i, $2 \leq i \leq k-1$. Since $|E(Y, C_i)| \leq 10$ by the choice of C_1 and Claim 8, $|E(x_2, C_i)| \geq 2$. On the other hand, $|E(x_2, C_i)| \leq 3$ by Lemma 1.

Case A $|E(x_2, C_i)| = 3$.

In this case, we have $|C_i| = 3$ by Lemma 1. Furthermore, we have $|E(v, Y)| \leq 2$ for any $v \in V(C_i)$, since otherwise we can find two shorter cycles than C_1 and C_i in $\langle H \cup C_1 \cup C_i \rangle$. Then

$$f \leq 2 \times 3 + 2 \times 3 = 12,$$

a contradiction.

Case B $|E(x_2, C_i)| = 2$.

In this case, $|C_i| \leq 4$ by Lemma 1. Since $f \geq 13$, we have

$$|E(Y, C_i)| \geq 9. \tag{2}$$

Hence $d_{C_i}(y) \geq 3$ for some $y \in Y$. Without loss of generality, we may assume that $d_{C_i}(y_1) \geq 3$. Moreover, $d_{C_i}(y_1) = 3$ by the choice of C_1. Let $Y' = \{y_2, y_3, y_4\}$.

Case B.1 $|C_i| = 4$.

Let $C_i = v_1v_2v_3v_4v_1$. We may assume that $v_1, v_2, v_3 \in N_{C_i}(y_1)$. Then $|E(v_j, Y')| \leq 1$ for $j \in \{1, 3, 4\}$ and $|E(v_2, Y')| \leq 2$ since otherwise we can find two shorter cycles than C_1 and C_i in $\langle C_1 \cup C_i \rangle$. Hence

$$|E(Y, C_i)| = d_Y(\{v_1, v_2, v_3, v_4\}) \leq 2 + 3 + 2 + 1 = 8,$$

but this contradicts (2).

Case B.2 $|C_i| = 3$.

Let $C_i = v_1v_2v_3v_1$. In this case, $N_{C_i}(y_1) = \{v_1, v_2, v_3\}$ and we may assume that $N_{C_i}(x_2) = \{v_1, v_2\}$. Then $|E(v_j, Y')| \leq 2$ for $1 \leq j \leq 2$ and $|E(v_3, Y')| \leq 1$, since otherwise we can find two shorter cycles than C_1 and C_i in $\langle C_1 \cup C_i \cup \{x_2\} \rangle$. Hence

$$|E(Y, C_i)| = d_Y(\{v_1, v_2, v_3\}) \leq 3 + 3 + 2 = 8,$$

but this contradicts (2). Hence Subclaim 9.1 is proved. □

Since each of $\{x_2, y_1, y_3\}$ and $\{x_2, y_2, y_4\}$ is independent,

$$\begin{aligned} 2(6k-2) &\leq 2d_G(x_2) + d_G(Y) \\ &\leq 12(k-2) + 4 + 8 + |E(Y, H - \{x_2\})| \end{aligned}$$

by Subclaim 9.1. Hence $|E(Y, H - \{x_2\})| \geq 8$. Since $|H| \leq 4$, $|E(Y, x)| \geq 3$ holds for some $x \in V(H) - \{x_2\}$, but this contradicts the minimality of L by Lemma 1. Hence the proof of Claim 9 is completed. □

First, we consider the case $|H| = 4$ and $H \simeq 2K_2$.

CASE 2.1 $H \simeq 2K_2$.

By Claim 9, $E(x, C_1) \neq \emptyset$ for all $x \in V(H)$. Then it is easy to see that $|C_1| \leq 6$. Let $x_1 y \in E(G)$ for $y \in V(C_1)$. We give an orientation to C_1 so that $x_2 y^- \notin E(G)$ if it is possible. Since at least one of x_3 and x_4 is not adjacent to y^-, we may assume that $x_3 y^- \notin E(G)$. Then $Z = \{x_1, x_3, y^-\}$ is independent. Let $H' = \langle H \cup C_1 \rangle$.

Claim 10 $|E(Z, C_i)| \leq 6$ *for any* i, $2 \leq i \leq k-1$.

Proof. Suppose that $|E(Z, C_i)| \geq 7$ for some i, $2 \leq i \leq k-1$. We consider the following two cases.

