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Magic labellings of infinite graphs over infinite groups

D. Combe

School of Mathematics
University of New South Wales
Sydney NSW 2052
Australia
diana@unsw.edu.au

A. M. Nelson

School of Mathematics and Statistics
The University of Sydney
NSW 2006
Australia
adriann@maths.usyd.edu.au

Abstract

A *total labelling* of a graph over an abelian group is a bijection from the set of vertices and edges onto the set of group elements. A labelling can be used to define a *weight* for each edge and for each vertex of finite degree. A labelling is *edge-magic* if all the edges have the same weight and *vertex-magic* if all the vertices have finite degree and the same weight. We exhibit magic labellings for various countable graphs.

For countably infinite stars, countable galaxies and forests, and for an arbitrary abelian group, we determine when there is an edge-magic labelling and what group elements occur as constants. In the particular case where the group is the integers, we determine when there is a vertex-magic labelling, and which integers occur as constants. We also give explicit edge-magic labellings of various frieze graphs over the integers.

1 Introduction

A detailed survey of many types of graph labellings can be found in the dynamic survey by Gallian [8]. A *vertex labelling* of a graph is an assignment of labels to the vertices. An *edge labelling* is an assignment of labels to the edges. A *total labelling* is an assignment of labels to the combined set of vertices and edges. The set of labels is commonly a subset of the integers, so a labelling can be used to define a *weight* for each vertex and edge. For a vertex the weight is the sum of the label of the vertex and the labels of its incident edges. For an edge, the weight is the sum of the label of the edge and the labels of its end vertices. A *magic labelling* of a finite graph with v vertices and e edges is a total labelling of the graph by the integers $1, 2, 3, \ldots, v + e$ with constant edge or vertex weights. Magic labellings have been considered by various authors: for example vertex-magic labellings by MacDougall et al [11], edge-magic labellings by Kotzig and Rosa [10], and by Baskaro et al [3], and labellings which are both vertex-magic and edge-magic by Exoo et al [6]. There is an extensive list of references about magic labellings of finite graphs in the book on Magic Graphs by Wallis [12].

A labelling of a finite graph over n consecutive integers can be viewed as a labelling over a cyclic group of order n. Combe, Nelson and Palmer [2] introduced magic labellings of finite graphs, where the labels are the elements of an arbitrary abelian group, of order $v+e$. Previously some authors considered labelling graphs by groups, but none were considering total labellings with magic properties. Fukuchi [7] and Egawa [5] considered vertex labellings of a graph by an elementary abelian group such that the connected components of the graph had constant weight. Gimbel [9] and Edelman and Saks [4] considered the relationship between vertex labellings and edge labellings over abelian groups.

In this paper we consider labellings of countably infinite graphs by countably infinite abelian groups. The idea of extending magic graph labellings over groups from the finite to the infinite situation was suggested by Alan Beardon at the 2002 meeting of the Australian Mathematical Society in Newcastle. We are indebted to Alan for this idea and for many enthusiastic communications about magic labellings. Beardon [1] has shown that many countable graphs have uncountably many edge-magic labellings over the integers.

In this paper we are motivated by labellings over the integers. We give inductively defined labellings (over the integers and over arbitrary countable groups) which are different to those of Beardon, and in which we emphasise symmetry.

For particular countable graphs, and families of countable graphs, we are interested in which countable groups can be used to give a magic labelling and we are also interested in determining the set of group elements which occur as magic constants. In an earlier paper [2], on magic labelling of finite graphs over abelian groups, we dealt completely with the example of (finite) star graphs. Here we continue with this theme and consider edge-magic labellings of countable stars and countable galaxies over arbitrary abelian groups, and vertex-magic labellings over the integers. We extend these arguments to magic labellings of countable forests. Finally we use modulus classes to construct explicit edge-magic labellings of frieze graphs over the integers.

2 $\mathbb{A}$-labellings of graphs

2.1 Definitions

Throughout this paper we use *countable* to mean *countable infinite.* By a *graph* G we mean a *countable graph* with no loops and no multiple edges. The graph need not be connected. The vertex set is V and the edge set E is a (possibly empty) set of unordered pairs of vertices. The set $V \cup E$ is the set of *graph elements.* When we say a graph is countable we mean that the set of graph elements is countable, and hence that the vertex set is countable and the edge set is finite or countable. If $x, y \in V$, then $x \sim y$ means there is an edge between x and y, and the edge is denoted xy (or yx). Although we are considering infinite graphs, if the graph is not connected it may have finite components.

In this paper the group $\mathbb{A}$ is always a countable abelian group (with identity element denoted by 0). Since we are often considering the integers, $\mathbb{Z}$, it is convenient to consider our groups additively. We use $\mathbb{G}$ to denote a *finite* abelian group.

Let G be a (countable) graph, and $\mathbb{A}$ a (countable abelian) group. A *total $\mathbb{A}$-labelling* of G, or a *total labelling of G over $\mathbb{A}$*, is a bijection from $V \cup E$ to $\mathbb{A}$. Our $\mathbb{A}$-labellings will always be total, and not only of the vertices or only of the edges, so, without risk of confusion, we will refer to them as *$\mathbb{A}$-labellings of G* or *labellings of G over $\mathbb{A}$*.