Case A $5 \leq |C_1| \leq 6$, or $|C_1| = 4$ and there exists a cycle of order 4 containing $x_1 x_2$ in H'.

In this case, we may assume that $x_2 y' \in E(G)$ for $y' = y^{(|C_1|-3)+}$. If we take $C'_1 = x_1 C_1[y, y'] x_2 x_1$ and $C'_j = C_j$ for $2 \leq j \leq k-1$, then $\{C'_1, \ldots, C'_{k-1}\}$ is minimal. Note that y^- does not lie in C'_i. By Lemma 1, $d_{C_j}(y^-) \leq 3$ for $2 \leq j \leq k-1$ and $d_{C_j}(y^-) = 3$ implies $|C_j| = 3$. Since $|E(Z, C_i)| \geq 7$, $d_{C_i}(z) \geq 3$ for some $z \in Z$, and we have $|C_i| = 3$. Let $C_i = v_1 v_2 v_3 v_1$.

Suppose that $d_{C_i}(x_1) = 3$. Since $d_{C_i}(\{x_3, y^-\}) \geq 4$, $d_{C_i}(x_3) \geq 1$ and we may assume that $x_3 v_3 \in E(G)$. Take $C'_i = x_1 v_1 v_2 x_1$ and $C'_j = C_j$ for $j \neq i$. Then $\{C'_1, \ldots C'_{k-1}\}$ is minimal and $G - \bigcup_{j=1}^{k-1} V(C'_j)$ is connected or $G - \bigcup_{j=1}^{k-1} V(C'_j) \simeq P_3 \cup K_1$. By Lemma 6, this contradicts the choice of cycles (1). Therefore, $d_{C_i}(x_1) \leq 2$. Similarly, we have $d_{C_i}(x_3) \leq 2$.

Hence $d_{C_i}(y^-) = 3$ and $d_{C_i}(x_1) = d_{C_i}(x_3) = 2$. Without loss of generality, we may assume that $x_3 v_3 \in E(G)$. Taking $C'_1 = x_1 C_1[y, y'] x_2 x_1$, $C'_i = y^- v_1 v_2 y^-$ and $C'_j = C_j$ for $j \neq 1, i$, then $\{C'_1, \ldots C'_{k-1}\}$ is minimal and $G - \bigcup_{j=1}^{k-1} V(C'_j)$ is connected, or $G - \bigcup_{j=1}^{k-1} V(C'_j) \simeq P_3 \cup K_1$. But this contradicts the choice of cycles (1) by Lemma 6.

Case B $|C_1| = 4$ and there exists no cycle of order 4 containing $x_1 x_2$ in H'.

By symmetry, we may assume that there exists no cycle of order 4 containing x_3x_4 in H'. In this case, $x_2y^{2+} \in E(G)$ and $d_{C_1}(x) = 1$ for all $x \in V(H)$.

By the choice of C_1, $d_{C_i}(y^-) \le 3$ holds. Then $d_{C_i}(\{x_1, x_3\}) \ge 4$, and we have $d_{C_i}(x_1) \ge 2$ or $d_{C_i}(x_3) \ge 2$. By Lemma 1, we have $|C_i| \le 4$.

Case B.1 $|C_i| = 4$.

Let $C_i = v_1v_2v_3v_4v_1$. By Lemma 1, $d_{C_i}(x_1) \le 2$ and $d_{C_i}(x_3) \le 2$. Hence we have $d_{C_i}(y^-) = 3$ and $d_{C_i}(x_1) = d_{C_i}(x_3) = 2$.

Since $d_{C_i}(y^-) = 3$ and $d_{C_i}(x_3) = 2$ without loss of generality, we may assume that $v_1, v_2, v_3 \in N_{C_i}(y^-)$ and $v_3 \in N_{C_i'}(x_3)$. If we take $C_1' = x_1C_1[y, y^{2+}]x_2x_1$, $C_i' = y^-v_1v_2y^-$ and $C_j' = C_j$ for $j \ne 1, i$, then $\{C_1', \ldots, C_{k-1}'\}$ is minimal and $G - \bigcup_{j=1}^{k-1} V(C_j')$ is connected. This also contradicts the choice of cycles (1).

Case B.2 $|C_i| = 3$.