Let λ be a labelling of G over a group $\mathbb{A}$. The *weight*, $\omega = \omega_\lambda$, of an edge $xy \in E$ is the sum of the label of xy and the labels of x and y, that is

$$\omega(xy) = \lambda(x) + \lambda(xy) + \lambda(y).$$

The labelling λ is an *edge-magic $\mathbb{A}$-labelling* of G if there is an element k of $\mathbb{A}$ such that for every $xy \in E$, $\omega(xy) = k$. The element k is the *edge constant.*

Let $x \in V$ be a vertex of finite degree, then the *weight* of x is the sum of the label of x and the labels of the edges incident with x, that is

$$\omega(x) = \lambda(x) + \sum_{y \in V\,:\; x \sim y} \lambda(xy).$$

The labelling λ is a *vertex-magic $\mathbb{A}$-labelling* of G if every element of G has finite degree and there is an element h of $\mathbb{A}$ such that for every $x \in V$, $\omega(x) = h$. The element h is the *vertex constant.*

2.2 Translations and edge-magic labellings

Let G be a graph and λ an $\mathbb{A}$-labelling of G. For $a \in \mathbb{A}$, define $a + \lambda$, the *translation of λ by a*, to be the $\mathbb{A}$-labelling of G such that for $t \in V \cup E$, $(a + \lambda)(t) = a + \lambda(t)$. Define $-\lambda$, the *negative* of λ, to be the $\mathbb{A}$-labelling of G such that for $t \in V \cup E$, $(-\lambda)(t) = -\lambda(t)$.

The following straightforward lemma is useful as we are interested in which group elements can occur as magic constants. The first part implies that the set of group elements which occur as edge-magic constants of the $\mathbb{A}$-labellings of a graph G is a union of cosets modulo the subgroup $3\mathbb{A} = \{3a : a \in \mathbb{A}\}$.

Lemma 1. *Let G be a graph with an edge-magic labelling λ over a group $\mathbb{A}$ with edge constant k, and let $a \in \mathbb{A}$. Then*

(i) *$a + \lambda$ is an edge-magic labelling with constant $k + 3a$;*

(ii) *$-\lambda$ is an edge-magic labelling with constant $-k$.*

Proof. Immediate. □

Corollary 2. *For a graph G with an edge-magic $\mathbb{Z}$-labelling λ the following hold.*

(i) *There is an edge-magic $\mathbb{Z}$-labelling of G with edge constant $k = 0$ or 1.*

(ii) *If G has an edge-magic $\mathbb{Z}$-labelling with edge constant $k = 0$, then G has an edge-magic $\mathbb{Z}$-labelling with constant k for all $k \equiv 0 \pmod 3$.*

(iii) *If G has an edge-magic $\mathbb{Z}$-labelling with constant $k = 1$, then G has an edge-magic $\mathbb{Z}$-labelling with edge constant k for any $k \not\equiv 0 \pmod 3$.*

Proof. Immediate. □

2.3 Example: Star graphs

For $n \geq 0$, the finite star with n rays, T_n, has a central vertex and n non-central vertices. There are n edges, one from the central vertex to each of the other vertices. For $n \geq 1$, there are various edge-magic labellings (with labels $1, 2, \ldots, 2n + 1$) and the edge constants which occur are $2n+4$, $3n+3$ and $4n+2$, see for example Wallis, [12]. Combe, Nelson and Palmer [2] show that for any finite abelian group $\mathbb{G}$ of order $2n + 1$ there are edge-magic $\mathbb{G}$-labellings of T_n, and that elements which occur as edge constants make up the subgroup $3\mathbb{G}$. Therefore, unless 3 divides $2n + 1$, (the order of the group), each group element occurs as edge constant for some edge-magic labelling. Consider now the case of the countable star.

Theorem 3. *Let $T_{\mathbb{N}}$ be the countable star with vertex set $V = \{v_0, v_1, v_2, \ldots\}$ and edge set $E = \{v_0v_i : i = 1, 2, \ldots\}$.*

(i) *There is an edge-magic $\mathbb{Z}$-labellings of $T_{\mathbb{N}}$ with edge constant 0.*

(ii) *$T_{\mathbb{N}}$ does not have an edge-magic $\mathbb{Z}$-labelling with edge constant 1.*

(iii) *The elements of $\mathbb{Z}$ which occur as edge constants of edge-magic $\mathbb{Z}$-labellings of $T_{\mathbb{N}}$ are the elements of $3\mathbb{Z}$.*

Proof. (i) The map $\lambda : V \cup E \longrightarrow \mathbb{Z}$ given by setting

$$\lambda(v_i) = i, \qquad i = 0, 1, 2, \ldots$$

$$\lambda(v_0v_i) = -i, \quad i = 1, 2, \ldots$$

is an edge-magic $\mathbb{Z}$-labellings of $T_{\mathbb{N}}$ with edge constant 0.

(*ii*) Suppose λ is an edge-magic labelling with edge constant 1. Then $\lambda(v_0) = z$ for some $z \in \mathbb{Z}$. By Lemma 1 the $\mathbb{Z}$-labelling $(-z + \lambda)$ is edge-magic, with edge constant $k = 1 - 3z$ and with the central vertex labelled $(-z + \lambda)(v_0) = 0$. Now $k \in \mathbb{Z}$, $k \neq 0$, so k must label a non-central vertex v_i or an edge v_0v_i. However k cannot label v_i since that would not leave a possible label for v_0v_i to preserve edge-weight of k. Similarly k cannot label v_0v_i since that would not leave a possible label for v_i. Therefore $T_{\mathbb{N}}$ does not have an edge-magic labelling over $\mathbb{Z}$ with edge constant 1.

(*iii*) Since there is an edge-magic $\mathbb{Z}$-labellings of $T_{\mathbb{N}}$ with edge constant 0, by Corollary 2 the integers which occur as edge constants of edge-magic $\mathbb{Z}$-labellings of $T_{\mathbb{N}}$ include all of $3\mathbb{Z}$. Since there is no edge-magic $\mathbb{Z}$-labelling with edge constant 1, by Lemma 1, there is no edge-magic $\mathbb{Z}$-labelling with edge constant -1, and hence no edge-magic $\mathbb{Z}$-labelling with edge constant any $k \not\equiv 0 \pmod 3$. Therefore the set of edge constants is precisely $3\mathbb{Z}$. □

More generally, suppose $\mathbb{A}$ is a countable abelian group in which there are no elements of order 2. Then we can take a sequence of non-zero elements $a_1, a_2, \ldots$ in the group such that the set of non-zero elements $\mathbb{A} \setminus \{0\}$ is the disjoint union of the set of these elements and their set of inverses. Label the central vertex of $T_{\mathbb{N}}$ by 0, and then for each $i = 1, 2, 3, \ldots$ label the ith vertex a_i and its incident edge $-a_i$. This gives an edge-magic $\mathbb{A}$-labelling of $T_{\mathbb{N}}$ with edge constant $k = 0$. Then similar arguments to those in the proof above show:

Theorem 4. *Let $\mathbb{A}$ be a countable abelian group with no elements of order 2. Then the elements of $\mathbb{A}$ which occur as edge constants of edge-magic $\mathbb{A}$-labellings of the countable star $T_{\mathbb{N}}$ make up all $3\mathbb{A}$. Furthermore, for any $a \in \mathbb{A}$, any edge-magic labelling of $T_{\mathbb{N}}$ which labels the central vertex a, has edge constant $3a$.*

Finally we note the situation when $\mathbb{A}$ has elements of order 2.

Theorem 5. *Let $\mathbb{A}$ be a countable abelian group which contains an element of order 2. Then there are no edge-magic $\mathbb{A}$-labellings of the countable star $T_{\mathbb{N}}$.*

Proof. Let $i \in \mathbb{A}$ be an element of order 2. Suppose there is an edge-magic $\mathbb{A}$-labelling of $T_{\mathbb{N}}$. Then by translation there is an edge-magic labelling λ which labels the central vertex by 0. Let k be the edge constant corresponding to λ. If $k \neq 0$, then by the above arguments there is no graph element which could be labelled by k. If $k = 0$ then there is no graph element which could be labelled by i. Therefore there are no edge-magic $\mathbb{A}$-labellings of $T_{\mathbb{N}}$. □

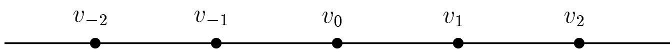
Note that an infinite star has a vertex of infinite degree and hence cannot have a vertex-magic labelling.

3 Inductively defined labellings

In this section we give examples of inductively defined $\mathbb{Z}$-labellings.

3.1 Example: Infinite path

Denote by P the infinite path which has $V = \{v_i : i \in \mathbb{Z}\}$, and $E = \{v_i v_{i+1} : i \in \mathbb{Z}\}$.



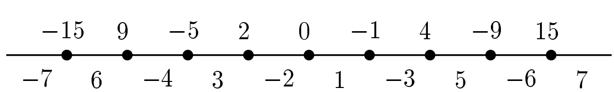
Example 6. *For any $k \in 3\mathbb{Z}$, the infinite path P has an edge-magic $\mathbb{Z}$-labelling which has edge constant k.*

Proof. It is sufficient to define an edge-magic $\mathbb{Z}$-labelling λ of P with edge constant $k = 0$, and we do so inductively. We label first v_0, then label the next edge and vertex to the right, then next edge and vertex to the left, then the next edge and vertex to the right, and so on.

List the integers in the order: 0, 1, -1, 2, -2, etc. Define initially $\lambda(v_0) = 0$; then $\lambda(v_0 v_1) = 1$ and hence $\lambda(v_1) = -1$; then $\lambda(v_{-1} v_0) = -2$ and hence $\lambda(v_{-1}) = 2$. After these initial choices the subsequent choices of label $\lambda(v_1 v_2) = -3$, $\lambda(v_2) = 4$; $\lambda(v_{-1} v_{-2}) = 3$, $\lambda(v_{-2}) = -5$; $\ldots$ are determined, for $j = 1, 2, \ldots$, as follows.

The label for the edge $v_j v_{j+1}$ is chosen to be the first integer in the list which has not yet used as a label, and which is of the same sign as $\lambda(v_j)$. The label for its vertex v_{j+1} will then be determined in order to make the edge weight 0 . Similarly the edge $v_{-j} v_{-(j+1)}$ takes label the first integer in the list which has not yet used as a label, and which is of the same sign as $\lambda(v_{-j})$. The label for its vertex $v_{-(j+1)}$ will then be determined in order to make the edge weight 0.

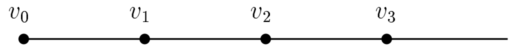
Moving outward left, or outward right, from v_0 the edge labels alternate in sign, as do the vertex labels. The choices of edge label ensure that each element of $\mathbb{Z}$ is an edge label. So λ is surjective. The edge labels are chosen to be distinct, and in addition, each vertex label is, inductively, larger in absolute value than any previously used label. So λ is injective. Hence λ is bijective. By construction it gives every edge weight 0.



It is clear from the diagram, and straightforward to prove, that very soon a lovely symmetry appears:that is for $i \geq 3$, $\lambda(-v_i) = -\lambda(v_i)$, and $\lambda(v_{-i} v_{-(i+1)}) = -\lambda(v_i v_{i+1})$. □

3.2 Example: Semi-infinite path

We denote by S the semi-infinite path with $V = \{v_i : i \in \mathbb{N}\}$ and $E = \{v_i v_{i+1} : i \in \mathbb{N}\}$.