Let $C_i = v_1v_2v_3v_1$. Since $d_{C_i}(y^-) \le 3$, we have $d_{C_i}(\{x_1, x_3\}) \ge 4$. Suppose that $d_{C_i}(x_1) = 3$. Then $d_{C_i}(x_3) \ge 1$ and without loss of generality, we may assume that $v_3 \in N_{C_i}(x_3)$. If we take $C_i' = x_1v_1v_2x_1$ and $C_j' = C_j$ for $j \ne i$, then $\{C_1', \ldots C_{k-1}'\}$ is minimal and $G - \bigcup_{j=1}^{k-1} V(C_j')$ is connected, or $G - \bigcup_{j=1}^{k-1} V(C_j') \simeq P_3 \cup K_1$. By Lemma 6, this contradicts the choice of cycles (1). Hence $d_{C_i}(x_1) \le 2$. Similarly, we have $d_{C_i}(x_3) \le 2$. Then $d_{C_i}(y^-) = 3$ and $d_{C_i}(x_1) = d_{C_i}(x_3) = 2$, and we may assume that $v_1, v_2 \in N_{C_i}(x_1)$. If $v_3 \in N_{C_i}(x_3) \cap N_{C_i}(x_4)$, then $C_i' = x_1v_1v_2x_1$, $C_k' = x_3x_4v_3x_3$ and $C_j' = C_j$ for $j \ne i$ are k vertex-disjoint cycles in G. Hence $v_3 \notin N_{C_i}(x_1) \cap N_{C_i}(x_3)$. If $x_2v_3 \notin E(G)$, then $C_i' = x_1v_1v_2x_1$ and $C_j' = C_j$ for $j \ne i$ are $k-1$ minimal vertex-disjoint cycles and $G - \bigcup_{j=1}^{k-1} V(C_j') \simeq K_2 \cup 2K_1$ or $P_3 \cup K_1$ since $v_3 \notin N_{C_i}(x_1) \cap N_{C_i}(x_3)$. By Lemma 6, this contradicts the choice of cycles (1). Therefore, $x_2v_3 \in E(G)$. Since $d_{C_i}(y^-) = 3$, $y^-v_3 \in E(G)$. Furthermore, since there is no cycle of order 4 containing x_3x_4 in H' and the minimality of L, $E(\{x_3, x_4\}, \{y, y^+\}) \ne \emptyset$. If we take $C_1' = x_2v_3y^-y^{2+}x_2$, $C_i' = x_1v_1v_2x_1$ and $C_j' = C_j$ for $j \ne 1, i$, then $\{C_1', \ldots, C_{k-1}'\}$ is minimal and $G - \bigcup_{j=1}^{k-1} V(C_j')$ is connected. But this contradicts the choice of cycles (1). This completes the proof of Claim 10. □

Claim 11 $|E(Z, H')| \le 9$.

Proof. Suppose that $|E(Z, H')| \ge 10$. Since $d_{H'}(y^-) \le 4$, we have $d_{H'}(\{x_1, x_3\}) \ge 6$. On the other hand, $d_{H'}(\{x_1, x_3\}) \le 6$ since $d_{C_1}(x) \le 2$ for $x \in \{x_1, x_3\}$. Hence $d_{H'}(\{x_1, x_3\}) = 6$ and $d_{H'}(y^-) = 4$. Especially, $d_{C_1}(x_1) = d_{C_1}(x_3) = 2$. Then we have $x_1y^{2+}, x_3y, x_3y^{2+} \in E(G)$. Since $d_{H'}(y^-) = 4$, we have $x_2y^-, x_4y^- \in E(G)$. By the choice of an orientation of C_1, $x_2y^+ \in E(G)$. But this gives two vertex-disjoint cycles in H', $x_1x_2y^+yx_1$ and $x_3x_4y^-y^{2+}x_3$, a contradiction. □

By Claims 10 and 11, we have

$$6k - 2 \le d_G(Z) \le 6(k-2) + 9 = 6k - 3,$$

a contradiction. This completes the proof of CASE 2.1.

In the following, we consider the case $3 \leq |H| \leq 4$ and H is connected. Other than the assumptions we put at the beginning of Case 2, we may further assume that $x_2x_3 \in E(G)$ if $|H| = 4$. By Claim 9, we may assume that $x_2y \in E(G)$ for some $y \in V(C_1)$. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y^-, y, y^+\}$.

Claim 12 $g = |E(X \cup Y, C_i)| \leq 12$ *for any* i, $2 \leq i \leq k-1$.