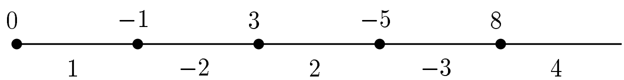
Example 7. *For any $k \in \mathbb{Z}$, the semi-infinite path S has an edge-magic $\mathbb{Z}$-labelling which has edge constant k.*

Proof. It is sufficient to show that there are edge-magic $\mathbb{Z}$-labellings with edge constants $k = 0, 1$.

We define, inductively, an edge-magic $\mathbb{Z}$-labelling λ_0 of S, with edge constant 0.

List the integers in the order of the previous example. Set initially $\lambda_0(v_0) = 0$, $\lambda_0(v_0v_1) = 1$ and hence $\lambda_0(v_1) = -1$. Continue labelling edges and vertices, for $i = 1, 2, \ldots,$ as follows.

Label the edge v_iv_{i+1} by the first integer which has not already been used as a label and which is of the same sign as $\lambda_0(v_i)$. The constant edge weight condition then determines the label of the next vertex v_{i+1}.



Similarly an edge-magic $\mathbb{Z}$-labelling λ_1 of S, with edge constant 1, can be defined using the same inductive step, but starting with $\lambda_1(v_0) = 0$, $\lambda_1(v_0v_1) = -1$, and hence $\lambda_1(v_1) = 2$.

For both of these labellings the edge labels alternate in sign. So do the vertex labels. In addition, each vertex label will be (by induction) larger in absolute value than any previously used label. □

Note that each of the infinite and semi-infinite paths is a countably infinite tree which contains a semi-infinite path. Beardon [1] has shown that such a tree has edge-magic $\mathbb{Z}$-labellings with edge-constant $k = 0$. His algorithm works equally well for any $k \in \mathbb{Z}$. We demonstrate this in the proof of the next example:

Example 8. *For any $k \in \mathbb{Z}$, the infinite path P has an edge-magic $\mathbb{Z}$-labelling which has edge constant k.*

Proof. Recall that, for the infinite path, $V = \{v_i : i \in \mathbb{Z}\}$, and $E = \{v_iv_{i+1} : i \in \mathbb{Z}\}$. By Corollary 2 and using Example 6, it is enough to show that there is an edge-magic labelling with edge constant 1. List the integers in some order. Label v_0 with the first integer in the list. Label v_2 and v_{-2} with the next two integers. There are infinitely many ways of simultaneously labelling the vertices v_1 and v_{-1} and the connecting edges v_2v_1, v_1v_0, v_0v_{-1} and $v_{-1}v_{-2}$ with unused labels such that the edges each have weight 1. Choose one such labelling. Next label v_4 and v_{-4} with the first two integers in the list not yet used as labels. There are infinitely many ways of simultaneously labelling the vertices v_3 and v_{-3} and the connecting edges v_4v_3, v_3v_2, $v_{-2}v_{-3}$ and $v_{-3}v_{-4}$ with unused labels such that the edges all have weights 1. Continue in this way labelling the vertices which are at distance 2 from the labelled part of the graph, and then labelling the graph components which connect them to the labelled section of the graph.

Clearly this produces an edge-magic $\mathbb{Z}$-labelling with edge constant 1.

□

3.3 Example: Edge-magic $\mathbb{Z}$-labellings of countable galaxies.

We call a graph a *countable galaxy* if it has countably many connected components, each of which is a finite or countable star.

Theorem 9. *Let G be a countable galaxy and $k \in \mathbb{Z}$. Then we can construct an edge-magic $\mathbb{Z}$-labelling of G with edge constant k.*

Proof. Each component star of the galaxy has a central vertex. (In a T_1 component designate one vertex central and the other non-central). Each central vertex has a finite or countable number of incident edges. Call the central vertices and the edges *galaxy elements*. List the set of galaxy elements in some order such that each edge occurs in the list after the central vertex of the star in which it occurs. List the elements of $\mathbb{Z}$ in some order. The first entry in the list of galaxy elements is a central vertex. Label this vertex with the first integer in the list of integers. We now proceed along the list of galaxy elements, attaching labels according to the recipe below.

(i) If the current galaxy element is a central vertex label it with the first listed integer not already used as a label.

(ii) If the current galaxy element is an edge then the central vertex of the star to which it belongs will already have been labelled, z say. Let X be the set of integers already used as labels, and $c = k - z$. Then we can find distinct $a, b \in \mathbb{A} \setminus X$ such that $a + b = c$. Label the edge b and the non-central vertex of the edge a.

In this way we label the graph elements of G by $\mathbb{Z}$. The choices in case (i) ensure the labelling is surjective. The choices in case (ii) ensure both that it is injective, and that each edge weight equals k.

Note the existence of suitable a and b as needed in case (ii) above is easy to see, but is also a particular case of the Lemma 10 to follow. □

Lemma 10. *Let X be a finite subset of an infinite abelian group $\mathbb{A}$ and $c \in \mathbb{A}$.*

(i) *Suppose $2\mathbb{A} \neq \{0\}$. Then there exist distinct $a, b \in \mathbb{A} \setminus X$ such that $a + b = c$.*

(ii) *Suppose $2\mathbb{A} = \{0\}$. Then there exist distinct $a, b \in \mathbb{A} \setminus X$ such that $a + b = c$ if (and only if) $c \neq 0$.*

Proof. For every c in an infinite abelian group $\mathbb{A}$ the equation $a + b = c$ has infinitely many solutions, one for each $a \in \mathbb{A}$. Whenever we can show that for infinitely many of these a we have $a \neq b$ we are done. For then, since only finitely many such solutions can have $a \in X$, and only finitely many can have $b = c - a \in X$, there are in fact infinitely many $a + b = c$ with $a \neq b$ and $a, b \in \mathbb{A} \setminus X$.