Proof. Suppose that $g \geq 13$ for some i, $2 \leq i \leq k-1$. Then we have $|E(X, C_i)| \geq 5$ since $|E(Y, C_i)| \leq 8$ holds by the choice of C_1 and Claim 8. Hence $d_{C_i}(x) \geq 2$ for some $x \in X$ and we have $|C_i| \leq 4$ by Lemma 1.

Subclaim 12.1 $|E(Y, C_i)| \leq 7$ *if* $|C_i| = 4$.

Proof. Suppose that $|C_i| = 4$ and $|E(Y, C_i)| \geq 8$. Then $d_{C_i}(y') \geq 3$ for some $y' \in Y$. On the other hand, $d_{C_i}(y') \leq 3$ for any $y' \in Y$ by the choice of C_1. Hence $d_{C_i}(y') = 3$ for some $y' \in Y$. Let $C_i = v_1v_2v_3v_4v_1$.

Suppose that $d_{C_i}(y^-) = 3$. Without loss of generality, we may assume that $v_1, v_2, v_3 \in N_{C_i}(y^-)$. Since $|E(Y, C_i)| \geq 8$, we have $|E(\{y, y^+\}, C_i)| \geq 5$ and $N_{C_i}(y) \cap N_{C_i}(y^+) \neq \emptyset$. But this implies that we can find two shorter cycles than C_1 and C_i in $\langle C_1 \cup C_i \rangle$, a contradiction.

Hence $d_{C_i}(y^-) \leq 2$. Similarly, we have $d_{C_i}(y^+) \leq 2$. But this means that $|E(Y, C_i)| \leq 7$, a contradiction. □

Suppose that $|C_i| = 4$ and let $C_i = v_1v_2v_3v_4v_1$. Since $|E(Y, C_i)| \leq 7$ by Subclaim 12.1, we have $|E(X, C_i)| \geq 6$. On the other hand, $|E(X, C_i)| \leq 6$ holds by Lemma 1. Hence $|E(X, C_i)| = 6$ and $|E(Y, C_i)| = 7$. Without loss of generality, we may assume that $\{v_1, v_3\} = N_{C_i}(x_1) = N_{C_i}(x_3)$ and $\{v_2, v_4\} = N_{C_i}(x_2)$. Note that there exists a cycle of order 4 in $\langle (H \cup C_i) - \{v_j, v_{j+1}\} \rangle$ for any j, $1 \leq j \leq 3$. Since $|E(Y, C_i)| = 7$, $d_{C_i}(y') = 3$ for some $y' \in Y$ and $\{v_j, v_{j+1}\} \subseteq N_{C_i}(y')$ for some j, $1 \leq j \leq 3$. This means that we can find a triangle and a cycle of order 4 in $\langle H \cup C_1 \cup C_i \rangle$. This contradicts the minimality of L.

Hence we may assume that $|C_i| = 3$. Let $C_i = v_1v_2v_3v_1$ and $H'' = \langle H \cup C_1 \cup C_i \rangle$.

Suppose that $d_{C_i}(y^-) = 3$. Then $N_{C_i}(x_1) \cap N_{C_i}(x_2) = \emptyset$ and $N_{C_i}(x_2) \cap N_{C_i}(x_3) = \emptyset$, since otherwise we can find two vertex-disjoint triangles in H''. Hence $|E(X, C_i)| \leq 6$. Also $N_{C_i}(y) \cap N_{C_i}(y^+) = \emptyset$, since otherwise we can find two vertex-disjoint triangles in $\langle C_1 \cup C_i \rangle$. Then $|E(\{y, y^+\}, C_i)| \leq 3$ and we get $g \leq 12$, a contradiction.

Hence $d_{C_i}(y^-) \leq 2$ and we have $d_{C_i}(y^+) \leq 2$, similarly. Furthermore, since we do not use the existence of the path $C_1[y^+, y^-]$ in the above argument, we have also $d_{C_i}(x_1) \leq 2$ and $d_{C_i}(x_3) \leq 2$ by the same argument. Therefore, $|E(\{x_1, x_3, y^-, y^+\}, C_i)| \leq 8$ and this implies that $|E(\{x_2, y\}, C_i)| \geq 5$.