Case (i): The condition $2\mathbb{A} \neq \{0\}$ implies $\mathbb{B} = \{a \in \mathbb{A} : 2a = 0\}$ is a proper subgroup of $\mathbb{A}$. The set $\mathbb{C} = \{a \in \mathbb{A} : 2a = c\}$ is either empty or a coset of $\mathbb{B}$ in $\mathbb{A}$. Then $\mathbb{A} \setminus \mathbb{C}$ is non-empty disjoint union of cosets of $\mathbb{B}$. If $\mathbb{B}$ is finite this will be a non-finite union, and if $\mathbb{B}$ is not finite then each coset is infinite. In either case $\mathbb{A} \setminus \mathbb{C}$ is infinite. Hence there are infinitely many solutions to $a + b = c$ with a and b distinct. This settles case (i).

Case (ii): When $2\mathbb{A} = \{0\}$, every solution to $a + b = c$, $c \neq 0$, has $a \neq b$, (and every solution to to $a + b = 0$ has $a = b$). □

3.4 Example: Edge-magic $\mathbb{A}$-labellings of countable forests.

A star is an example of a *tree*, a graph with no circuits in which any two vertices are connected by a finite path. We call a graph a *countable forest* if it has countably many connected components, each of which is a finite or countable tree. We extend the algorithm of Theorem 9 to countable forests and give the complete story for edge-magic labellings of countable galaxies and forests by countable abelian groups.

Theorem 11. *Let G be a countable forest, $\mathbb{A}$ a countable abelian group, and $k \in \mathbb{A}$.*

(i) *If $2\mathbb{A} \neq \{0\}$ then we can construct an edge-magic $\mathbb{A}$-labelling of G which has edge constant k.*

(ii) *If $2\mathbb{A} = \{0\}$ and G has no isolated vertices then G has no edge-magic $\mathbb{A}$-labellings.*

(iii) $2\mathbb{A} = \{0\}$ *and G has an isolated vertex then we can construct an edge-magic $\mathbb{A}$-labelling of G which has edge constant k.*

Proof. For each tree designate a vertex to be called the *central vertex*. For any edge xy in the tree there is a unique (finite) sequence of distinct edges forming a path from the central vertex to xy. Call the the central vertices and the edges of the graph the *forest elements*. List the forest elements in some order such that each edge occurs after the central vertex of its tree and after all the other vertices and edges of the path connecting it to the central vertex of the tree it is in. List the elements of $\mathbb{A}$ in some order.

The algorithm of Theorem 9 can be applied directly in the case $2\mathbb{A} \neq \{0\}$. When the current item is an edge, exactly one of its end vertices must already be labelled. Then Lemma 10 allows us to use rule (ii) of the algorithm to label the edge and its other vertex.

Suppose now $2\mathbb{A} = \{0\}$. If $a, b, c, \in \mathbb{A}$ and $a + b + c = k$, then a, b and c are distinct if and only none of them equals k. Hence if G has no isolated vertices it cannot have an edge-magic labelling with edge constant k, because no edge or vertex can be labelled k. However if G does have isolated vertices then the extended algorithm works if we take the first central vertex in the list of forest elements to be an isolated vertex and the first group element in the list to be k. □

3.5 Example: vertex-magic labellings of countable galaxies.

Note the following observations about vertex-magic labellings of countable galaxies.

(i) If a countable galaxy includes any infinite stars it cannot have a vertex-magic labelling over any group.

(ii) If one of the connected components of a graph is the star T_1, then there is no vertex-magic labelling over any group.

(iii) If the graph has any isolated vertex then any vertex-magic labelling must label the isolated vertex by the vertex-magic constant.

(iv) If a countable galaxy has vertex-magic $\mathbb{A}$-labelling with constant $h = 0$, the identity of the group, then 0 must label a central vertex. For if 0 labels a noncentral vertex there is no possible label for the incident edge, and if 0 labels an edge there is no possible label for noncentral vertex of that edge.

(v) A countable union of stars T_2 cannot have a vertex-magic labelling with vertex constant 0, for if it did then 0 must label the central vertex of one of the T_2 components. The two edges of this star must be labelled a and $-a$ for some $a \in \mathbb{A}$, forcing the two noncentral vertices to be labelled $-a$ and a, which is impossible.

Therefore, a countable galaxy with a vertex-magic labelling over some group $\mathbb{A}$ must have connected components which are all finite, include at most one T_0, include no T_1, and if the constant is $h = 0$ there must be at least one T_n with $n \neq 2$.

The next theorem shows for $\mathbb{Z}$-labellings of countable galaxies which vertex-magic constants are possible.

Theorem 12. *Suppose G is a countable galaxy of finite stars, at most one of which is T_0, none of which is T_1. Then:*
(i) *There is a vertex-magic labelling with $h = 0$ if and only if there is at least one connected component T_n with $n \neq 2$.*
(ii) *For $h \in \mathbb{Z}$, $h \neq 0$, there is a vertex-magic labelling with vertex constant h, except for the galaxy with one T_0 component and each other component a T_2.*

Proof. The "only if" part of the theorem follows from the preceding observations.

Let G be a countable galaxy of finite stars, at most one of which is T_0, none of which is T_1, and let $h \geq 0$. Assume that (i) if $h = 0$, then there is at least one connected component T_n with $n \neq 2$ and (ii) if $h \neq 0$, then G is not the galaxy with one T_0 component and the remainder T_2 components.

We define inductively a vertex-magic labelling with vertex-magic constant h. Taking the negative of this labelling gives a vertex-magic labelling with constant $-h$.

If G has a T_0 component we label its vertex h. If there is no T_0 component or $h \neq 0$ we will not yet have used the label 0. In this case there must then be a T_n component with $n \geq 3$. Choose one, and label its central vertex with 0, and its edges by (distinct) integers $x_1, x_2, \ldots, x_n$, with sum h and the non-central vertices of the edges by respectively $h - x_1, h - x_2, \ldots, h - x_n$, with $x_1, x_2, \ldots, x_n, h - x_1, h - x_2,$

$\ldots$, $h - x_n$, all distinct and none equal to 0 or h. For example we can take

$$\begin{aligned} x_1 &= h+1, \\ x_2 &= h+2, \\ &\vdots \\ x_{n-1} &= h+n-1, \\ x_n &= -(n-2)h - \tfrac{1}{2}(n-1)n. \end{aligned}$$

We have now labelled any T_0 component and we have used the label 0. All labels used so far are distinct and give all vertices of stars labelled so far weight h.

Assume we have labelled the vertices and edges of a finite number of the component stars of the galaxy with (distinct) integer labels, and that each vertex has weight h. We assume also that these stars include any T_0 component, and that 0 has been used as a label. Then, by our assumptions, all remaining component stars have $n \geq 2$ edges. Consider such a star. Assume y_0 is an unused label. Choose (distinct) integers y_1, y_2, $\ldots$, y_n such that

$$y_0 + y_1 + \cdots + y_n = h,$$

and y_0, y_1, y_2, $\ldots$, y_n, $h - y_1$, $h - y_2$, $\cdots$, $h - y_n$ are distinct and not already used as labels.

For example, let $K > 0$ be an integer larger than the size of any previously used label and greater than $|y_0|$. Since y_0 is an unused label $y_0 \neq 0$.

If $y_0 > 0$, set

$$\begin{aligned} y_1 &= K+h+1, \\ y_2 &= K+h+2, \\ &\vdots \\ y_{n-1} &= K+h+n-1, \\ y_n &= -(n-1)K - (n-2)h - \tfrac{1}{2}(n-1)n - y_0. \end{aligned}$$

If $y_0 < 0$, set

$$\begin{aligned} y_1 &= -K-1, \\ y_2 &= -K-2, \\ &\vdots \\ y_{n-1} &= -K-(n-1), \\ y_n &= (n-1)K + h + \tfrac{1}{2}n(n-1) - y_0. \end{aligned}$$

Now we can extend the labelling to include the new star, give each vertex of the new star weight h, and use the label y_0, if we label the central vertex of this star by y_0 and the edges by y_1, y_2, $\ldots$, y_n, and the corresponding non-central vertices, $h - y_1$, $h - y_2$, $\ldots$, $h - y_n$.

Hence, having made our initial labellings, we can list the remaining stars in some order and extend the labelling inductively to a labelling of the whole galaxy by a subset of $\mathbb{Z}$, giving each vertex weight h. If at each stage we choose y_0 to be a minimal unused label then the set of labels will be all $\mathbb{Z}$. Thus we will have a vertex-magic $\mathbb{Z}$-labelling of the galaxy, with vertex-magic constant h. □

4 $\mathbb{Z}$-Labellings using modulus classes

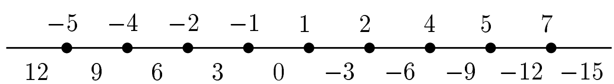
In this section we give examples of explicit edge-magic $\mathbb{Z}$-labellings of some graphs which we refer to as "frieze graphs" because they can be drawn as frieze patterns. We refer to these labellings as edge-magic *modulus labellings*.

4.1 Example: The Infinite Path

Recall the infinite path P has $V = \{v_i : i \in \mathbb{Z}\}$, and $E = \{v_i v_{i+1} : i \in \mathbb{Z}\}$.

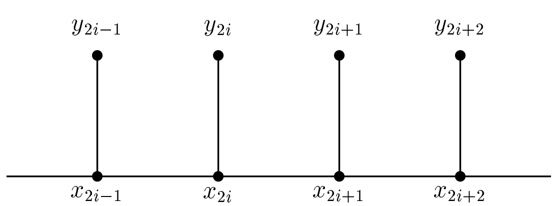
Label the vertices $\lambda(v_{2i}) = -1+3i$, $\lambda(v_{2i+1}) = 1+3i$, and the edges $\lambda(v_{2i-1}v_{2i}) = 3-6i$, $\lambda(v_{2i}v_{2i+1}) = -6i$, $i \in \mathbb{Z}$.

This defines an edge-magic $\mathbb{Z}$-labelling of the infinite path with edge constant 0.



The even indexed vertices are labelled by integers congruent to −1 modulo 3. The odd indexed vertices are labelled by integers congruent to 1 modulo 3. The integers divisible by 3 fall into two classes modulo $2 \times 3 = 6$: those congruent to 0 label the edges $v_i v_{i+1}$ with i even, and those and those congruent to 3 modulo 6 label the edges $v_i v_{i+1}$ with i odd.

4.2 Example: Picket Fence



$$V = \{x_i, y_i : i \in \mathbb{Z}\}, \quad E = \{x_i y_i,\, x_i x_{i+1} : i \in \mathbb{Z}\}.$$

Labelling vertices

$$\lambda(x_{2i}) = 4-6i, \quad \lambda(y_{2i}) = 1-6i, \quad \lambda(x_{2i+1}) = -6i, \quad \lambda(y_{2i+1}) = -1-6i,$$

and edges

$$\lambda(x_{2i}y_{2i}) = -4 + 12i, \quad \lambda(x_{2i}x_{2i+1}) = -3 + 12i,$$

$$\lambda(x_{2i+1}y_{2i+1}) = 2 + 12i, \quad \lambda(x_{2i+1}x_{2i+2}) = 3 + 12i,$$

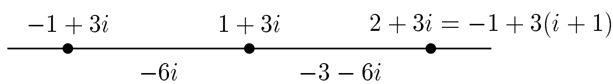
defines an edge-magic labelling with edge constant 1.