Suppose that $d_{C_i}(y) = 3$. Since $|E(Y, C_i)| \leq 7$, we have $|E(X, C_i)| \geq 6$. Also, since $d_{C_i}(x_1) \leq 2$ and $d_{C_i}(x_3) \leq 2$, we have $d_{C_i}(x_2) \geq 2$ and this implies that $N_{C_i}(x_1) \cap N_{C_i}(x_2) \neq \emptyset$. Then we can find two vertex-disjoint triangles in H'', a contradiction. Hence $d_{C_i}(y) \leq 2$. Again, we do not use the existence of the path

$C_1[y^+, y^-]$, then we have $d_{C_i}(x_2) \leq 2$ by the same argument. But this means that $g \leq 12$, a contradiction. This completes the proof of Claim 12. □

Since each of $\{x_1, x_3, y\}$ and $\{x_2, y^+, y^-\}$ is independent,

$$\begin{aligned} 2(6k-2) &\leq d_G(X \cup Y) \\ &\leq 12(k-2) + 10 + (|H| - 3) + |E(X, C_1)| + |E(Y, H)|. \end{aligned}$$

Hence

$$13 \leq |H| + |E(X, C_1)| + |E(Y, H)|. \tag{3}$$

We consider the following two cases.

CASE 2.2 $H \simeq P_4$.

By (3), $9 \leq |E(X, C_1)| + |E(Y, H)|$ and at least one of $|E(X, C_1)| \geq 5$ and $|E(Y, H)| \geq 5$ hold.

Let $H' = \langle H \cup C_1 \rangle$. Note that there is no triangle in H' by the minimality of L.

Claim 13 $|C_1| = 4$.

Proof. Suppose that $|E(X, C_1)| \geq 5$. Then $d_{C_1}(x) \geq 2$ for some $x \in X$ and we have $|C_1| = 4$ by Lemma 1.

Next, suppose that $|E(Y, H)| \geq 5$. This inequality implies that $d_Y(x) \geq 2$ for some $x \in H$ and also means that $|C_1| = 4$ by Lemma 1. □

Let $C_1 = yy^+y'y^-y$. By symmetry of x_2 and x_3, we have $E(x_3, C_1) \neq \emptyset$ by Claim 9.

Suppose that $x_3y' \in E(G)$. Then $d_{C_1}(x_2) = d_{C_1}(x_3) = 1$ since otherwise we can find a triangle in H'. If $x_1y^-, x_4y^+ \in E(G)$, then $x_1x_2yy^-x_1$ and $x_3x_4y^+y'x_3$ are two vertex-disjoint cycles in H', and we have k vertex-disjoint cycles of G, a contradiction. If $x_1y^+, x_4y^- \in E(G)$, then $x_1x_2yy^+x_1$ and $x_3x_4y^-y'x_3$ are two vertex-disjoint cycles in H'. Hence $|E(G) \cap \{x_1y^-, x_4y^+\}| \leq 1$ and $|E(G) \cap \{x_1y^+, x_4y^-\}| \leq 1$. But this implies that $|E(X, C_1)| + |E(Y, H)| \leq 8$, a contradiction.

Hence $N_{C_1}(x_3) \subset \{y^-, y^+\}$. By symmetry of y^+ and y^-, we may assume that $x_3y^+ \in E(G)$. By replacing C_1 with $x_2x_3y^+yx_2$, we may assume that $\{x_1, x_4, y^-, y'\}$ induces P_4. Since $x_1x_4 \notin E(G)$, we have either $\{x_1y^-, y'x_4\} \subset E(G)$ or $\{x_1y', y^-x_4\} \subset E(G)$. However, in the former case, $\langle H \cup C_1 \rangle$ has two vertex-disjoint cycles $x_1x_2yy^-x_1$ and $x_3x_4y'y^+x_3$, a contradiction. Thus, the latter case occurs. We have already seen $y'x_3 \notin E(G)$. By symmetry, we also have $x_2y^- \notin E(G)$. Then since $\langle H \cup C_1 \rangle$ has no triangle, we deduce $E(H, C_1) = \{x_1y', x_2y, x_3y^+, x_4y\}$. However, this implies that $|E(X, C_1)| + |E(Y, H)| \leq 6$. This is a contradiction and completes the proof of CASE 2.2.