4.3 Templates

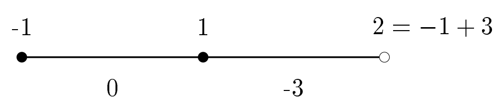
We use these two examples, the infinite path and the picket fence, to illustrate an elegant representation of these labellings of friezes which avoids excessive notation.

4.3.1 The Infinite Path

The infinite path, together with its labels, can be represented as follows.



However, we can reconstruct the diagram above, and hence the infinite path together with the given labelling, from the following template:

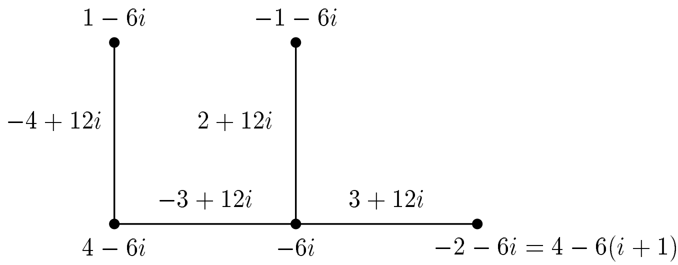


The template consists of the two vertices in bold and the two solid edges. The labelling was constructed " +mod 3". This means that to produce the labelling *add* $3i$ to the labels on the vertices and *subtract* $2 \times 3i = 6i$ from the labels on the edges. The label on the open vertex indicates not only the signed modulus, but how copies of the template fit together to form the original graph.

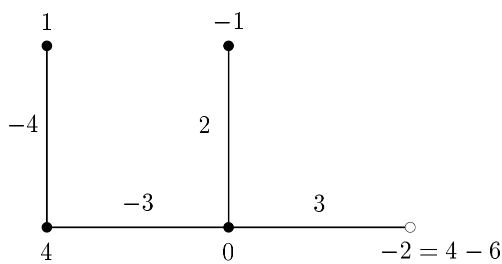
In this example, and in each of the examples which follow, the labelling is optimal in the sense that it is not possible to give a labelling of this type with a smaller template (and hence produced with a smaller modulus).

4.3.2 The Picket Fence ($k = 1$)

The picket fence together with its labelling can be represented as follows:



We can reconstruct the diagram above, and hence the picket fence together with the given labelling, from the following template:

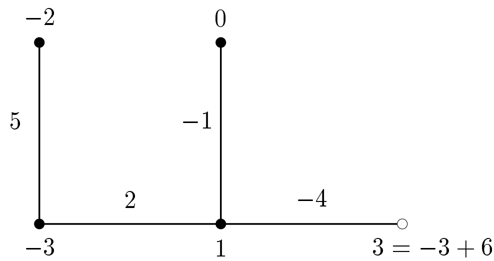


Note that there are $\mathbf{v}=4$ vertices and $\mathbf{e}=4$ edges in the template. This labelling is produced " $-$ mod 6" (note that $6=\mathbf{v}+\frac{1}{2}\mathbf{e}$). To produce the labelling *subtract* $6i$ from the vertex labels, and *add* $12i$ to the edge labels. The label $-2=4-6$ on the open vertex in this example indicates this labelling is produced " $-$ mod 6", and also how the copies of the template fit together to form the picket fence.

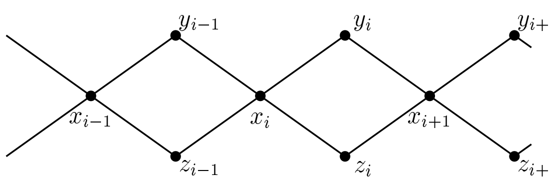
In general, when the template has $\mathbf{v}$ (bold) vertices and $\mathbf{e}$ edges, the modulus m for the labelling is $m=\mathbf{v}+\frac{1}{2}\mathbf{e}$. This is because each congruence class appears either once (as a vertex label), or twice (as edge labels). Therefore, if a graph has an edge-magic modulus labelling then the number of edges in the template is even. Whether the labelling is produced " $+$ mod m" or " $-$ mod m" is made explicit by the labelling(s) on the non-bold vertices in the template. These labels also indicate how the copies of the template fit together to from the original frieze graph. If the labelling is produced " $+$ mod m" then it can be determined by adding positive multiples of m to the vertex labels and subtracting positive multiples of $2m$ from the edge labels. If the labelling is produced " $-$ mod m" then it can be determined by subtracting positive multiples of m from the vertex labels and adding positive multiples of $2m$ to the edge labels.

4.3.3 The Picket Fence ($k=0$)

The diagram below is a template for an edge-magic labelling of a picket fence with edge constant $k=0$. It uses congruence classes modulo 6.



4.4 Example: The Diamond Frieze



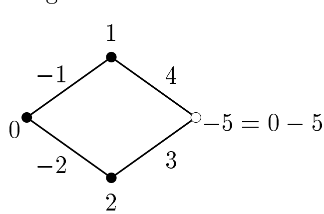
$$V = \{x_i, y_i, z_i \,:\, i \in \mathbb{Z}\}, \quad E = \{x_iy_i,\, x_iz_i,\, y_ix_{i+1},\, z_ix_{i+1} \,:\, i \in \mathbb{Z}\}.$$

Label the edges:
$\lambda(x_iy_i) = 10i - 1,\ \lambda(x_iz_i) = 10i - 2,\ \lambda(y_ix_{i+1}) = 10i + 4,\ \lambda(z_ix_{i+1}) = 10i + 3.$

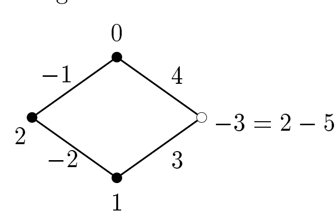
With this edge labelling, an edge-magic labelling with edge constant 0 can be obtained by labelling vertices $\lambda(x_i) = -5i, \quad \lambda(y_i) = -5i + 1, \quad \lambda(y_i) = -5i + 2.$

Keeping the same edge labels, an edge-magic labelling with edge constant 1 can be obtained by labelling vertices $\lambda(x_i) = -5i + 2, \quad \lambda(y_i) = -5i, \quad \lambda(y_i) = -5i + 1.$
We can represent these labellings, each of which is " − mod 5", using templates:

For edge constant $k = 0$

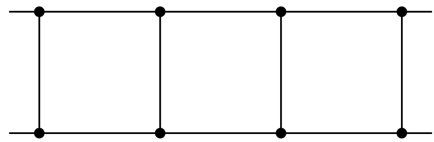


For edge constant $k = 1$

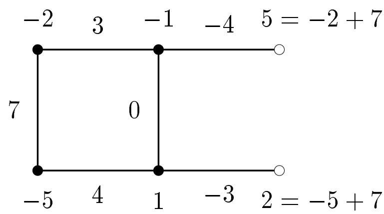


4.5 Example: Ladders

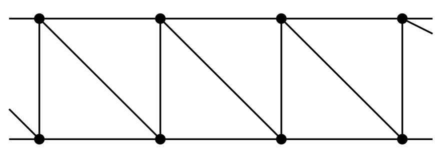
Ladder 1



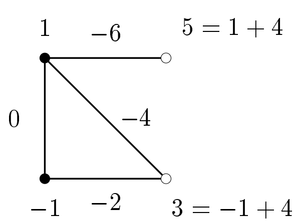
Ladder 1 has an edge-magic $\mathbb{Z}$-labelling with edge constant $k = 0$, represented by the following template, " + mod 7".



Ladder 2 (with "diagonal reinforcements")

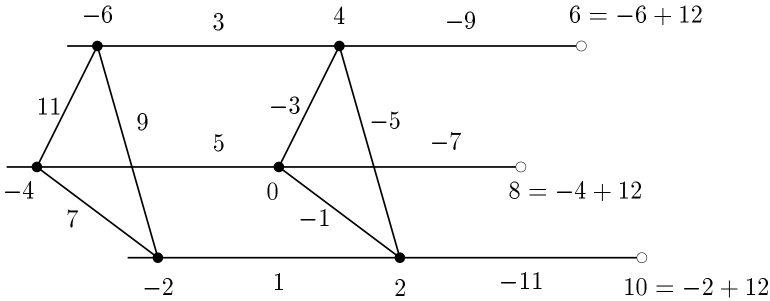


Ladder 2 has an edge-magic $\mathbb{Z}$-labelling with edge constant $k = 0$, represented by the following template, " + mod 4".



4.6 Triangular Prism

The diagram below is a template for an edge-magic labelling of a frieze graph based on a triangular prism. There are $\mathbf{v} = 6$ vertices, and $\mathbf{e} = 12$ edges. It uses congruence classes modulo 12 and has edge constant $k = 1$,

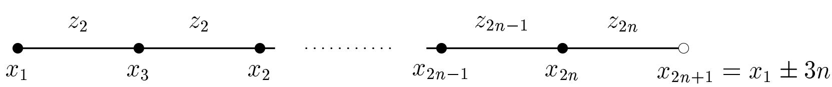


4.7 A non-existence result for modulus labellings.

In a graph with an edge-magic $\mathbb{Z}$-labelling with edge constant k there is not necessarily an edge-magic modulus labelling with edge constant k.

Theorem 13. *There is no modulus labelling of the infinite path with edge magic constant $k = 1$.*

Proof. Suppose we had a modulus labelling of the infinite path with edge magic constant $k = 1$. A template for this labelling would have an an even number of edges and hence an even number of consecutive vertices. Suppose the template has $\mathbf{v} = 2n$ vertices and hence $\mathbf{e} = 2n$ edges. Then the modulus class labelling has modulus $m = \mathbf{v} + \frac{1}{2}\mathbf{e} = 3n$. Suppose the template is



Let the sum of the vertex labels $x_1 + x_2 + \cdots + x_{2n} = X$ and the sum of the edge labels $z_1 + z_2 + \cdots + z_{2n} = Z$.

By assumption, for $i = 1, 2, \ldots, 2n$, each edge weight $x_i + z_i + x_{i+1} = 1$. Adding these relations gives

$$2X + Z \equiv 2n \pmod{3n}.$$

However, in a modulus class labelling, each congruence class occurs either once (as a vertex label) or twice (as an edge labels). Hence $X + \frac{1}{2}Z$ is congruent, modulo $3n$, to the sum of the distinct modulus classes, 0, 1, ..., $3n - 1$. This implies

$$X + \tfrac{1}{2}Z \equiv 0 + 1 + \cdots + 3n - 1 \equiv \frac{3n(3n-1)}{2} \pmod{3n}.$$

Therefore,

$$2X + Z \equiv 0 \pmod{3n}.$$

This contradicts the first displayed congruence above. Hence we cannot have a modulus labelling of the infinite line with edge magic constant $k = 1$.

□

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