CASE 2.3 $|H| = 3$

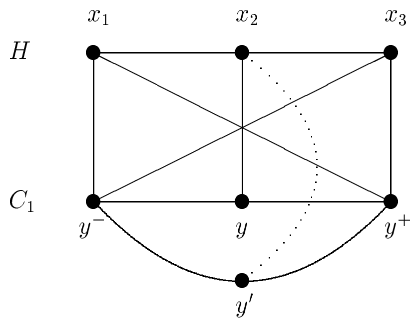

Figure 2: The configuration between H and C_1

By (3), we have $10 \leq |E(X, C_1)| + |E(Y, H)|$. Since there is no triangle in $\langle H \cup C_1 \rangle$ because of the minimality of L, $d_H(y) = 1$ and $d_H(y_0) \leq 2$ for $y_0 \in \{y^-, y^+\}$. Hence $|E(Y, H)| \leq 5$, and this implies that $|E(X, C_1)| \geq 5$. Then $|C_1| = 4$ by Lemma 1 since $d_{C_1}(x) \geq 2$ for some $x \in X$.

On the other hand, we have $|E(X, C_1)| \leq 6$ by Lemma 1. This implies that $|E(X, C_1)| + |E(Y, H)| \leq 11$.

Since each of $\{x_1, x_3, y\}$ and $\{x_2, y^-, y^+\}$ is independent,

$$\begin{aligned} 2(6k-2) &\leq d_G(X \cup Y) \\ &\leq \sum_{i=2}^{k-1} |E(X \cup Y, C_i)| + 10 + |E(X, C_1)| + |E(Y, H)| \\ &\leq 12(k-2) + 10 + 11 \\ &\leq 12k - 3 \end{aligned}$$

by Claim 12. Therefore,

$$11 \leq |E(X \cup Y, C_i)| \leq 12 \tag{4}$$

holds for any i, $2 \leq i \leq k-1$.

Let $C_1 = yy^+y'y^-y$. Then we may assume that $\{x_1y^-, x_1y^+, x_2y, x_3y^-, x_3y^+\} \subset E(G)$ since $|E(X, C_1)| \geq 5$ (see Figure 2). Let $Z = \{x_1, x_3, y\}$ and $Z' = \{x_2, y^-, y^+\}$.

Claim 14 $|E(Z, C_i)| \leq 6$ *for any* i, $2 \leq i \leq k-1$.

Proof. Suppose that $|E(Z, C_i)| \geq 7$ for some i. If we take $C_1' = x_1x_2x_3y^-x_1$ and $C_j' = C_j$ for $2 \leq j \leq k-1$, then $\{C_1', \ldots, C_{k-1}'\}$ is minimal. By Lemma 1, $d_{C_j}(y) \leq 3$ for $2 \leq j \leq k-1$ and $d_{C_j}(y) = 3$ implies $|C_j| = 3$. Since $|E(Z, C_i)| \geq 7$ and $d_{C_i}(z) \geq 3$ for some $z \in Z$. Then $|C_i| = 3$, and let $C_i = v_1v_2v_3v_1$.

Suppose that $d_{C_i}(x_1) = 3$. If $d_{C_i}(y) = 2$, then $d_{C_i}(z') = 0$ for any $z' \in Z'$ since otherwise we can find two vertex-disjoint triangles in $\langle H \cup C_1 \cup C_i \rangle$. Then $|E(Z \cup Z', C_i)| = |E(X \cup Y, C_i)| \leq 9$, but this contradicts (4). Hence $d_{C_i}(y) \leq 1$, and we have $d_{C_i}(x_3) = 3$. In this case, we have also $d_{C_i}(z') = 0$ for any $z' \in Z'$ and this means that $|E(X \cup Y, C_i)| \leq 7$, a contradiction.

Hence $d_{C_1}(x_1) \leq 2$. Similarly, we have $d_{C_i}(x_3) \leq 2$. This means that $d_{C_i}(y) = 3$ and $d_{C_i}(x_1) = d_{C_i}(x_3) = 2$. In this case, we have $d_{C_i}(z') = 0$ for all $z' \in Z'$ again, and this implies that $|E(X \cup Y, C_i)| \leq 7$, a contradiction. This completes the proof of Claim 14. □

Since Z is independent, we have

$$6k - 2 \leq d_G(Z) \leq 6(k-2) + 9 = 6k - 3$$

by Claim 14, but this is a contradiction. This completes the proofs of CASE 2.3 and Theorem 4.

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